Michigan Math. J. 50 (2002)

Lagrangian Helicoids

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1. Introduction

In 1744, Euler showed that a catenoid is a minimal surface; in 1766, Meusnier showed that a right helicoid is a minimal surface. The converse—that the catenoid is the only surface of revolution that is minimal—is due to Meusnier in 1785 (see e.g. [4]). That the helicoid is the only ruled minimal surface (aside from the plane) was proved by Catalan in 1842. The reader is also referred to the historical paragraphs in [9, pp. 207–208]). A remarkable feature of these surfaces is that they are locally isometric. In fact, one can easily construct a family of minimal surfaces depending on a parameter λ such that $\lambda = 0$ is a helicoid and $\lambda = 1$ is a catenoid (see e.g. [3, pp. 213, 221–224]).

In [1] the author showed that a conformally flat, minimal hypersurface M^n ($n \geq 4$) of Euclidean space $\mathbb{R}^{n+1}$ is either totally geodesic or a hypersurface of revolution $S^{n-1} \times \gamma(s)$, where the profile curve γ is determined by its curvature as a function of arc length by $\kappa = (1-n)/\nu^n$ and $s = \int\big(\nu^{n-1}/\sqrt{A\nu^{2n-2}-1}\big)\,d\nu$ (here A is a constant). For hypersurfaces of dimension $n \geq 4$ in Euclidean space, conformal flatness implies quasi-umbilicity and hence is a natural generalization of a surface of revolution. For $n = 3$, replacing conformal flatness by quasi-umbilicity gives the same result with the same proof. For $n = 2$, the profile curve is a catenary and hence these hypersurfaces are called *generalized catenoids*. Jagy [12] gave an independent study of this question by assuming that the minimal hypersurface is foliated by spheres from the outset.

In [2], Vanstone and the author showed that a complete minimal hypersurface M^n of $\mathbb{R}^{n+1}$ that admits a codimension-1 foliation by Euclidean $(n-1)$-spaces is either totally geodesic or a product $M^2 \times \mathbb{R}^{n-2}$, where M^2 is a helicoid in $\mathbb{R}^3$. In contrast to the classical case, this generalized helicoid is not locally isometric to the generalized catenoid just described because such an isometric deformation would preserve the conformal flatness.

In [5], Castro and Urbano (see also Castro [4]) introduced the Lagrangian catenoid. The manifold itself was introduced by Harvey and Lawson [11] as an example of a minimal Lagrangian submanifold and is defined by

$$M_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{C}^n : \\ |x|y = |y|x,\ \Im(|x| + i|y|)^n = 1,\ |y| < |x|\tan(\pi/n)\}.$$

Received February 27, 2001. Revision received October 11, 2001.
Research supported in part by NSF Grant no. INT-9903302.

Topologically, M_0 is $\mathbb{R} \times S^{n-1}$. To describe it precisely, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and view a point $p \in S^{n-1}$ as an n-tuple in $\mathbb{R}^n$ giving its coordinates. Define a map $\phi_0 \colon \mathbb{R} \times S^{n-1} \to \mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ by

$$\phi_o(u, p) = \cosh^{1/n}(nu) e^{i\beta(u)} p,$$

where $\beta(u) = \frac{\pi}{2n} - \frac{2}{n} \arctan\big(\tanh \frac{nu}{2}\big) \in \big(0, \frac{\pi}{n}\big)$ and the multiplication $e^{i\beta}p$ multiplies each coordinate of p by $e^{i\beta}$ and lists the real and imaginary parts as a $2n$-tuple in $\mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$. Let g_0 be the standard metric of constant curvature 1 on S^{n-1}; then the metric induced on $\mathbb{R} \times S^{n-1}$ by ϕ_0 is

$$ds^2 = \cosh^{2/n}(nu)(du^2 + g_0), \tag{1.1}$$

which clearly is conformally flat. This Lagrangian submanifold defined by the mapping $\phi_0 \colon \mathbb{R} \times S^{n-1} \to \mathbb{C}^n$ together with its induced metric (1.1) is known as the *Lagrangian catenoid*. The main result of [5] is the following.

THEOREM. *Let $\phi \colon M^n \to \mathbb{C}^n$ be a minimal* (*nonflat*) *Lagrangian immersion. Then M^n is foliated by pieces of round $(n-1)$-spheres in $\mathbb{C}^n$ if and only if ϕ is congruent* (*up to dilations*) *to an open subset of the Lagrangian catenoid.*

For future use we give the coordinate expression for the case $n = 2$. Writing p as $(\cos\theta, \sin\theta)$, we have

$$\phi_o(u, p) = \sqrt{\cosh 2u}\,(\cos\beta\cos\theta, \cos\beta\sin\theta, \sin\beta\cos\theta, \sin\beta\sin\theta).$$

Now $\beta = \frac{\pi}{4} - \arctan(\tanh u)$ and

$$\cos\beta = \frac{\sqrt{2}}{2}\frac{1+\tanh u}{\sqrt{1+\tanh^2 u}}, \quad \sin\beta = \frac{\sqrt{2}}{2}\frac{1-\tanh u}{\sqrt{1+\tanh^2 u}},$$
$$\frac{\sqrt{\cosh 2u}}{\sqrt{1+\tanh^2 u}} = \cosh u,$$

giving

$$\phi_o(u, p) = \left(\frac{e^u}{\sqrt{2}}\cos\theta, \frac{e^u}{\sqrt{2}}\sin\theta, \frac{e^{-u}}{\sqrt{2}}\cos\theta, \frac{e^{-u}}{\sqrt{2}}\sin\theta\right). \tag{1.2}$$

In view of the conformal flatness of the Lagrangian catenoid and the author's result on conformally flat minimal hypersurfaces in $\mathbb{R}^{n+1}$, it is natural to ask what are the conformally flat minimal Lagrangian submanifolds in $\mathbb{C}^n$. This seems to be a difficult question and is a continuing effort of A. Carriazo and the author. We will see, however, that the only minimal Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by pieces of $(n-1)$-planes are pieces of n-planes. Thus we drop the minimality and study Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by Euclidean $(n-1)$-planes.

