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Principal Bundles over Chains or Cycles of Rational Curves

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The objects of study in this paper are Zariski locally trivial principal bundles over chains or cycles of rational curves. A chain of rational curves is either $\mathbb{P}^1$ or a connected reduced complex curve $X = X_1 \cup \cdots \cup X_k$ with $k \geq 2$ irreducible components X_i $(1 \leq i \leq k)$, all of them smooth rational curves, such that X_i intersects X_{i+1} in a nodal singular point q_i for all $i = 1, \dots, k-1$. A cycle of rational curves is either an irreducible reduced rational nodal complex curve with exactly one singularity or a connected reduced complex curve $X = X_1 \cup \cdots \cup X_k$ with $k \geq 2$ irreducible components X_i $(1 \leq i \leq k)$, all of them smooth rational curves, such that (a) X_i intersects X_{i+1} in a nodal singular point q_i for $i = 1, \dots, k-1$ and (b) X_k intersects X_1 in a nodal singular point q_k. We are interested in classifying Zariski locally trivial principal G-bundles over chains and cycles when G is a connected reductive complex algebraic group. The motivation is to generalize the classification (see [1]) of vector bundles over a chain or a cycle of rational curves. As shown there, one can hope to get a complete classification of all the vector bundles over a connected nodal curve only when the curve is either a chain or a cycle of rational curves. In the terminology of [1], the classification problem for vector bundles is of finite type for a chain of rational curves and of tame type for a cycle of rational curves. For all the other nodal curves, the classification problem is of wild type, meaning that the problem is at least as hard as the classification of the representations of finitely generated complex algebras. We will see that the problem for principal bundles seems to have the same trichotomy. For a chain, the classification of principal bundles is determined by discrete parameters. In the case of a cycle, the classification seems to depend on a finite-dimensional space of parameters.

This paper is divided into four parts. In the first part, we take the case of a smooth rational curve X. We recall (Theorem 1.1) the classification (see [3]) of principal G-bundles over X, which says that the structure of any principal G-bundle can be reduced to a maximal torus T of G. This result can be reformulated by saying that any principal G-bundle can be obtained from the principal $\mathbb{C}^*$-bundle $\mathbb{C}^2 - \{0\} \to \mathbb{P}^1$ by extending the structure group using a homomorphism $\lambda : \mathbb{C}^* \to T$. Let $X_*(T)$ be the abelian group of homomorphisms from $\mathbb{C}^*$ to T, and let P_λ be the principal G-bundle over X obtained as before from $\lambda \in X_*(T)$.

In the second part, we consider the case of a chain of rational curves. We prove (Theorem 2.2) that the structure group of any principal G-bundle on a chain of

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rational curves can be reduced to T. The classification of all principal G-bundles is obtained in Theorem 2.11. The classification data is discrete and consists of a k-tuple $(\lambda_1, \dots, \lambda_k)$ of homomorphisms from $\mathbb{C}^*$ to T, representing the restrictions of the principal bundle P to the components X_i. The starting point of the proof of these results is Proposition 2.1, which states that any principal bundle on X is obtained from principal bundles on the components plus some gluing data at each of the singular points of X.

In the third part, we consider the case of an irreducible nodal singular curve X with exactly one singularity $q \in X$. If $\lambda \in X_*(T)$ is dominant and if $g \in G$, then we construct a principal G-bundle on X from the G-bundle P_λ over $\mathbb{P}^1$ by identifying the fibers over two points of $\mathbb{P}^1$ using the isomorphism of G given by the left multiplication with g. Two pairs (λ_1, g_1), (λ_2, g_2) will define isomorphic G-bundles over X if and only if $\lambda_1 = \lambda_2$ and $g_2 = (a_1 b_2)^{-1} g_1 (a_1 b_1)$, where $a_1 \in Z(\lambda_1)$ and $b_1, b_2 \in U(\lambda_1)$. Here $Z(\lambda) \subset G$ denotes the centralizer of the dominant element $\lambda \in X_*(T)$, and $U(\lambda)$ denotes the unipotent radical of the parabolic $P(\lambda)$ associated to λ. Thus the classification of principal G-bundles on X is equivalent to the classification of all equivalence classes of the equivalence relation $\cong$ on G given by $g_1 \cong g_2$ if and only if $g_2 = (a_1 b_2)^{-1} g_1 (a_1 b_1)$ for some $a_1 \in Z(\lambda_1)$ and $b_1, b_2 \in U(\lambda_1)$.

In the fourth part, we consider the case of a cycle of rational curves with at least two irreducible components. The classification data for principal G-bundles over a cycle of rational curves consists of $\lambda_1, \dots, \lambda_k \in X_*(T)$ dominant cocharacters (which represent the restrictions of the principal bundle to the components X_i), gluing data $n_1, \dots, n_{k-1} \in N(T)$ for the points q_i $(1 \le i \le k-1)$, and an equivalence class $\hat{g}$ of some equivalence relation on G, which represents the gluing data at q_k.

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Part I

In this part, we are interested in principal bundles over $X = \mathbb{P}^1$, the complex projective space of dimension 1. The following theorem gives the classification of the principal G-bundles over X.

THEOREMS 1.1 [3]. *Let G be a complex connected reductive algebraic group, $T \subset G$ a maximal torus, and W the Weyl group. We denote by $X_*(T)$ the abelian group of homomorphisms from $\mathbb{C}^*$ to T, called the group of cocharacters of T.*

(1) *The structure group of any principal G-bundle over $X = \mathbb{P}^1$ can by reduced to the maximal torus $T \subset G$. Moreover,* $\mathrm{H}^1(X, T(\mathcal{O}_X))/W = \mathrm{H}^1(X, G(\mathcal{O}_X))$.

