CERTAIN SUBGROUPS OF THE HOMOTOPY GROUPS

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1. INTRODUCTION

In euclidean $(n+1)$-space E^{n+1}, let S^n be the unit sphere with center at the origin. Let W be the union of two unit spheres S_1^n and S_2^n with centers at $(0, \cdots, 0, 1)$ and $(0, \cdots, 0, -1)$, respectively. Let $t: S^n \to W$ be defined by the condition that $t(y)$ is to be the point of intersection of W with the ray from the origin through y. Let $y_0 = (1, 0, \cdots, 0)$. Let ρ_1 and ρ_2 be orientation-preserving affine transformations

$$\rho_1 : (S_1^n, \theta) \to (S^n, y_0),$$

$$\rho_2 : (S_n^n, \theta) \to (S^n, y_0),$$

where θ is the origin in E^{n+1}.

If x_0 is a point in a space X (the word 'space' will mean arcwise connected separable metric space), we let $F_n(X, x_0)$ be the set of all (continuous) mappings of the pair (S^n, y_0) into the pair (X, x_0). We define an operation on $F_n(X, x_0)$ in the following manner. Given f and g in $F_n(X, x_0)$, let $fg: (S^n, y_0) \to (X, x_0)$ be defined by the rule that

$$fg(y) = \begin{cases} f\rho_1 t(y) & \text{if } y \in t^{-1}(S_1^n), \\ g\rho_2 t(y) & \text{if } y \in t^{-1}(S_2^n). \end{cases}$$

If $\pi_n(X, x_0)$ is the set of homotopy classes of $F_n(X, x_0)$ obtained by using homotopies with fixed base point, then the operation on $F_n(X, x_0)$ induces an operation on $\pi_n(X, x_0)$ in such a manner that $\pi_n(X, x_0)$ becomes the nth homotopy group of X based at x_0. (Other operations on $F_n(X, x_0)$ would give the same group operation for $\pi_n(X, x_0)$, but we need a specific operation for later use in the paper.)

If additional requirements are put on the maps $f: (S^n, y_0) \to (X, x_0)$, then the subset of $F_n(X, x_0)$ so defined determines, under the natural map $\psi: F_n(X, x_0) \to \pi_n(X, x_0)$, a subset of $\pi_n(X, x_0)$. Such subsets need not be subgroups, but we shall see that they are, in many cases. This paper is devoted to a preliminary study of the subgroups which one obtains by using mappings that are restricted by conditions imposed on the inverse images of points. It is important to note that these subgroups, unlike the homotopy groups themselves, are not invariants of homotopy type, but are more sensitive to some set-theoretic properties.

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2. G-CLASSES OF MAPPINGS

Let ψ: $F_n(X, x_0) \rightarrow \pi_n(X, x_0)$ be the natural map. We say that a subset S of $F_n(X, x_0)$ is a G-*class* if $\psi(S)$ is a subgroup of $\pi_n(X, x_0)$. The special G-classes which we consider here can be described as follows. We define ϕ: $S^n \rightarrow S^n$ by letting $\phi(z_1, \cdots, z_n, z_{n+1}) = (z_1, \cdots, z_n, -z_{n+1})$. Then, a nonempty subset S of $F_n(X, x_0)$ is a G-class if it is closed under the operation on $F_n(X, x_0)$ and is closed under composition with ϕ.

We consider two types of G-classes.

Type I. A condition is imposed on each inverse image of a point.

Type II. A condition is imposed on all but a finite number of the inverse images of points.

It is easy to see what kind of conditions can be used to obtain G-classes of type II. In particular, let P be a property which is meaningful for closed subsets of S^n, and such that

(1) if two disjoint closed subsets A and B of S^n each have property P, then $A \cup B$ has property P; and

(2) if A is a closed subset of S^n such that A has property P, then $\phi(A)$ has property P.

We use such a property to define a G-class, by taking all maps f in $F_n(X, x_0)$ such that $f^{-1}(x)$ has property P except for a finite number of $x \in X$. We give three examples of such a property:

(i) A has property P if and only if $\text{Dim } A \leq k$.

(ii) A has property P if and only if $H_k(A) = 0$ (or $H_i(A) = 0$ for all $i \leq k$, or $H_i(A) = 0$ for all $i > k$.)