The program in this paper will be the following five points.

THEOREM 1. *Let M^n be a complete Lagrangian submanifold of $\mathbb{C}^n$ that is foliated by $(n-1)$-planes. Then M^n is either totally geodesic, flat H-umbilical, or the* *product of a ruled (foliated by lines) Lagrangian surface in* $\mathbb{C}^2$ *and a Lagrangian* $(n-2)$*-plane in* $\mathbb{C}^{n-2}$.

Flat H-umbilical Lagrangian submanifolds of $\mathbb{C}^n$ were completely classified by Chen in [8] and they are foliated by $(n-1)$-planes. Thus our main point here is to consider the case $n = 2$ in detail.

THEOREM 2. *Let* M^2 *be a nonflat Lagrangian submanifold in* $\mathbb{C}^2$ *that is foliated by lines. Then there exist local coordinates* (t, x) *such that the induced metric takes the form* $ds^2 = f^2 dt^2 + dx^2$, *where* f^2 *is a positive function that is quadratic in* x. *The Weingarten maps* A_1 *and* A_2 *corresponding to the normals* $\zeta_1 = \frac{1}{f} J \frac{\partial}{\partial t}$ *and* $\zeta_2 = J \frac{\partial}{\partial x}$ *are given by* $A_1 = \left(\begin{smallmatrix} b & a \\ a & 0 \end{smallmatrix}\right)$ *and* $A_2 = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, *where* $a = A(t)/f^2$, $-4A(t)^2$ *is the discriminant of* f^2, *and*

$$b = \frac{1}{f}\left[\int \frac{A'(t) f^2 - A(t)(f^2)_t}{f^4}\, dx + B(t)\right]$$

for some function $B(t)$. *Conversely, let* M^2 *be a simply connected domain in the* (t, x)*-plane and let*

$$f^2 = F(t)x^2 + G(t)x + H(t)$$

be a positive quadratic function on M^2. *Then there exists an isometric Lagrangian immersion of* M^2 *into* $\mathbb{C}^2$ *that is foliated by line segments whose first and second fundamental forms are as given in this theorem.*

We now suppose there is a 1-parameter family of Lagrangian surfaces in $\mathbb{C}^2$ connecting a ruled Lagrangian surface to the Lagrangian catenoid. Let $(v^1 + iv^2, v^3 + iv^4)$ denote the coordinates on $\mathbb{C}^2$ and let $\mathbf{v}$ be the mapping $\mathbf{v}\colon M^2 \to \mathbb{C}^2$ given by $v^i = v^i(t, x)$.

THEOREM 3. *Suppose there exists a* 1*-parameter family of Lagrangian surfaces in* $\mathbb{C}^2$ *connecting a ruled Lagrangian surface* M^2 *to the Lagrangian catenoid. Then* M^2 *is given by*

$$v^1 = k(\cos t)x + \beta_1(t), \quad v^2 = l(\cos t)x + \beta_2(t),$$
$$v^3 = k(\sin t)x + \beta_3(t), \quad v^4 = l(\sin t)x + \beta_4(t),$$

where k *and* l *are constants satisfying* $k^2 + l^2 = 1$. *The quadratic then becomes* $x^2 + G(t)x + H(t)$, *and the* β_i *are determined by*

$$\beta_1' = -\left(\frac{kG(t)}{2} + lA(t)\right)\sin t, \quad \beta_2' = \left(-\frac{lG(t)}{2} + kA(t)\right)\sin t,$$
$$\beta_3' = \left(\frac{kG(t)}{2} + lA(t)\right)\cos t, \quad \beta_4' = \left(\frac{lG(t)}{2} - kA(t)\right)\cos t,$$

where $-4A(t)^2 = G(t)^2 - 4H(t)$.

We call a surface given as in Theorem 3 a *Lagrangian helicoid*.

Our next result is that, even though the Lagrangian helicoids can be connected to a Lagrangain catenoid through a family of Lagrangian surfaces, the Lagrangian submanifolds of Theorem 1 cannot be locally isometric to a Lagrangian catenoid.

THEOREM 4. *Let M^n be a Lagrangian submanifold of $\mathbb{C}^n$ that is foliated by $(n-1)$-planes. Then M^n is not locally isometric to a Lagrangian catenoid.*

Finally, one can raise the question of whether it is possible to have Lagrangian submanifolds of $\mathbb{C}^n$ that are "doubly ruled"—that is, for which there exist two foliations by $(n-1)$-planes. The answer is negative except for the totally geodesic case.

PROPOSITION 5. *If a Lagrangian submanifold M^n of $\mathbb{C}^n$ admits two foliations by $(n-1)$-planes, then M^n is totally geodesic.*

2. Preliminaries

If $(v^1 + iv^2, \ldots, v^{2n-1} + iv^{2n})$ are the coordinates on $\mathbb{C}^n$, then an n-dimensional submanifold M^n of $\mathbb{C}^n$ is said to be *Lagrangian* if the restriction of the canonical symplectic form $\Omega = \sum_{i=1}^n dv^{2i-1} \wedge dv^{2i}$ to M^n vanishes.