(2) *Let P_λ be the G-bundle obtained from the principal $\mathbb{C}^*$-bundle $\mathbb{C}^2 - \{0\} \to \mathbb{P}^1$ extending the structure group using the homomorphism $\lambda \in X_*(T)$. Any principal G-bundle over $X = \mathbb{P}^1$ is isomorphic with P_λ for some $\lambda \in X_*(T)$. Two principal G-bundles P_λ, $P_{\lambda'}$ are isomorphic if and only if λ, λ' belong to the same orbit under the Weyl group, that is, iff there is a $w \in W$ such that $\lambda = w \cdot \lambda'$.*

OBSERVATIONS 1.2. (1) Because, under the action of the Weyl group, any orbit in $X_*(T)$ contains a unique dominant element, any principal G-bundle over $\mathbb{P}^1$ is isomorphic with P_λ for some (unique) dominant cocharacter $\lambda \in X_*(T)$.

(2) If q_0, q_1 are two different points of X, we consider $X = V_0 \cup V_1$ the open affine covering of X with two copies ($V_0 = X - \{q_1\}$ and $V_1 = X - \{q_0\}$) of $\mathbb{C}$. The principal $\mathbb{C}^*$-bundle $\mathbb{C}^2 - \{0\} \to \mathbb{P}^1$ has trivializations over V_0 and V_1 such that the cocycle with respect to these trivializations is a group isomorphism $V_0 \cap V_1 = \mathbb{C}^* \to \mathbb{C}^*$. It follows from Theorem 1.1(2) that any principal bundle over X has trivializations over V_0 and V_1 such that the cocycle with respect to these trivializations is a dominant group homomorphism $\lambda\colon V_0 \cap V_1 = \mathbb{C}^* \to T \subset G$. Such a cocharacter $\lambda \in X_*(T)$ is determined uniquely.

(3) If $\lambda \in X_*(T)$ then we can define the G-bundle P_λ in the following way. Let q_0, q_1 be two different points of $X = \mathbb{P}^1$. Let $V_1 = X - \{q_0\}$ and $V_0 = X - \{q_1\}$. The two sets define an open affine covering of X. We define a principal G-bundle P_λ over X by gluing the trivialized G-bundles $\pi_0\colon V_0 \times G \to V_0$ and $\pi_1\colon V_1 \times G \to V_1$ over $V_0 \cap V_1$. We define the gluing $\pi_0^{-1}(V_0 \cap V_1) \to \pi_1^{-1}(V_0 \cap V_1)$ by the rule $(x, g) \to (x, \lambda(x)g)$, where $\lambda\colon \mathbb{C}^* = V_0 \cap V_1 \to T \subset G$.

(4) In a similar way, we can define a principal T-bundle T_λ over X by gluing over $V_0 \cap V_1$ the trivialized T-bundles $\pi_0\colon V_0 \times T \to V_0$ and $\pi_1\colon V_1 \times T \to V_1$. We define the gluing as a morphism $\pi_0^{-1}(V_0 \cap V_1) \to \pi_1^{-1}(V_0 \cap V_1)$ defined by the rule $(x, t) \to (x, \lambda(x)t)$. Clearly, the principal G-bundle P_λ is obtained from the principal T-bundle T_λ by extending the group structure using the inclusion homomorphism $T \to G$.

For future use we recall the following result about the automorphism group of a principal G-bundle over the projective line. Let $\lambda \in X_*(T)$ be dominant. We fix a maximal torus T and a Borel subgroup $T \subset B \subset G$ of the algebraic group G. Let $N(T)$ be the normalizer of T in G and let $W = N(T)/T$ be the Weyl group of G. Let $P(\lambda)$ be the parabolic associated to λ generated by T and by the root groups U_α with $(\lambda, \alpha) \geq 0$. The centralizer $Z(\lambda)$ of the subgroup of T determined by λ is a connected reductive group and $P(\lambda) = Z(\lambda)U(\lambda)$, where $U(\lambda)$ is the unipotent radical of $P(\lambda)$. For the Lie algebras we have $\mathfrak{z}(\lambda) = \mathfrak{t} \oplus \sum_{(\lambda,\alpha)=0} \mathfrak{u}_\alpha$ and $\mathfrak{u}(\lambda) = \sum_{(\lambda,\alpha)>0} \mathfrak{u}_\alpha$.

THEOREM 1.3 [4]. *Let $\lambda \in X_*(T)$ be a dominant cocharacter and P_λ the corresponding principal G-bundle on the projective line. Then the group of automorphisms of P_λ (acting trivially on the base) is isomorphic to*

$$\mathrm{Aut}(P_\lambda) = Z(\lambda) \times \prod_{(\lambda,\alpha)>0} \mathrm{H}^0(\mathbb{P}^1, T_\lambda(\mathfrak{u}_\alpha)),$$

where T_λ is the principal T-bundle induced from λ and $T_\lambda(\mathfrak{u}_\alpha) := T_\lambda \times_T \mathfrak{u}_\alpha$ is the associated line bundle.

Since $T_\lambda(\mathfrak{u}_\alpha)$ is a line bundle on $\mathbb{P}^1$ of degree (λ, α), we have the following corollary.

COROLLARY 1.4. *Let $\lambda \in X_*(T)$ be a dominant cocharacter and let q_0, q_1 be two different points of X. Let P_λ be the principal G-bundle on $\mathbb{P}^1$ associated to λ, as in Observation 1.2(3).*

(1) *Let $g \in Z(\lambda)$ and $a, b \in U(\lambda)$. Then there is an automorphism $\phi \in \mathrm{Aut}(P_\lambda)$ such that $\phi_{q_0} = ga$ and $\phi_{q_1} = gb$ in the corresponding trivializations.*
(2) *Let $b \in B$. Then there is an automorphism $\phi \in \mathrm{Aut}(P_\lambda)$ such that $\phi_{q_0} = b$ in the corresponding trivialization around q_0.*

OBSERVATION 1.5. If $\lambda \in X_*(T)$ and $n \in N(T)$, let $\lambda' = n\lambda n^{-1} \in X_*(T)$. The element λ' depends only on the class of n in $W = N(T)/T$. Since $P_{\lambda'} = P_\lambda \times_\rho G$, where $\rho\colon G \to G$ is the inner conjugation in G with $n \in N(T)$, we have an isomorphism $\phi_n\colon P_\lambda \to P_{\lambda'}$ of principal G-bundles. If q_0, q_1 are different points of X and if P_λ, $P_{\lambda'}$ are defined as in Observation 1.2(3), then the isomorphism ϕ_n is represented by the multiplication on the left with n in the trivializations used to define P_λ and $P_{\lambda'}$.