(iii) A has property P if and only if $\pi_k(A, a_0) = 0$ for each $a_0 \in A$.

We shall study example (i) in the following sections.

The kind of property Q which determines a G-class of Type I is a little more complicated. Let U and L be the upper and lower closed hemispheres of S^n, and let $S^{n-1} = U \cap L$ be the equator. Let t be the map of S^n defined in the Introduction. Let Q be a property which is meaningful for closed subsets of S^n, and which is such that:

(1) if $A \subset U$ and $B \subset L$ are closed sets, and t(A) and t(B) both have property Q, then $A \cup B \cup S^{n-1}$ has property Q; and

(2) if $A \subset S^n$ and A has property Q, then $\phi(A)$ has property Q.

We mention two examples of such a property:

(i) A has property Q if and only if $\text{Dim } A \geq k$;

(ii) A has property Q if and only if $H_k(A)$ is finitely generated.

The first example is clear, and it is proved in Section 5 that (ii) gives a property Q, with mild restrictions on the homology theory used.

3. THE k-LIGHT GROUPS

Definition. A map $f: Y \to W$ is said to be *k-light* if $\operatorname{Dim} f^{-1}(w) \leq k$ for all but a finite number of $w \in W$.

It follows from elementary properties of dimension that this is a property P and hence defines a G-class of Type II; this G-class determines a subgroup $D_n^k(X, x_0)$ of $\pi_n(X, x_0)$. We call this subgroup the k-light subgroup of $\pi_n(X, x_0)$. The following example shows that the concept is not vacuous.

Let X_1 be the space consisting of two disjoint 2-spheres joined by an arc having only its end points on the spheres, and let x_1 be one of the end points of the arc. Let X_2 be the space consisting of two 2-spheres with a single point x_2 in common. It is immediately clear that X_1 and X_2 have the same homotopy type and hence the same homotopy groups. Also, it is easily seen that

$$D_2^0(X_2, x_2) = \pi_2(X_2, x_2)$$

while

$$D_2^0(X_1, x_1) \neq \pi_2(X_1, x_1).$$

Let x' be an interior point of the arc in X_1. Then $D_2^0(X_1, x') = 0$, whereas $D_2^0(X_1, x_1)$ is infinite cyclic. The dependence of these groups on the base point is considered in the next section.

The following proposition shows that there does not exist an example in which $D_1^0(X, x_0) \neq \pi_1(X, x_0)$.

PROPOSITION. *For any space* X, $D_1^0(X, x_0) = \pi_1(X, x_0)$.

Proof. We must show that every homotopy class of $F_1(X, x_0)$ contains a 0-light map. The constant map is 0-light, and therefore we need to consider only nontrivial homotopy classes. Let $f: (S^1, y_0) \to (X, x_0)$ be an element of $F_1(X, x_0)$ which does not represent the identity element of $\pi_1(X, x_0)$. Let $f = hg$ be the monotone-light factorization of f. Then $g(S^1)$ is a 1-sphere, and we may assume without loss of generality that $g(S^1) = S^1$ and that g has degree 1. It follows that h is in the same homotopy class as f. Since h is light, and hence 0-light, the proposition is proved.

It seems intuitively clear from the definition that the k-light groups should vanish for indices higher than the dimension of X. The following proposition shows that this is indeed the case.

PROPOSITION. *If* X *is k-dimensional, then* $D_n^m(X, x_0) = 0$ *for* $n > m + k$.

Proof. Let $f: (S^n, y_0) \to (X, x_0)$ with $n > m + k$, and suppose that there are only a finite number of points $x_1, \cdots, x_p$ such that $\operatorname{Dim} f^{-1}(x_i) > m$. Then

$$U = S^n - \bigcup_{i=1}^{p} f^{-1}(x_i)$$

is an open set, and it is nonempty; for S^n is connected, and $[x_1, \cdots, x_p]$ is not connected unless $p = 1$ (in which case f is trivial). Take a nonempty open set V such that the closure of V is contained in U. Application of Theorem V 17 on page 91 of [3] to the map $f: \overline{V} \to X$ gives the existence of $x \in X$ such that $f^{-1}(x) \cap \overline{V}$ has dimension at least $n - k > m$. Hence $f^{-1}(x)$ has dimension greater than m, and x is not a point of the set $[x_1, \cdots, x_p]$. This shows that there are no essential m-light maps in $F_n(X, x_0)$, whence $D_n^m(X, x_0) = 0$, as was asserted.