For an isometrically immersed submanifold (M, g) of $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$, the Levi–Civita connection ∇ of g and the second fundamental form σ are related to the ambient Levi–Civita connection $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{2.1}$$

For a normal vector field ν, we denote by A_ν the corresponding Weingarten map and we denote by D the connection in the normal bundle; in particular, A_ν and D are defined by

$$\bar{\nabla}_X \nu = -A_\nu X + D_X \nu.$$

The Gauss equation is

$$R(X, Y, Z, W) = \langle\sigma(Y, Z), \sigma(X, W)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle.$$

Defining the covariant derivative of σ by

$$(\nabla'\sigma)(X, Y, Z) = D_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

the Codazzi equation is

$$(R_{XY}Z)^\perp = (\nabla'\sigma)(X, Y, Z) - (\nabla'\sigma)(Y, X, Z).$$

For a Lagrangian submanifold, the almost complex structure J of the Kähler manifold $\mathbb{C}^n$ is an isometry between the tangent bundle and the normal bundle and hence the equation of Ricci–Kühn, $R^\perp(X, Y, \nu, \zeta) = g([A_\nu, A_\zeta]X, Y)$, gives no further information.

A non–totally geodesic Lagrangian submanifold of a Kähler manifold is said to be *Lagrangian H-umbilical* ([6] or [9, p. 333]) if there exist functions λ and μ and a local orthonormal basis $\{e_1, \ldots, e_n\}$ with respect to which the second fundamental form is given by

$$\sigma(e_1, e_1) = \lambda J e_1,\ \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1,$$

$$\sigma(e_1, e_j) = \mu J e_j,\ \sigma(e_j, e_k) = 0, \quad j \neq k,\ \ j, k = 2, \ldots, n.$$

In [8], Chen gave a classification of flat H-umbilical Lagrangian submanifolds of $\mathbb{C}^n$ for which we need some ideas about Legendre curves. Let $z\colon I \to S^{2n-1} \subset \mathbb{C}^n$ be a unit-speed Legendre curve in the unit hypersphere centered at the origin; that is, $z = z(s)$ is a unit-speed curve in S^{2n-1} satisfying $\langle z'(s), iz(s)\rangle = 0$. Moreover, since $z(s)$ is spherical, it follows that $\langle z(s), z'(s)\rangle = 0$. Thus $z(s), iz(s), z'(s), iz'(s)$ are orthonormal vector fields defined along the Legendre curve, and we can extend to a full orthonormal frame field along the curve by normal fields $P_3(s), iP_3(s), \dots, P_n(s), iP_n(s)$. The Legendre curve is called a *special Legendre curve* if the P_j are parallel along the curve and if

$$z''(s) = i\lambda(s)z'(s) - z(s) - \sum_{j=3}^{n} a_j(s)P_j(s)$$

for some functions $\lambda, a_3, \dots, a_n$. We can now state the classification of Chen.

Theorem. *Let $\lambda, b, a_3, \dots, a_n$ be n functions defined on an open interval I with $n \geq 2$ and λ nowhere zero, and let $z\colon I \to S^{2n-1} \subset \mathbb{C}^n$ be a special Legendre curve. Set*

$$f(t, u_2, \dots, u_n) = b(t) + u_2 + \sum_{j=3}^{n} a_j(t)u_j.$$

Let M^n be the product manifold $I \times \mathbb{R}^{n-1}$ with the twisted product metric $g = f^2dt^2 + du_2^2 + \cdots + du_n^2$. Then M^n is a flat Riemannian manifold and

$$L(t, u_2, \dots, u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j P_j(t) + \int^t b(t)z'(t)\,dt \tag{2.2}$$

defines a Lagrangian H-umbilical isometric immersion of M^n into $\mathbb{C}^n$.

Conversely, up to rigid motions of $\mathbb{C}^n$, locally every flat Lagrangian H-umbilical submanifold in $\mathbb{C}^n$ without totally geodesic points is either a Lagrangian cylinder over a curve or a Lagrangian submanifold obtained in the manner just described.

We also recall the following existence theorem of Chen (see e.g. [7, p. 292]) for Lagrangian isometric immersions into complex space forms.

Theorem. *Let M^n be a simply connected Riemannian manifold. Let α be a TM-valued symmetric bilinear form on M^n satisfying:*

(1) $g(\alpha(X, Y), Z)$ *is totally symmetric*;
(2) $(\nabla\alpha)(X, Y, Z) = \nabla_X\alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$ *is totally symmetric*;
(3) $R_{XY}Z = \alpha(\alpha(Y, Z), X) - \alpha(\alpha(X, Z), Y)$.

Then there exists a Lagrangian isometric immersion of M^n into $\mathbb{C}^n$ whose second fundamental form is given by $\sigma(X, Y) = J\alpha(X, Y)$.

Finally, we recall a reduction theorem of Erbacher [10] (see also [9, pp. 206–207]).

Theorem. *Let M^n be a submanifold of a complete simply connected Riemannian manifold $\tilde{M}^m(c)$ of constant curvature c. If there exists a normal subbundle E*

of rank l that is parallel in the normal bundle and if the first normal space (span of the second fundamental form) at each point $x \in M^n$ is contained in E_x, then M^n is contained in an $(n+l)$-dimensional totally geodesic submanifold of $\tilde{M}^m(c)$.