Part II

In this part we study Zariski locally trivial principal G-bundles over a reducible chain of rational curves. Such a chain is a connected reduced complex curve $X = X_1 \cup \cdots \cup X_k$ with $k \geq 2$ irreducible components X_i $(1 \leq i \leq k)$, all of them smooth rational curves, such that X_i intersects X_{i+1} in a nodal singular point q_i for all $1 \leq i \leq k-1$. We choose $q_0 \in X_1$ different from q_1 and $q_k \in X_k$ different from q_{k-1}. The base curve X can be viewed as coming from k copies $X_1, \ldots, X_k$ of a smooth rational curve by gluing $X_i - \{q_{i-1}\}$ to $X_{i+1} - \{q_{i+1}\}$ at the point q_i for $1 \leq i \leq k-1$. The gluing can be thought of in this way: $X_i - \{q_{i-1}\}$ is an affine space $\mathrm{Spec}(A)$, and the point $q_i \in X_i - \{q_{i-1}\}$ corresponds to a homomorphism $f\colon A \to \mathbb{C}$; similarly, $X_{i+1} - \{q_{i+1}\}$ is an affine space $\mathrm{Spec}(B)$, and the point $q_i \in X_{i+1} - \{q_{i+1}\}$ corresponds to a homomorphism $g\colon B \to \mathbb{C}$. The result of gluing $X_i - \{q_{i-1}\}$ to $X_{i+1} - \{q_{i+1}\}$ at q_i is $\mathrm{Spec}(C)$, where C is the algebra $C = \{(a, b) \in A \oplus B \mid f(a) = g(b)\}$.

Let G be a connected reductive complex algebraic group. The starting point of the study of principal bundles over X is Proposition 2.1, which claims that any principal G-bundle over X is obtained from principal G-bundles P_i over X_i by gluing the fibers over the common points q_i, $1 \leq i \leq k-1$.

PROPOSITION 2.1. *Let $Y = Y_1 \cup \cdots \cup Y_k$ be a reduced curve with $k \geq 2$ irreducible components Y_i such that* (a) *Y_i intersects Y_{i+1} in a point q_i for $1 \leq i \leq k-1$ and* (b) *the structure sheaf $\mathcal{O}_Y$ is given by $(f_1, \ldots, f_k) \in \mathcal{O}_{Y_1} \oplus \cdots \oplus \mathcal{O}_{Y_k}$, with $f_i(q_i) = f_{i+1}(q_i)$ for $1 \leq i \leq k-1$. Let j_i be the inclusion of q_i in Y_i and let j_i' be the inclusion of q_i in Y_{i+1}. Let $\mathcal{C}_k$ be the category of tuples $(P_1, A_1, P_2, A_2, \ldots, A_{k-1}, P_k)$, where P_i is a principal G-bundle over Y_i and $A_i\colon j_i^* P_i \to j_i'^* P_{i+1}$ is a morphism of G-bundles over q_i. A morphism in $\mathcal{C}_k$ between $(P_1, A_1, P_2, A_2, \ldots, A_{k-1}, P_k)$ and $(Q_1, B_1, Q_2, B_2, \ldots, B_{k-1}, Q_k)$ is given by a tuple $(\phi_1, \ldots, \phi_k)$, where $\phi_i\colon P_i \to Q_i$ is a G-morphism over Y_i $(1 \leq i \leq k)$ such that $B_i \circ j_i^*\phi_i = j_i'^*\phi_{i+1} \circ A_i$. There is an equivalence of categories between the category of* *principal G-bundles on Y and the category $\mathcal{C}_k$. Under this equivalence, the bundles $P_1, \dots, P_k$ are the pull-backs of the bundle P.*