The following theorem, together with the preceding proposition, implies that for manifolds the k-light groups give no new topological invariants.

THEOREM. *If* X *is a* k-*manifold, then* $D_n^m(X, x_0) = \pi_n(X, x_0)$ *for* $n \leq m + k$.

Proof. Since X is a manifold, it has the property that nearby maps are homotopic. More precisely, corresponding to each $f \in F_n(X, x_0)$ there is an $\varepsilon > 0$ such that if $g \in F_n(X, x_0)$ and $\rho(f(y), g(y)) < \varepsilon$ for all $y \in S^n$, then f and g represent the same element of $\pi_n(X, x_0)$. It is a consequence of a theorem in [2] that the m-light maps of S^n into X are dense in the space of maps of S^n into X if and only if $n \leq m + k$. These two statements combine to give the conclusion of the theorem.

The preceding theorem is false if X is required only to be an algebraic variety. For example, the space X consisting of the union of a 2-sphere and a 1-sphere which have a single common point x_0 is a real algebraic variety, and $D_2^0(X, x_0)$ is infinite cyclic, whereas $\pi_2(X, x_0)$ is not. It is hoped that these groups may be a useful vehicle for stating results about the types of singularities in algebraic varieties.

4. DEPENDENCE ON BASE POINT

If x_0 and x_1 are points in X, then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic, since X is arcwise connected. We have seen, however, that $D_n^m(X, x_0)$ and $D_n^m(X, x_1)$ need not be isomorphic unless some more restrictive assumptions are made about X. In this section we note that the obvious conditions are actually sufficient.

Definition. X is k-*lightly* n-*connected* if for any two points a and b of X there is a k-light map $f: S^n \to X$ such that $f(y) = a$ and $f(\phi(y)) = b$, where

$$y = (0, \cdots, 0, 1) \in E^{n+1}.$$

PROPOSITION. *If* X *is* n-*lightly* n-*connected, then* X *is* k-*lightly* n-*connected for* $k = 0, 1, \cdots, n$.

PROPOSITION. X *is* (n - 1)-*lightly* n-*connected if and only if it is arcwise connected.*

PROPOSITION. *If* X *is* k-*lightly* n-*connected, then* $D_n^k(X, x_0)$ *is independent of the base point* x_0.

The proofs of the three propositions above are trivial. The connectedness condition localizes in the obvious way, and one sees that arcwise connectedness together with the local property gives the property in the large. The somewhat natural conjecture that if X is arcwise connected and locally k-lightly n-connected, then $D_n^k(X, x_0) = \pi_n(X, x_0)$, is seen to be false by a simple example.

5. THE k-MONOTONE SUBGROUPS

Definition. A map $f: Y \to W$ is k-*monotone* if each group $H_k(f^{-1}(w))$ is finitely generated.

We require the following conditions on any homology theory used in the definition above:

(1) the homology theory satisfies the Eilenberg-Steenrod axioms,

(2) the coefficient group is finitely generated,

(3) the homology groups are invariant under relative homeomorphisms (see [1, p. 266]).

THEOREM. *The property of having a finitely generated* k-*dimensional homology group is a property* Q.

Proof. The condition (2) for property Q is automatic in this case, and we need only consider condition (1).

Let $f, g \in F_n(X, x_0)$, and let $A' = f^{-1}(x_0)$, $B' = g^{-1}(x_0)$. Then $A = t^{-1}(A')$ is in U, and $B = t^{-1}(B')$ is in L. By hypothesis, $H_k(A')$ and $H_k(B')$ are finitely generated. Since $t(A)$ and $t(B)$ have only one point in common, $H_k(t(A \cup B \cup S^{n-1}))$ is finitely generated. We must show that $H_k(A \cup B \cup S^{n-1})$ is finitely generated.