3. Proof of Theorem 1

Since the given submanifold M^n is foliated by $(n-1)$-planes, there exists a unit vector field U tangent to M^n and orthogonal to the leaves of the foliation. The foliation is by Euclidean spaces and hence we can complete U to an orthonormal basis by vector fields $e_2, \dots, e_n$ such that $\bar{\nabla}_{e_i} e_j = 0$; moreover, we may choose local coordinates $(t, x^2, \dots, x^n)$ such that $e_i = \frac{\partial}{\partial x^i}$ and $U = \frac{1}{f}\frac{\partial}{\partial t}$, $f > 0$. Let $p = -\ln f$. Then $[U, e_i] = -p_i U$, where $p_i = \frac{\partial p}{\partial x^i}$. With respect to these coordinates, the induced metric on M^n is given by

$$ds^2 = f^2 dt^2 + (dx^2)^2 + \cdots + (dx^n)^2.$$

From this form of the metric we have immediately that

$$\nabla_U U = \sum_{i=2}^{n} p_i e_i, \quad \nabla_{e_i} U = 0, \quad \nabla_U e_i = -p_i U, \quad \nabla_{e_i} e_j = 0. \tag{3.1}$$

Introduce local normal fields by $\zeta_1 = JU$ and $\zeta_i = Je_i$, and denote the corresponding Weingarten maps by A_1 and A_i. Then

$$-A_1 X + D_X \zeta_1 = \bar{\nabla}_X \zeta_1 = J\bar{\nabla}_X U = J(\nabla_X U + \sigma(X, U)).$$

Now, taking X equal to U and (respectively) e_i, we obtain

$$A_1 U = -J\sigma(U,U), \quad D_U \zeta_1 = \sum_{i=2}^{n} p_i \zeta_i, \quad A_1 e_i = -J\sigma(e_i, U), \quad D_{e_i}\zeta_1 = 0.$$

Similarly,

$$-A_j U + D_U \zeta_j = \bar{\nabla}_U \zeta_j = J\bar{\nabla}_U e_j = J(-p_j U + \sigma(U, e_j)),$$

whence

$$A_j U = -J\sigma(U, e_j) = A_1 e_j \quad \text{and} \quad D_U \zeta_j = -p_j \zeta_1.$$

Finally,

$$-A_j e_i + D_{e_i} \zeta_j = \bar{\nabla}_{e_i} \zeta_j = J\bar{\nabla}_{e_i} e_j = 0,$$

giving

$$A_j e_i = 0 \quad \text{and} \quad D_{e_i} \zeta_j = 0.$$

In particular, we see that the matrices of the Weingarten maps are of the form

$$A_1 = \begin{pmatrix} b & a & \cdots & a \\ a & & & \\ \vdots & & 0 & \\ a & & & \end{pmatrix}, \qquad A_j = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}. \tag{3.2}$$

We pause to remark that one now readily obtains the result mentioned in the introduction: that the only minimal Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by pieces of $(n-1)$-planes are pieces of n-planes (i.e., the totally geodesic case).

We now utilize the Codazzi equation: For any index $k \geq 1$,

$$\nabla_X A_k Y - A_k \nabla_X Y + \sum_{l=1}^{n} \langle D_X \zeta_l, \zeta_k \rangle A_l Y = \nabla_Y A_k X - A_k \nabla_Y X + \sum_{l=1}^{n} \langle D_Y \zeta_l, \zeta_k \rangle A_l X.$$

Applying this to U and the e_i $(i \geq 2)$, for indices $i, j \geq 2$ we have

$$\nabla_{e_i} A_1 U = \nabla_U A_1 e_i + A_1(p_i U), \tag{3.3}$$

$$\nabla_{e_i} A_j U = A_j(p_i U) + p_j A_1 e_i, \tag{3.4}$$

$$\nabla_{e_i} A_1 e_j = \nabla_{e_j} A_1 e_i. \tag{3.5}$$

Using the matrices in (3.2) for the Weingarten maps, these three equations imply the following, which we will need both here and in our later work. Equation (3.5) implies $e_i a = e_j a$ for $i, j \geq 2$. Similarly, equation (3.4) implies $e_i a = a(p_i + p_k)$ for $i, k \geq 2$. From these we have

$$e_i a = 2 a p_i \tag{3.6}$$

and hence either

$$p_2 = \cdots = p_n \quad \text{or} \quad a = 0.$$

In the same manner, equation (3.3) implies

$$e_i b = U a + b p_i. \tag{3.7}$$

We deal first with the case $p_2 = \cdots = p_n$. Let

$$V = \frac{1}{\sqrt{n-1}} \sum_{j=2}^{n} e_j \quad \text{and} \quad W_k = \frac{1}{\sqrt{2}}(e_k - e_2), \; k = 3, \ldots, n.$$

Then

$$\nabla_U V = -\sqrt{n-1}\, p_2 U \quad \text{and} \quad \nabla_V U = 0$$

and hence $[U, V] = -\sqrt{n-1}\, p_2 U$. We also have $[W_k, W_l] = 0$. Therefore, M^n is locally a product $M^2 \times M^{n-2}$. Moreover,

$$\nabla_U U = \sum_{i=2}^{n} p_i e_i = p_2 \sqrt{n-1}\, V \quad \text{and} \quad \nabla_V V = 0;$$

hence M^2 is totally geodesic in M^n. Also, since $\bar{\nabla}_{W_k} W_l = 0$, it follows that M^{n-2} is totally geodesic in $\mathbb{C}^n$ and thus (by completeness) M^{n-2} is Euclidean $(n-2)$-space.