Proof. We prove the proposition for $k = 2$; the general case is similar. Let P be a principal G-bundle over Y. Let $l_i : Y_i \to Y$ be the inclusion morphism and let $P_i = l_i^* P$ be the pull-back G-bundle on Y_i. Since $l_1 \circ j_1 = l_2 \circ j_1'$, there is a canonical isomorphism $A_1 \colon j_1^* P_1 \to j_1'^* P_2$ of principal G-bundles over q_1. We define a functor S from the category of principal G-bundles on Y to the category $\mathcal{C}_2$, which on objects associates to a principal G-bundle P the triple (P_1, A_1, P_2). We prove that S defines an equivalence of categories by constructing an inverse functor T from the category $\mathcal{C}_2$ to the category of principal G-bundles on Y. If (P_1, A_1, P_2) is an object of $\mathcal{C}_2$, then we define a principal G-bundle $T(P_1, A_1, P_2)$ on Y as follows. We choose trivializations of P_1 over an open covering $\mathcal{U}^1 = \{U_\alpha^1 \mid \alpha \in I\}$ of Y_1 such that there is exactly one open set $U_{\alpha_0}^1 \in \mathcal{U}^1$ that contains q_1. We denote the cocycle of P_1 with respect to $\mathcal{U}^1$ by $g_{\alpha,\alpha'}^1$. Similarly, we choose trivializations of P_2 over an open covering $\mathcal{U}^2 = \{U_\beta^2 \mid \beta \in J\}$ of Y_2 such that there is exactly one open set $U_{\beta_0}^2 \in \mathcal{U}^2$ that contains q_1. The trivialization of P_1 on $U_{\alpha_0}^1$ induces a trivialization of $j_1^* P_1$ on q_1, and the trivialization of P_2 on $U_{\beta_0}^2$ induces a trivialization of $j_1'^* P_2$ on q_1. In these trivializations, the morphism A_1 is the multiplication on the left with an element of G. We can choose a trivialization of P_2 on $U_{\beta_0}^2$ such that the morphism A_1, in the given trivialization at q_1, is the identity. We denote the cocycle of P_2 with respect to $\mathcal{U}^2$ by $g_{\beta,\beta'}^2$. Let $U = U_{\alpha_0}^1 \cup U_{\beta_0}^2$. We construct a G-bundle $T(P_1, A_1, P_2)$ on Y from trivial G-bundles over U_α^1 $(\alpha \neq \alpha_0)$, U_β^2 $(\beta \neq \beta_0)$, and U by gluing according to the cocycles $g_{\alpha,\alpha'}^1$ and $g_{\beta,\beta'}^2$. A choice of other trivializations will, as before, define an isomorphic G-bundle on Y. We can extend this construction on morphisms. Thus we have a functor T from $\mathcal{C}_2$ to the category of principal G-bundles on Y. If $Q = T(P_1, A_1, P_2)$ then there are natural isomorphisms $h_1 \colon P_1 \to l_1^* Q$ and $h_2 \colon P_2 \to l_2^* Q$. If $C \colon j_1^* l_1^* Q \to j_1'^* l_2^* Q$ is the canonical isomorphism, we have $C \circ j_1^* h_1 = A_1 \circ j_1'^* h_2$. These observations show that T is an inverse, in both orders, of S. □

The next theorem shows that the classification of principal G-bundles over a chain of rational curves is similar to the classification of principal bundles over a smooth rational curve. See also Theorem 2.11 for a description of the classification data.

THEOREM 2.2. *Let G be a complex connected reductive group. The structure of a principal G-bundle over a chain of rational curves can be reduced to a maximal torus.*

COROLLARY 2.3. *A vector bundle over a chain of rational curves is a direct sum of line bundles.*

We now fix some notation and we recall some results about connected reductive complex algebraic groups. Let G be a complex reductive algebraic group, and let T be a maximal torus. Let W be the Weyl group of G. Let $T \subset B$ be a Borel subgroup, and let $X_*(T)$ be the abelian group of homomorphisms from $\mathbb{C}^\times$ to T.

If $r = \operatorname{rank}(G)$ is the rank of G, then $T \cong (\mathbb{C}^\times)^r$ and we have a group isomorphism $X_*(T) \cong \mathbb{Z}^r$ that associates to $(a_1, \dots, a_r) \in \mathbb{Z}^r$ the homomorphism $z \to (z^{a_1}, \dots, z^{a_r})$ from $\mathbb{C}^\times$ to T. (*Note:* $X_*(T)$ *is the standard notation in the literature,* and there is no connection with the base curve X.) Let $\lambda \in X_*(T)$ be a dominant element; that is, let $(\lambda, \alpha) \geq 0$ for all positive roots α of G, where positivity is defined with respect to a fixed Borel subgroup. Let $P(\lambda)$ be the parabolic associated to λ generated by T and by the root groups U_α with $(\lambda, \alpha) \geq 0$. (Here we are using the standard notation $P(\lambda)$ from the literature for the parabolic subgroups; the parabolic subgroup $P(\lambda)$ should not be confused with the principal G-bundle P_λ defined in the previous part.) The centralizer in G of the subgroup $\operatorname{im}(\mathbb{C}^\times \xrightarrow{\lambda} T)$ of T is denoted by $Z(\lambda)$. The centralizer $Z(\lambda)$ is a connected reductive group generated by T and the root groups U_α for which α satisfies $(\lambda, \alpha) = 0$. The unipotent radical of $Z(\lambda)$ is denoted $U(\lambda)$. The group $U(\lambda)$ is generated by the root groups U_α with $(\lambda, \alpha) > 0$. We have $P(\lambda) = Z(\lambda)U(\lambda)$. If $\mathfrak{u}_\alpha$ is the Lie algebra of the root group U_α and if $\mathfrak{t}$ is the Lie algebra of T, then $\mathfrak{z}(\lambda) = \mathfrak{t} \oplus \sum_{(\lambda,\alpha)=0} \mathfrak{u}_\alpha$ and $\mathfrak{u}(\lambda) = \sum_{(\lambda,\alpha)>0} \mathfrak{u}_\alpha$.

We recall the Bruhat decomposition.

THEOREM 2.4. *There is a bijective correspondence between the set $B\backslash G/B$ and the elements of the Weyl group W. More precisely, for any $g \in G$, there are elements $b_1, b_2 \in B$ and $n \in N(T)$ such that $g = b_2 n b_1^{-1}$. The element n is unique up to the multiplication by an element of T. That is, if $n, n' \in N(T)$ and $b_1, b_2, b_1', b_2' \in B$ are such that $b_2 n b_1^{-1} = b_2' n' b_1'^{-1}$, then there is an element $t \in T$ such that $n' = nt$.*

The next result is an immediate corollary of the Bruhat decomposition.

COROLLARY 2.5. *If $n \in N(T)$, let $B' = nBn^{-1}$. There is a bijective correspondence between the set $B\backslash G/B'$ and the elements of the Weyl group W. More precisely, for any $g \in G$, there are elements $b_1 \in B'$, $b_2 \in B$, and $n \in N(T)$ such that $n = b_2 g b_1^{-1}$. The element n is unique up to the multiplication by an element of T. That is, if $n, \tilde{n} \in N(T)$, $b_1, \tilde{b}_1 \in B'$, and $b_2, \tilde{b}_2 \in B$ are such that $n = b_2 g b_1^{-1}$ and $\tilde{n} = \tilde{b}_2 g \tilde{b}_1^{-1}$, then there is an element $t \in T$ such that $\tilde{n} = nt$.*

We will need the next propositions in the proof of Theorem 2.2.