By the invariance under relative homeomorphisms,

$$H_k(t(A \cup B \cup S^{n-1}), t(S^{n-1})) \approx H_k(A \cup B, (A \cup B) \cap S^{n-1});$$

and since $t(S^{n-1})$ is a point, this gives

$$H_k(t(A \cup B \cup S^{n-1})) \approx H_k(A \cup B, (A \cup B) \cap S^{n-1}).$$

Now, by using the excision

$$(A \cup B, (A \cup B) \cap S^{n-1}) \to (A \cup B \cup S^{n-1}, S^{n-1}),$$

we obtain the relation

$$H_k(A \cup B, (A \cup B) \cap S^{n-1}) \approx H_k(A \cup B \cup S^{n-1}, S^{n-1}).$$

Finally, considering the exact homology sequence of the pair $(A \cup B \cup S^{n-1}, S^{n-1})$, and using the fact that $H_k(S^{n-1}) = 0$ except for $k = n - 1$, we obtain

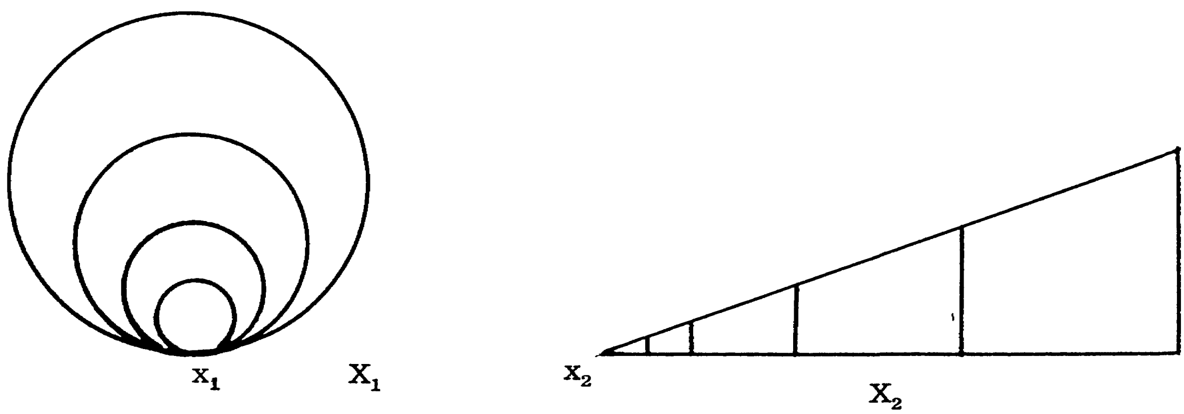
$$H_k(A \cup B \cup S^{n-1}) \approx H_k(t(A \cup B \cup S^{n-1})).$$

Also, $H_{n-1}(A \cup B \cup S^{n-1})$ is isomorphic with a subgroup of $H_n(A \cup B \cup S^{n-1}, S^{n-1})$; the latter group is isomorphic with $H_n(t(A \cup B \cup S^{n-1}))$, whence all $H_k(A \cup B \cup S^{n-1})$ are finitely generated. The theorem is proved.

COROLLARY. *The* k-*monotone maps constitute a* G-*class in* $F_n(X, x_0)$.

Definition. The subgroup determined by the G-class of k-monotone maps is called the k-*monotone subgroup* of $\pi_n(X, x_0)$, and it is denoted by $M_n^k(X, x_0)$.

Example. Let X_1 and X_2 be the one-dimensional spaces shown in the figure. (There are an infinite number of circles for X_1 and an infinite number of trapezoids



for X_2.) We can verify that $\pi_1(X_1, x_1)$ is uncountably generated, while $M_1^0(X_1, x_1)$ is countably generated. Both $\pi_1(X_2, x_2)$ and $M_1^0(X_2, x_2)$ are uncountably generated. Clearly, X_1 and X_2 have the same homotopy type.

The k-monotone subgroups are of interest only for somewhat pathological spaces, as the following theorem indicates.

THEOREM. *If* X *is a finite polyhedron, then* $M_n^k(X, x_0) = \pi_n(X, x_0)$.

Proof. By the simplicial approximation theorem, each f in $F_n(X, x_0)$ can be approximated by a simplicial map g which is homotopic to f. Each $g^{-1}(x)$ is a finite polyhedron, so that all of its homology groups are finitely generated.

Remark. Unlike the k-light groups, the k-monotone groups are independent of base point.

REFERENCES

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