Now consider M^2 as a submanifold in $\mathbb{C}^n$ as well as the bundle over M^2 spanned by ζ_1 and $\nu = JV$. Let D' and σ' denote the connection in the normal bundle and the second fundamental form of M^2 in $\mathbb{C}^n$. We have easily that

$$D'_U \zeta_1 = p_2 \sqrt{n-1}\, \nu, \quad D'_U \nu = -p_2 \sqrt{n-1}\, \zeta_1, \quad D'_V \zeta_1 = 0, \quad D'_U \nu = 0.$$

Thus the bundle $\{\zeta_1, \nu\}$ is a parallel subbundle of the normal bundle of M^2 in $\mathbb{C}^n$. Furthermore,

$$\langle \sigma'(X,Y), JW_k\rangle = -\langle \bar{\nabla}_X JW_k, Y\rangle = \frac{-1}{\sqrt{2}}\langle \bar{\nabla}_X Je_k - \bar{\nabla}_X Je_2, Y\rangle$$
$$= \frac{-1}{\sqrt{2}}\langle A_k X - A_2 X, Y\rangle = 0.$$

This, together with the fact that M^2 is totally geodesic in M^n, implies that the first normal space of M^2 in $\mathbb{C}^n$ is contained in the bundle $\{\zeta_1, \nu\}$. Thus, by the theorem of Erbacher [10] stated in Section 2, M^2 lies in some $\mathbb{C}^2$ with the $\mathbb{C}^2$ being a totally geodesic submanifold of $\mathbb{C}^n$.

If $a = 0$, then the second fundamental form simplifies to

$$\sigma(U,U) = b\zeta_1, \quad \sigma(U, e_i) = 0, \quad \sigma(e_i, e_j) = 0;$$

in particular, M^n is H-umbilical in $\mathbb{C}^n$. Moreover, the Gauss equation gives immediately that M^n is flat. The result now follows from the classification of flat H-umbilical Lagrangian submanifolds given in [8] and stated in Section 2. Note that equation (2.2) is linear in the u_i and so flat H-umbilical Lagrangian submanifolds of $\mathbb{C}^n$ are foliated by $(n-1)$-planes.

4. Proof of Theorem 2

Here we consider a ruled Lagrangian surface M^2 in $\mathbb{C}^2$. In light of the foregoing development we have local coordinates (t, x), and the information from the Codazzi equations (3.6) and (3.7) becomes

$$\frac{\partial a}{\partial x} = 2a\frac{\partial p}{\partial x}, \qquad \frac{\partial b}{\partial x} = e^p\frac{\partial a}{\partial t} + b\frac{\partial p}{\partial x}.$$

Moreover, computing both sides of the Gauss equation

$$g(R_{e_2 U}U, e_2) = \langle \sigma(U,U), \sigma(e_2, e_2)\rangle - |\sigma(U, e_2)|^2$$

yields

$$e_2 p_2 - (p_2)^2 = -a^2.$$

Returning to the function $f = e^{-p}$ and denoting differentiation by subscripts, our equations become

$$a_x = -\frac{2a}{f}f_x, \quad b_x = \frac{1}{f}a_t - \frac{b}{f}f_x, \quad f_{xx} = fa^2. \tag{4.1}$$

Differentiating the third and using the first gives

$$f_{xxx} = -3\frac{f_{xx}}{f}f_x \quad \text{or} \quad (ff_{xx} + f_x^2)_x = 0.$$

Therefore

$$(ff_x)_x = ff_{xx} + f_x^2 = F(t)$$

and, in turn,

$$\left(\tfrac{1}{2}f^2\right)_x = ff_x = F(t)x + \tfrac{1}{2}G(t), \qquad f^2 = F(t)x^2 + G(t)x + H(t).$$

Writing the first equation of (4.1) as $(\ln a + \ln f^2)_x = 0$, we see that a is of the form $a = A(t)/f^2$. Since we began by differentiating the third equation of (4.1), some information was lost and we return to $ff_{xx} + f_x^2 = F(t)$:

$$a^2 = \frac{f_{xx}}{f} = \frac{F}{f^2} - \frac{f_x^2}{f^2} = \frac{4FH - G^2}{4f^4};$$

hence $-4A(t)^2$ is the discriminant of f^2. Writing the second equation of (4.1) as $(bf)_x = a_t = A'(t)/f^2 - (A/f^4)(f^2)_t$ and then integrating, we have

$$b = \frac{1}{f}\left[\int \frac{A'(t)f^2 - A(t)(f^2)_t}{f^4}\,dx + B(t)\right], \tag{4.2}$$

where the integral is elementary and can be computed as desired.

Conversely, if M^2 is a simply connected domain in the (t, x)-plane and if

$$f^2 = F(t)x^2 + G(t)x + H(t)$$

is a positive quadratic function on M^2, let $a = A(t)/f^2$ (where $4A^2 = 4FH - G^2$) and let b be a function of the form given by (4.2). Then a, b, and f satisfy the Gauss and Codazzi equations (4.1). Set $U = \frac{1}{f}\frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial x}$ and, with respect to this basis, define linear transformation fields on M^2 by

$$A_1 = \begin{pmatrix} b & a \\ a & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Now define a tangent bundle–valued symmetric bilinear form by

$$\alpha(X, Y) = g(A_1X, Y)U + g(A_2X, Y)e_2.$$

It is easy to check that $\langle \alpha(X, Y), Z\rangle$ is symmetric in X, Y, Z. Using the first two equations of (4.1) (Codazzi equations), direct computation shows that the second condition of Chen's existence theorem is satisfied. Similarly, using the third equation of (4.1) (Gauss equation), direct computation shows that the third condition of Chen's existence theorem is satisfied. Thus, by the Chen existence theorem (see [7] or Section 2), there exists a Lagrangian isometric immersion of M^2 with metric $ds^2 = f^2dt^2 + dx^2$ into $\mathbb{C}^2$ whose second fundamental form is given by $J\alpha(X, Y)$.

