THE n-CUBE AS A PRODUCT SEMIGROUP

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If a topological space S with a "boundary" is a semigroup with identity, then the boundary seems to play a special role. For example, if S is a compact manifold with boundary, then the identity and all elements with inverses lie on the boundary [1], [3]. If in addition the boundary is a connected Lie group, then the multiplication in S may be described by the introduction of generalized "polar coordinates," that is, a unit interval coordinate and a boundary coordinate [2]. The importance of the boundary was first pointed out to us in conversations with A. D. Wallace.

In this note we show that if the boundary of a semigroup S on the n-cell is isomorphic to the boundary of a product semigroup, then S is a product semigroup.

By a *semigroup* S we mean a topological semigroup, that is, a Hausdorff space with a continuous associative multiplication. If A and B are subsets of S, then AB denotes the set of all products ab with $a \in A$ and $b \in B$. An ideal is a subset A such that $AS \subset A$ and $SA \subset A$.

An (I)-*semigroup* is a semigroup on the unit interval in which 0 is a zero, $0x = x0 = 0$ for all x, and 1 is an identity. Such a semigroup must be abelian [4]; but there are infinitely many nonisomorphic (I)-semigroups. A classification is given in [2].

$A \setminus B$ denotes set-theoretic difference.

We are now ready to state our main result.

THEOREM. *Let* S *be a semigroup on the* n-*cell* $(n > 1)$, *with boundary* B. *Let* $J_1, \cdots, J_n$ *be* (I)-*semigroups, and let* $T = J_1 \times \cdots \times J_n$ *be the product semigroup on the* n-*cell with boundary* C. *If* ϕ *is a homeomorphism of* C *onto* B *such that* $\phi(c)\phi(d) = \phi(cd)$ *whenever* c,d *and* cd *are in* C, *then* S *is isomorphic to* T.

In other words, if the boundaries are isomorphic, then the isomorphism may be extended throughout the interiors.

Proof. The points of T may be represented by coordinates $(x_1, \cdots, x_n)$ $(0 \leq x_i \leq 1)$, with the boundary consisting of those points which have 0 or 1 for at least one coordinate. It follows that $(0, \cdots, 0)$ is the zero for T and $(1, \cdots, 1)$ is the identity. Let $0 = \phi(0, \cdots, 0)$, $1 = \phi(1, \cdots, 1)$; these points are respectively a zero and an identity for B.

(i) If Q is any arc in B from 0 to 1, then $QB = S$. Indeed, assume that there is a $z \in S$ such that $z \notin QB$. Then $z \notin qB$ for each $q \in Q$, and the index ("winding number") of the mapping qB with respect to the point z is defined for each $q \in Q$ and is therefore constant. But for $q = 1$ the index is one, and for $q = 0$ the index is zero; the contradiction shows that $QB = S$.

In C, let $I_1 = (x_1, 1, \cdots, 1)$ $(0 \leq x_1 \leq 1)$, $I_2 = (1, x_2, 1, \cdots, 1)$ $(0 \leq x_2 \leq 1)$, and so forth. The sets I_k are those edges of T that have $(1, \cdots, 1)$ as a vertex, and $T = I_1 I_2 \cdots I_n$. Let C_1 be the set of those points of C that have at least one coordinate zero. Then C_1 is an ideal in T and therefore in C. Let $J_i = \phi(I_i)$ and $B_1 = \phi(C_1)$. Then the J_i are (I)-semigroups and B_1 is an ideal in B.

Received November 12, 1956.

(ii) We now show that $J_1 J_2 \cdots J_n = S$ in a one-to-one manner, that is to say, that if $j_1 j_2 \cdots j_n = k_1 k_2 \cdots k_n$ $(j_i, k_i \in J_i)$, then $j_i = k_i$ $(i = 1, 2, \cdots, n)$. Let Q be an arc from 1 to 0 which follows along J_1 to B_1 and then stays in B_1. Let $Q_1 = Q \setminus J_1$, so that $Q_1 \subset B_1$. Now $S = QB$, by paragraph (i); but $Q_1 B \subset B_1$, and therefore $S = J_1 B$. In C one verifies easily that $I_1(C \setminus I_2 I_3 \cdots I_n) \subset C$; the mapping ϕ carries this over to B, that is, $J_1(B \setminus J_2 J_3 \cdots J_n) \subset B$. Therefore $S = J_1 J_2 \cdots J_n$.

Assume that $x_i, y_i \in J_i$ $(i = 1, \cdots, n)$, and that

$$x_1 \cdots x_n = y_1 \cdots y_n \,. \tag{1}$$

Let u be the endpoint of J_1 which is a zero for J_1 $(u = \phi(0, 1, \cdots, 1))$. Then $ux_1 = u = uy_1$, and $u \in B_1$. Multiply (1) on the left by u; then

$$ux_2 \cdots x_n = uy_2 \cdots y_n \,. \tag{2}$$

In C, the product $I_2 \cdots I_n$ is one-to-one and is contained in C, and the product $(0, 1, \cdots, 1)I_2 \cdots I_n$ is one-to-one. Therefore this carries over to B, and from (2) we deduce that $x_i = y_i$ $(i = 2, 3, \cdots, n)$.

Similarly, if we multiply (1) on the right by v, the zero of J_n, we get the relations $x_i = y_i$ $(i = 1, 2, \cdots, n - 1)$; in particular, $x_1 = y_1$.

(iii) The elements of J_i, J_k commute with each other for every i, k. The proof will be given for J_1, J_2. If $n > 2$, the result is immediate, for then $I_1 I_2 \subset C$; in C the elements commute, and so this carries over to B. If $n = 2$, then by paragraph (ii), $J_1 J_2 = S$. Similarly, $J_2 J_1 = S$. Let $j_i \in J_i$ $(i = 1, 2)$ be given, and let $t_i \in J_i$ $(i = 1, 2)$ be such that

$$j_1 j_2 = t_2 t_1 \,. \tag{3}$$

Let $u = \phi(0, 1)$ be that endpoint of J_1 which is a zero for J_1. Then $uj_1 = ut_1 = u$. Also, $u \in B_1$ and therefore u commutes with all elements of B. Multiplying (3) by u, we deduce that $uj_2 = ut_2$. But in C the product $(0, 1)I_2$ is one-to-one, and therefore $j_2 = t_2$. Similarly $j_1 = t_1$, and therefore $j_1 j_2 = j_2 j_1$.

By combining paragraphs (ii) and (iii), we are able to introduce coordinates $(x_1, \cdots, x_n)$ $(x_i \in J_i)$ in S, and the multiplication is coordinatewise; this completes the proof.

REFERENCES

1. P. S. Mostert and A. L. Shields, *Semigroups with identity on a manifold*, to appear.

2. ———, *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. (2) 65 (1957), 117-144.

3. A. D. Wallace, *Inverses in euclidean mobs*, Math. J. Okayama Univ. 3 (1953), 23-28.

4. W. M. Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. 6 (1955), 741-747.

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