PROPOSITION 2.6. *Let G be a connected reductive algebraic group, $T \subset G$ a maximal torus, and B a Borel subgroup of G containing T. Let $\lambda \in X_*(T)$ be dominant and let $n \in N(T)$. Then there is an element $m \in N(T)$ such that*

$$mBm^{-1} \subset (Z(\lambda) \cap nBn^{-1})U(\lambda).$$

Proof. We first claim that $(Z(\lambda) \cap nBn^{-1})^0$ is a Borel subgroup of $Z(\lambda)$, where $(Z(\lambda) \cap nBn^{-1})^0$ is the connected component of the identity of $Z(\lambda) \cap nBn^{-1}$. The group $(Z(\lambda) \cap nBn^{-1})^0$ is a connected closed solvable subgroup of $Z(\lambda)$. It is also maximal with these properties because its Lie algebra is

$$\mathfrak{t} \oplus \sum_{\alpha>0,\, w\alpha>0,\, (\lambda,\alpha)=0} \mathfrak{u}_\alpha \oplus \sum_{\alpha<0,\, w\alpha>0,\, (\lambda,\alpha)=0} \mathfrak{u}_\alpha,$$

where $w \in W = N(T)/T$ is the element defined by $n \in N(T)$. Using the previous claim, $(Z(\lambda) \cap nBn^{-1})^0$ and $(Z(\lambda) \cap B)^0$ are two Borel subgroups in the reductive group $Z(\lambda)$. Since any two Borel subgroups in a connected reductive group are conjugated, there is an element $m \in Z(\lambda)$ such that $(Z(\lambda) \cap nBn^{-1})^0 = m(Z(\lambda) \cap B)^0 m^{-1}$. Because the two Borel subgroups $(Z(\lambda) \cap nBn^{-1})^0$ and $(Z(\lambda) \cap B)^0$ contain T, the element m must normalize T, that is, $m \in N(T)$. Since $m \in Z(\lambda)$ normalizes $U(\lambda)$ and since $(Z(\lambda) \cap B)^0 U(\lambda) = B$, we obtain the result of the proposition. □

PROPOSITION 2.7. *Let $\nu \in X_*(T)$ be another dominant cocharacter. Then there is an element $m \in N(T)$ such that*

$$mBm^{-1} \subset (Z(\lambda) \cap nP(\nu)n^{-1})U(\lambda).$$

Proof. Since $B \subset P(\nu)$, the result follows from Proposition 2.6. □

PROPOSITION 2.8. *Let $\lambda_1, \dots, \lambda_i \in X_*(T)$ be dominant and let $n_1, \dots, n_{i-1} \in N(T)$. Define $A_1 = P(\lambda_1)$ and $A_{j+1} = (Z(\lambda_{j+1}) \cap n_j A_j n_j^{-1})U(\lambda_{j+1})$ for $1 \le j \le i-1$. Then there is an element $m \in N(T)$ such that $mBm^{-1} \subset A_i$.*

Proof. This follows from Proposition 2.7 by induction on i. □

COROLLARY 2.9. *Let $\lambda_1, \dots, \lambda_i \in X_*(T)$ be dominant. Let $n_1, \dots, n_{i-1} \in N(T)$ and let m be as in Proposition 2.8. Let $a \in mBm^{-1}$. Then there exist $a_j \in Z(\lambda_j)$ and $c_j \in U(\lambda_j)$ for $1 \le j \le i$ such that $(a_{j+1})n_j(a_j c_j)^{-1} = n_j$ $(1 \le j \le i-1)$ and $a = a_i c_i$.*

We are now ready for the proof of the Theorem 2.2.

Proof of Theorem 2.2. Let P be a principal G-bundle over X. Using Proposition 2.1, P is constructed from principal G-bundles over the components of X by gluing the fibers over the common points q_i. Let P_i be the pull-back of P to X_i. Applying Observation 1.2(2) to the bundle P_i over X_i and to the points $q_{i-1}, q_i \in X_i$, we obtain a dominant cocharacter $\lambda_i \in X_*(T)$ such that $P_i = P_{\lambda_i}$. We can view $P_i = P_{\lambda_i}$ as the principal G-bundle obtained from the trivial bundles $\pi : X_i - \{q_{i-1}\} \times G \to X_i - \{q_{i-1}\}$ and $\pi' : X_i - \{q_i\} \times G \to X_i - \{q_i\}$ by gluing $\pi^{-1}(X_i - \{q_{i-1}, q_i\}) \to \pi'^{-1}(X_i - \{q_{i-1}, q_i\})$ via the formula $(x, g) \to (x, \lambda_i(x)g)$, where we consider λ_i as a morphism $\lambda_i : X_i - \{q_{i-1}, q_i\} = \mathbb{C}^* \to T$. Using Proposition 2.1, the principal G-bundle corresponds to the object $(P_{\lambda_1}, A_1, P_{\lambda_2}, \dots, A_{k-1}, P_{\lambda_k})$ of $\mathcal{C}_k$. Using the trivialization of P_{λ_i} over $X_i - \{q_{i-1}\}$ and the trivialization of $P_{\lambda_{i+1}}$ over $X_{i+1} - \{q_{i+1}\}$, A_i can be considered as an element g_i of G. We fix these trivializations of P_i over $X_i - \{q_{i-1}\}$ and $X_{i+1} - \{q_{i+1}\}$ because we will refer to them in the proof. In this way, the gluing data is equivalent to a tuple $(g_1, \dots, g_{k-1}) \in G^{k-1}$.