5. Geometry of Ruled Lagrangian Surfaces and Proof of Theorem 3

We now discuss the geometry of ruled Lagrangian surfaces in $\mathbb{C}^2$ in more detail. Let $(v^1 + iv^2, v^3 + iv^4)$ be the coordinates of $\mathbb{C}^2$; we study the mapping $\mathbf{v}\colon M^2 \to \mathbb{C}^2$ given by $v^i = v^i(t, x)$ and adopt the notation $\partial_t = \frac{\partial v^i}{\partial t}\partial_{v^i}$, et cetera. Since x is the coordinate along the rulings, we have

$$0 = \bar{\nabla}_{\partial_x}\partial_x = \frac{\partial^2 v^i}{\partial x^2}\partial_{v^i}$$

and hence the v^i are linear in x, say,

$$v^i = \alpha_i(t)x + \beta_i(t).$$

From $ds^2 = f^2dt^2 + dx^2$ we obtain

$$f^2 = \sum(\alpha_i' x + \beta_i')^2, \quad \sum \alpha_i^2 = 1, \quad \sum \alpha_i(\alpha_i' x + \beta_i') = 0. \tag{5.1}$$

Because M^2 is Lagrangian, the restriction of $dv^1 \wedge dv^2 + dv^3 \wedge dv^4$ to M^2 vanishes, giving

$$(\alpha_1' x + \beta_1')\alpha_2 - (\alpha_2' x + \beta_2')\alpha_1 + (\alpha_3' x + \beta_3')\alpha_4 - (\alpha_4' x + \beta_4')\alpha_3 = 0;$$

therefore,

$$\alpha_1'\alpha_2 - \alpha_2'\alpha_1 + \alpha_3'\alpha_4 - \alpha_4'\alpha_3 = 0, \quad \beta_1'\alpha_2 - \beta_2'\alpha_1 + \beta_3'\alpha_4 - \beta_4'\alpha_3 = 0. \tag{5.2}$$

Next we compare

$$\begin{aligned}\sigma(\partial_x, U) &= JA_1\partial_x = J(aU)\\ &= \frac{a}{f}J\partial_t = \frac{a}{f}\left(\frac{\partial v^1}{\partial t}\partial_{v^2} - \frac{\partial v^2}{\partial t}\partial_{v^1} + \frac{\partial v^3}{\partial t}\partial_{v^4} - \frac{\partial v^4}{\partial t}\partial_{v^3}\right)\end{aligned}$$

with

$$\bar\nabla_{\partial_x}\frac{1}{f}\partial_t = -\frac{f_x}{f}\sum(\alpha_i' x + \beta_i')\partial_{v^i} + \frac{1}{f}\sum\alpha_i'\partial_{v^i}$$

and so obtain the following equations:

$$\begin{aligned}-\frac{f_x}{f}(\alpha_1' x + \beta_1') + \alpha_1' &= -a(\alpha_2' x + \beta_2'),\\ -\frac{f_x}{f}(\alpha_2' x + \beta_2') + \alpha_2' &= a(\alpha_1' x + \beta_1'),\\ -\frac{f_x}{f}(\alpha_3' x + \beta_3') + \alpha_3' &= -a(\alpha_4' x + \beta_4'),\\ -\frac{f_x}{f}(\alpha_4' x + \beta_4') + \alpha_4' &= a(\alpha_3' x + \beta_3').\end{aligned} \tag{5.3}$$

Multiplying each of these by its own right side and then adding and using (5.1) and (5.2) yields

$$af^2 = A = \beta_1'\alpha_2' - \beta_2'\alpha_1' + \beta_3'\alpha_4' - \beta_4'\alpha_3'. \tag{5.4}$$

Now turning to the proof of Theorem 3, suppose there is a 1-parameter family of Lagrangian surfaces in $\mathbb{C}^2$ connecting the Lagrangian ruled surface described above to the Lagrangian catenoid (1.2) of Castro and Urbano. In particular, consider the surfaces whose position vectors are

$$\begin{aligned}&P(\lambda)\big(\alpha_1(t)x + \beta_1(t), \alpha_2(t)x + \beta_2(t), \alpha_3(t)x + \beta_3(t), \alpha_4(t)x + \beta_4(t)\big)\\ &\qquad + Q(\lambda)\left(\frac{e^x}{\sqrt2}\cos t, \frac{e^{-x}}{\sqrt2}\cos t, \frac{e^x}{\sqrt2}\sin t, \frac{e^{-x}}{\sqrt2}\sin t\right),\end{aligned}$$

where for the parameter λ we have $P(0) = 1$ and $P(\Lambda) = 0$ as well as $Q(0) = 0$ and $Q(\Lambda) = 1$, with P and Q being continuous functions on an interval $[0, \Lambda]$.

Applying $dv^1 \wedge dv^2 + dv^3 \wedge dv^4$ to ∂_t and ∂_x shows that, if the surface is Lagrangian for each λ, then

$$\begin{aligned} \alpha_1' \cos t + \alpha_3' \sin t &= 0, \\ \alpha_2' \cos t + \alpha_4' \sin t &= 0, \\ \alpha_1 \sin t - \beta_1' \cos t - \alpha_3 \cos t - \beta_3' \sin t &= 0, \\ \alpha_2 \sin t + \beta_2' \cos t - \alpha_4 \cos t + \beta_4' \sin t &= 0. \end{aligned} \tag{5.5}$$

From equations (5.3) we have

$$\beta_1' = -\alpha_1' x + \frac{f f_x \alpha_1' + a f^2 \alpha_2'}{f_x^2 + a^2 f^2}, \quad \beta_2' = -\alpha_2' x + \frac{f f_x \alpha_2' - a f^2 \alpha_1'}{f_x^2 + a^2 f^2},$$
$$\beta_3' = -\alpha_3' x + \frac{f f_x \alpha_3' + a f^2 \alpha_4'}{f_x^2 + a^2 f^2}, \quad \beta_4' = -\alpha_4' x + \frac{f f_x \alpha_4' - a f^2 \alpha_3'}{f_x^2 + a^2 f^2}.$$

Using these and the first two equations of (5.5), we have $\beta_1' \cos t + \beta_3' \sin t = 0$; in turn, from the third equation of (5.5) we have $\alpha_1 \sin t - \alpha_3 \cos t = 0$. This, together with the first equation of (5.5), gives $\alpha_3/\alpha_1 = \tan t = -\alpha_1'/\alpha_3'$, from which $\alpha_1^2 + \alpha_3^2 = k^2$, a constant. Writing $\alpha_1 = k \cos \gamma(t)$ and $\alpha_3 = k \sin \gamma(t)$, from $\alpha_3/\alpha_1 = \tan t$ we may, in fact, take $\gamma(t) = t$. Using a similar argument for β_2 and β_4, we now have:

$$\alpha_1 = k \cos t, \quad \alpha_2 = l \cos t, \quad \alpha_3 = k \sin t, \quad \alpha_4 = l \sin t,$$
$$\beta_1' \cos t + \beta_3' \sin t = 0, \quad \beta_2' \cos t + \beta_4' \sin t = 0.$$

Returning to the second equation in (5.1) we see that $k^2 + l^2 = 1$; in turn, from the first equation in (5.1) we obtain

$$f^2 = x^2 - 2[k(\beta_1' \sin t - \beta_3' \cos t) + l(\beta_2' \sin t - \beta_4' \cos t)]x + \sum \beta_i'^2.$$

In particular, $F = 1$ in the quadratic $f^2 = Fx^2 + Gx + H$.

Now the equations for the β_i become

$$\beta_1' = \left(kx - \frac{k f f_x + l a f^2}{f_x^2 + a^2 f^2} \right) \sin t, \ldots.$$

Then from equation (5.4) we have

$$a f^2 = A = -l(\beta_1' \sin t - \beta_3' \cos t) + k(\beta_2' \sin t - \beta_4' \cos t) = \frac{a f^2}{f_x^2 + a^2 f^2},$$

giving $f_x^2 + a^2 f^2 = 1$. Note also that $x - f f_x = -G/2$. Thus the functions β_i are determined from G and A by

$$\beta_1' = -\left(\frac{kG}{2} + lA \right) \sin t, \quad \beta_2' = \left(-\frac{lG}{2} + kA \right) \sin t,$$
$$\beta_3' = \left(\frac{kG}{2} + lA \right) \cos t, \quad \beta_4' = \left(\frac{lG}{2} - kA \right) \cos t.$$

6. Proof of Theorem 4

Since the Lagrangian catenoid is not flat, it is enough to consider the nonflat case of Theorem 1. If $n \geq 4$ then the theorem is quite easy. Since the Lagrangian catenoid is conformally flat, if $M^2 \times \mathbb{R}^{n-2}$ is locally isometric to the Lagrangian catenoid then it, too, must be conformally flat. Let e_1, e_2 be orthonormal vectors tangent to M^2 and let ρ and τ denote (respectively) the Ricci tensor and scalar curvature of $M^2 \times \mathbb{R}^{n-2}$. Then, from the well-known form of the curvature tensor of a conformally flat manifold,

$$\begin{aligned} g(R_{XY}Z, W) = \frac{1}{n-2}\big(&g(Y,Z)\rho(X,W) - \rho(X,Z)g(Y,W) \\ &+ \rho(Y,Z)g(X,W) - g(X,Z)\rho(Y,W)\big) \\ &- \frac{\tau}{(n-1)(n-2)}\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big), \end{aligned}$$

the Gaussian curvature of M^2 is given by

$$\begin{aligned} K = g(R_{e_1e_2}e_2, e_1) &= \frac{1}{n-2}(\rho(e_1,e_1) + \rho(e_2,e_2)) - \frac{\tau}{(n-1)(n-2)} \\ &= \frac{2K}{n-2}\left(1 - \frac{1}{n-1}\right); \end{aligned}$$

this yields $K = 0$, which contradicts the nonflatness of M^2.

If $n = 3$ then we recall that, on a 3-dimensional conformally flat manifold,

$$(\nabla_X\rho)(Y,Z) - \frac{X\tau}{4}g(Y,Z) = (\nabla_Y\rho)(X,Z) - \frac{Y\tau}{4}g(X,Z).$$

By the Riemannian product structure on $M^2 \times \mathbb{R}$, we can choose e_1, e_2, e_3 to diagonalize ρ on $M^2 \times \mathbb{R}$ with e_1, e_2 tangent to M^2; then

$$(\nabla_{e_1}\rho)(e_3,e_3) - \frac{e_1\tau}{4} = (\nabla_{e_3}\rho)(e_1,e_3),$$

which yields $e_1K = 0$. Similarly $e_2K = 0$, and hence M^2 is of constant curvature. From the Gauss equation, $a = \text{const}$. By (3.6) this implies that either $a = 0$ or each $p_i = 0$, but if each $p_i = 0$ then the Gauss equation will give $a = 0$ as well. Thus, as in the proof of Theorem 1, if $a = 0$ then M^n is flat.