LEMMA 1. *We claim that, by using automorphisms of P, we can replace the tuple $(g_1, \dots, g_{k-1}) \in G^{k-1}$ with a tuple $(n_1, \dots, n_{k-1}) \in N(T)^{k-1}$ of elements of $N(T)$, the normalizer of T in G.*

Let's see first how an automorphism of P will affect the gluing data. If $\phi_i \in \mathrm{Aut}(P_i)$, $1 \leq i \leq k$, are automorphisms of P_i, then the tuple $(\phi_1, \ldots, \phi_k)$ is an automorphism of the object $(P_{\lambda_1}, A_1, P_{\lambda_2}, \ldots, A_{k-1}, P_{\lambda_k})$ of $\mathcal{C}_k$, which by Proposition 2.1 corresponds to an automorphism of P. The automorphism $(\phi_1, \ldots, \phi_k)$ affects the gluing data $(A_1, \ldots, A_{k-1})$ by replacing it with

$$\big(\phi_2(q_1) \circ A_1 \circ \phi_1(q_1)^{-1}, \phi_3(q_2) \circ A_2 \circ \phi_2(q_2)^{-1}, \\ \ldots, \phi_k(q_{k-1}) \circ A_{k-1} \circ \phi_{k-1}(q_{k-1})^{-1}\big),$$

where $\phi_i(q)$ denotes the automorphism of the pull-back of P_i over q induced by the automorphism ϕ_i of P_i. By identifying the gluing data $(A_1, \ldots, A_{k-1})$ with a tuple $(g_1, \ldots, g_{k-1}) \in G^{k-1}$ and by using the fixed trivializations of P_i over $X_i - \{q_{i-1}\}$ and $X_{i+1} - \{q_{i+1}\}$, we can use the automorphism $(\phi_1, \ldots, \phi_k)$ to replace the tuple $(g_1, \ldots, g_{k-1})$ with the tuple

$$(\phi_{2,q_1} \cdot g_1 \cdot \phi_{1,q_1}^{-1}, \phi_{3,q_2} \cdot g_2 \cdot \phi_{2,q_2}^{-1}, \ldots, \phi_{k,q_{k-1}} \cdot g_{k-1} \cdot \phi_{k-1,q_{k-1}}^{-1}),$$

where $\phi_{i,q_i}, \phi_{i+1,q_i} \in G$ represents the automorphisms $\phi_i(q_i)$ and $\phi_{i+1}(q_i)$ in the fixed trivializations of P_i and P_{i+1} over $X_i - \{q_{i-1}\}$ and $X_{i+1} - \{q_{i+1}\}$, respectively. Lemma 1 states that, by using automorphisms of type $(\phi_1, \ldots, \phi_k)$, we can replace the tuple $(g_1, \ldots, g_{k-1}) \in G^{k-1}$ with a tuple $(n_1, \ldots, n_{k-1}) \in N(T)^{k-1}$ of elements of $N(T)$. We prove Lemma 1 by induction. In the initial step (Lemma 2), we show that we can replace the tuple $(g_1, \ldots, g_{k-1})$ with an equivalent tuple in which the first component is an element of $N(T)$. The inductive step is presented in Lemma 3. We prove there that, if $(g_1, \ldots, g_i, g_{i+1}, \ldots, g_{k-1})$ is a tuple such that $g_j \in N(T)$ for $1 \leq j \leq i$, then by using an automorphism $(\phi_1, \ldots, \phi_k)$ we can replace the tuple $(g_1, \ldots, g_k)$ with a tuple $(g_1, \ldots, g_i, g'_{i+1}, \ldots, g'_{n-1})$, where $g'_{i+1} \in N(T)$. This will finish the proof of Lemma 1. □

LEMMA 2. *We can replace the tuple $(g_1, \ldots, g_{k-1})$ with an equivalent tuple in which the first component is an element of $N(T)$.*

This follows from the Bruhat decomposition (Theorem 2.4) as follows. There exist $b_1, b_2 \in B$ and $n \in N(T)$ such that $b_2 n b_1^{-1} = g_1$. Let $\phi_1 \in \mathrm{Aut}(P_{\lambda_1})$ as in Corollary 1.4(2) be such that, in the trivialization of P_{λ_1} around q_1, we have $\phi_{1,q_1} = b_1$. Let $\phi_2 \in \mathrm{Aut}(P_{\lambda_2})$ as in Corollary 1.4(2) be such that, in the trivialization of P_{λ_2} around q_2, we have $\phi_{2,q_1} = b_2$. Let $\mathrm{Id}_i \in \mathrm{Aut}(P_{\lambda_i})$ denote the identity automorphism of P_{λ_i}. Then the automorphism $(\phi_1, \phi_2, \mathrm{Id}_3, \ldots, \mathrm{Id}_k)$ replaces the tuple $(g_1, g_2, \ldots, g_{k-1})$ with the tuple $(n, g'_2, \ldots, g'_{k-1})$. This ends the proof of Lemma 2 and also the first step of the induction. □

LEMMA 3. *If $(g_1, \ldots, g_i, g_{i+1}, \ldots, g_{k-1})$ is a tuple such that $g_j \in N(T)$ for $1 \leq j \leq i$, then using an automorphism $(\phi_1, \ldots, \phi_k)$ allows us to replace the tuple $(g_1, \ldots, g_k)$ with a tuple $(g_1, \ldots, g_i, g'_{i+1}, \ldots, g'_{k-1})$, where $g'_{i+1} \in N(T)$.*