The proof is more difficult in dimension 2. We will show that the metric (1.1) on the Lagrangian catenoid, $ds^2 = \cosh 2u(du^2 + d\theta^2)$, and also the metric $ds^2 = f^2dt^2 + dx^2$, where $f^2 = F(t)x^2 + G(t)x + H(t)$, cannot be locally isometric. We suppose that $u = u(t,x)$ and $\theta = \theta(t,x)$ is a local isometry mapping one metric to the other and then seek a contradiction. Preservation of the metric implies that

$$f^2 = \cosh 2u(u_t^2 + \theta_t^2), \quad 0 = u_tu_x + \theta_t\theta_x, \quad 1 = \cosh 2u(u_x^2 + \theta_x^2).$$

The Gaussian cuvature of $ds^2 = \cosh 2u(du^2 + d\theta^2)$ is $K = -2/(\cosh^3 2u)$ and that of $ds^2 = f^2dt^2 + dx^2$ is $K = (G^2 - 4FH)/4f^4$. Thus, u as a function of t and x must be given by

$$\cosh^3 2u = \frac{8f^4}{4FH - G^2}.$$

Now $\theta_t^2 = f^2/(\cosh 2u) - u_t^2$, $\theta_x^2 = 1/(\cosh 2u) - u_x^2$, and $\theta_t^2\theta_x^2 = u_t^2 u_x^2$, from which we have $f^2 = f^2(\cosh 2u)u_x^2 + (\cosh 2u)u_t^2$. The first two equations, for θ_t^2 and θ_x^2, then yield

$$\theta_t = \pm f u_x \quad \text{and} \quad \theta_x = \mp \tfrac{1}{f} u_t.$$

Comparing θ_{tx} and θ_{xt}, we see that the integrability condition for these equations is

$$\tfrac{1}{2}(f^2)_x u_x + f^2 u_{xx} = \tfrac{1}{2}(\ln f^2)_t u_t - u_{tt}.$$

Setting $z = \cosh^3 2u$, the integrability condition becomes

$$\begin{aligned}\frac{1}{2}(f^2)_x z_x + f^2 z_{xx} - \frac{2f^2 z_x^2}{3z} - \frac{f^2 z_x^2}{3(z - z^{1/3})}&\\ = \frac{1}{2}(\ln f^2)_t z_t - z_{tt} + \frac{2z_t^2}{3z} + \frac{z_t^2}{3(z - z^{1/3})}&.\end{aligned}$$

For simplicity we set $\Phi = F(t)x^2 + G(t)x + H(t)$ and $D = 4F(t)H(t) - G(t)^2$ and we use $'$ to denote differentiation with respect to t. Then the integrability condition becomes

$$\begin{aligned}-24(2Fx + G)^2\Phi^3 - 2(2Fx + G)^2\Phi^{5/3}D^{2/3} + 96F\Phi^4 - 24F\Phi^{8/3}D^{2/3}&\\ = 72\Phi^2\Phi'^2 - 10\Phi^{2/3}\Phi'^2 D^{2/3} - 12\frac{\Phi^3\Phi' D'}{D} - 5\frac{\Phi^{5/3}\Phi' D'}{D^{1/3}} - 24\frac{\Phi^4 D'^2}{D^2}&\\ + 8\frac{\Phi^{8/3}D'^2}{D^{4/3}} - 48\Phi^3\Phi'' + 12\Phi^{5/3}D^{2/3}\Phi'' + 24\frac{\Phi^4 D''}{D} - 6\frac{\Phi^{8/3}D''}{D^{1/3}}&.\end{aligned}$$

In this equation, half the terms are polynomials in x of degree 8 and the other terms have expansions in x with highest exponent $\frac{16}{3}$. Thus, taking the corresponding coefficients yields

$$0 = 6F'^2 - \frac{FF'D'}{D} - 2\frac{F^2D'^2}{D^2} - 4FF'' + 2\frac{F^2D''}{D}$$

and

$$-32F^3 = -10F'^2 - 5\frac{FF'D'}{D} + 8\frac{F^2D'^2}{D^2} + 12FF'' - 6\frac{F^2D''}{D}.$$

Multiplying the first of these by 3 and adding to the second gives

$$-32F^3 = 2\left(2F' - \frac{FD'}{D}\right)^2,$$

contradicting the positivity of the quadratic $F(t)x^2 + G(t)x + H(t)$.

7. Proof of Proposition 5

If a Lagrangian submanifold M^n of $\mathbb{C}^n$ admits two foliations by $(n-1)$-planes, then there exists a vector field $V = \delta U + \sum \varepsilon_i e_i$ with $\delta \neq 0$ and satisfying $\bar{\nabla}_V V = 0$. Using (2.1) and (3.1), we have

$$\begin{aligned} 0 = \bar{\nabla}_V V &= \delta(U\delta)U + \delta^2\Big(\sum_i p_i e_i + \sigma(U,U)\Big) \\ &+ \delta\sum_i (U\varepsilon_i)e_i + \delta\sum_i \varepsilon_i(-p_i U + \sigma(U,e_i)) \\ &+ \sum_j \varepsilon_j\Big((e_j\delta)U + \delta\sigma(e_j,U) + \sum_i (e_j\varepsilon_i)e_i\Big). \end{aligned}$$

Then, taking the normal part and its inner product with JU and Je_j and using the matrices (3.2), we obtain

$$0 = b\delta^2 + 2a\delta\sum_i \varepsilon_i \quad \text{and} \quad 0 = a\delta^2,$$

from which both a and b vanish; hence we see that M^n is totally geodesic.

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