We apply Proposition 2.8 to $\lambda_1, \ldots, \lambda_i \in X_*(T)$ and $n_1, \ldots, n_{i-1} \in N(T)$. Let $m \in N(T)$ be as in that proposition. Using Corollary 2.5 for $m \in N(T)$ and $g_{i+1} \in G$, we can find elements $b_1 \in mBm^{-1}$ and $b_2 \in B$ such that $b_2 g_{i+1} b_1^{-1} = n \in N(T)$. Applying Corollary 2.9 to $b_1 \in mBm^{-1}$, there exist $a_j \in Z(\lambda_j)$ and $c_j \in U(\lambda_j)$

for $1 \le j \le i$ such that $(a_{j+1})n_j(a_jc_j)^{-1} = n_j$ $(1 \le j \le i-1)$ and $b_1 = a_ic_i$. Let $\phi_1 \in \mathrm{Aut}(P_{\lambda_1})$ as in Corollary 1.4(2) be such that, in the trivialization of P_{λ_1} around q_1, we have $\phi_{1,q_1} = a_1c_1$. Let $\phi_j \in \mathrm{Aut}(P_{\lambda_j})$ $(2 \le j \le i)$ as in Corollary 1.4(1) be such that we have $\phi_{j,q_{j-1}} = a_j$ in the trivialization of P_{λ_j} around q_{j-1} and $\phi_{j,q_j} = a_jc_j$ in the trivialization of P_{λ_j} around q_j. Let $\phi_{i+1} \in \mathrm{Aut}(P_{\lambda_{i+1}})$ as in Corollary 1.4(2) be such that, in the trivialization of $P_{\lambda_{i+1}}$ around q_i, we have $\phi_{i+1,q_i} = b_2$. The automorphism $(\phi_1, \dots, \phi_i, \phi_{i+1}, \mathrm{Id}_{i+2}, \dots, \mathrm{Id}_k)$ replaces the tuple $(g_1, \dots, g_k)$ with $(g_1, \dots, g_i, n, g'_{i+2}, \dots, g'_{k-1})$. This ends the proof of Lemma 3 and hence of Lemma 1. □

We have thus proved that P is isomorphic to a principal G-bundle P' on X obtained by gluing the fibers over q_i of the principal G-bundles P_{λ_i} with gluing data, which in some fixed trivializations is a tuple $(n_1, \dots, n_{k-1})$ of elements of $N(T)$. To finish the proof of the Theorem 2.2, it suffices to prove that the structure group of the P' can be reduced to T. Let $l_{i+1} = n_in_{i-1}\cdots n_1 \in N(T)$ for $1 \le i \le k-1$ and let $l_1 = e \in N(T)$, where e denotes the identity element of G. Let $\lambda'_i = l_i^{-1}\lambda_il_i$ for $1 \le i \le k$. Let $\phi_i = \phi_{l_i^{-1}}\colon P_{\lambda_i} \to P_{\lambda'_i}$ $(1 \le i \le k)$ be the isomorphism defined in Observation 1.5. Then $(\phi_1, \phi_2, \dots, \phi_k)$ is an isomorphism in $\mathcal{C}_k$ between the principal G-bundle P' given by $(P_{\lambda_1}, n_1, P_{\lambda_2}, \dots, n_{k-1}, P_{\lambda_k})$ and another principal G-bundle P'' given by $(P_{\lambda'_1}, e, P_{\lambda'_2}, \dots, e, P_{\lambda'_k})$. The structure group of P'' can be reduced to T because the principal G-bundle P'' is obtained from the principal T-bundle $(T_{\lambda'_1}, e, T_{\lambda'_2}, \dots, e, T_{\lambda'_k})$ by extending the group structure via the group inclusion $T \to G$. This ends the proof of the Theorem 2.2. □

OBSERVATION 2.10. The proof can be briefly explained like this. A reduction of the group structure G of the principal bundle P to the subgroup $T \subset G$ is equivalent to a section of the fiber bundle P/T. The proof constructs such a section over X from sections over the components X_i. (Sections over the components exist by Theorem 1.1.)

The data for classification of principal G-bundles over X is a tuple $(\lambda_1, \dots, \lambda_k) \in X_*(T)^k$. For any such tuple, we define a principal G-bundle $Q(\lambda_1, \dots, \lambda_k)$ on X as follows. Applying Observation 1.2(3) to $\lambda_i \in X_*(T)$ and to the points $q_{i-1}, q_i \in X_i$, we obtain a principal G-bundle P_{λ_i} on X_i. As in the proof of Theorem 2.2, using the trivialization of P_{λ_i} on $X_i - \{q_{i-1}\}$ and $X_i - \{q_i\}$ enables us to identify the gluing data with a $(k-1)$-tuple of elements of G. We consider the gluing data $(A_1, \dots, A_{k-1})$ that correspond to the tuple $(e, \dots, e) \in G^{k-1}$, where e is the identity element of G. Using Proposition 2.1, we define $Q(\lambda_1, \dots, \lambda_k)$ as the principal G-bundle on X that corresponds to $(P_{\lambda_1}, A_1, \dots, A_{k-1}, P_{\lambda_k})$. The next theorem, which follows from the proof of Theorem 2.2, gives the classification of principal G-bundles on X.

THEOREM 2.11.

(1) *Any principal G-bundle on X is isomorphic with $Q(\lambda_1, \dots, \lambda_k)$ for some tuple $(\lambda_1, \dots, \lambda_k) \in X_*(T)^k$.*
(2) *If $\lambda'_i = w \cdot \lambda_i$ for some $w \in W$, then $Q(\lambda_1, \dots, \lambda_k)$ and $Q(\lambda'_1, \dots, \lambda'_k)$ are isomorphic.*

The converse of the second part of Theorem 2.11 is equivalent to the following conjecture.

CONJECTURE 2.12. *Let G be a complex reductive algebraic group. Let $\lambda_1, \dots, \lambda_k \in X_*(T)$. Let $n_i', n_i \in N(T)$ for $1 \le i \le k-1$. If $n_i' = (a_{i+1}c_{i+1})n_i(a_ib_i)^{-1}$ $(1 \le i \le k-1)$ for some $a_i \in Z(\lambda_i)$ and $b_i, c_i \in U(\lambda_i)$, then for all $1 \le i \le k-1$ there exist $m_i \in N(T) \cap Z(\lambda_i)$ such that $n_i' = m_{i+1}n_im_i^{-1}$.*

Part III

In this part, X is an irreducible singular nodal rational curve with exactly one singular point $q \in X$. Let $\pi : \mathbb{P}^1 \to X$ be the normalization of X and let $q_0, q_1 \in \mathbb{P}^1$ be the preimages of $q \in X$ under the morphism π. We study Zariski locally trivial principal G-bundles over X. The next proposition says that any principal G-bundle Q on X is obtained from a principal G-bundle P on $\mathbb{P}^1$ by gluing the fibers over the points q_0 and q_1.

PROPOSITION 3.1. *Let $Z = \mathrm{Spec}(\mathbb{C})$. Let $j_0 \colon Z \to \mathbb{P}^1$ be the morphism that factorizes through the point q_0, and let $j_1 \colon Z \to \mathbb{P}^1$ be the morphism that factorizes through the point q_1. Let $\mathcal{D}$ be the category of pairs (P, A), where P is a principal G-bundle over $\mathbb{P}^1$ and $A \colon j_0^*P \to j_1^*P$ is a morphism of G-bundles on Z. A morphism in $\mathcal{D}$ from (P, A) to (P', A') is given by a morphism $\phi \colon P \to P'$ of principal G-bundles over $\mathbb{P}^1$ such that $A' \circ j_0^*\phi = j_1^*\phi \circ A$. There is an equivalence of category of principal G-bundles on X and the category of pairs $\mathcal{D}$. Under this equivalence, the principal G-bundle Q over X corresponds to a pair (P, A), where the bundle P over $\mathbb{P}^1$ is the pull-back of Q.*

Proof. Let Q be a principal G-bundle over X. Let $P = \pi^*Q$ be the pull-back G-bundle on $\mathbb{P}^1$. Since $\pi \circ j_0 = \pi \circ j_1$, there is a canonical isomorphism $A \colon j_0^*P \to j_1^*P$ of principal G-bundles over q_1. We define a functor S from the category of principal G-bundles on X to the category $\mathcal{D}$, which on objects associates to a principal G-bundle Q the pair (P, A). We prove that S defines an equivalence of categories by constructing an inverse functor T from the category $\mathcal{D}$ to the category of principal G-bundles on X. If (P, A) is an object of $\mathcal{D}$, we construct a principal G-bundle $T(P, A)$ on X as in [2].

Let $p \in X$ be a point different from the singular point $q \in X$. Let $U = X - \{p\}$ and $\tilde{U} = \mathbb{P}^1 - \pi^{-1}(p)$ and let $\pi : \tilde{U} \to U$ be the restriction of the normalization morphism π. We may assume $\tilde{U} \cong \mathrm{Spec}(\tilde{R})$, where $\tilde{R} = \mathbb{C}[t]$. Then $U = \mathrm{Spec}(R)$, where $R = \{f \in \tilde{R} \mid f(q_0) = f(q_1)\}$. Since any principal G-bundle over $\mathbb{C}$ is trivial, we have a trivialization $P|_{\tilde{U}} \cong \tilde{U} \times G$ of P over $\tilde{U}$. Using this trivialization, the isomorphism A is equivalent to an automorphism of G, the mapping $\phi \colon G \to G$ given by multiplication on the left with an element of G. If $G = \mathrm{Spec}(S)$, then $\phi \colon G \to G$ corresponds to a $\mathbb{C}$-algebra isomorphism $\phi^* \colon S \to S$. Let B be the finitely generated R-algebra given by $B = \{s \in S \otimes_{\mathbb{C}} \tilde{R} = S[t] \mid s(x) = \phi^*(s(y))\} \subset S[t]$. We consider $h \colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$, which will be the restriction of $T(P, A)$ on $X - \{p\}$. There is a natural right action of $G = \mathrm{Spec}(S)$

on $\mathrm{Spec}(B)$ induced from the right action of G on $\mathrm{Spec}(S[t])$. The morphism $h\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is G-invariant and Zariski locally trivial. Local triviality follows after a pull-back under the étale morphism $\mathrm{Spec}\, R_1 \to \mathrm{Spec}\, R$, where $R_1 = \{(f, g) \in \mathbb{C}[t] \oplus \mathbb{C}[t] \mid f(x) = g(y)\}$. Note that $B \otimes_R R_1 \cong B_1$, where $B_1 = \{(r, s) \in S[t] \oplus S[t] \mid r(x) = s(y)\}$. Since $\mathrm{Spec}(B_1) \to \mathrm{Spec}(R_1)$ is Zariski locally trivial, by faithful flatness we obtain that $h\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is Zariski locally trivial.

This construction depends on the trivialization of P over $\tilde{U}$. For another trivialization, the isomorphism A defines an automorphism $\phi'\colon G \to G$ and hence another R-algebra B' and a morphism $h'\colon \mathrm{Spec}(B') \to \mathrm{Spec}(R)$. We can prove that there is a R-algebra isomorphism $B \to B'$ compatible with the actions of G on $\mathrm{Spec}(B)$ and $\mathrm{Spec}(B')$. Thus we have a G-equivariant isomorphism $f\colon \mathrm{Spec}(B') \to \mathrm{Spec}(B)$ such that $h \circ f = h'$. There is a canonical morphism $B \otimes_R \tilde{R} \to \tilde{R} \otimes_{\mathbb{C}} S$, which is an isomorphism that is compatible with the actions of G. That the morphism is an isomorphism can be seen by tensoring with the faithfully flat morphism $R \to R_1$. This means that the base change of $h\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ using the base morphism $\pi : \mathrm{Spec}(\tilde{R}) \to \mathrm{Spec}(R)$ is the projection $\mathrm{Spec}(\tilde{R}) \times G \to \mathrm{Spec}(\tilde{R})$.

Let $p_1 \in X$ be another point of X different from p and q. As before, let $U_1 = X - \{p_1\}$ and $\tilde{U}_1 = \mathbb{P}^1 - \pi^{-1}(p_1)$ and let $\pi : \tilde{U}_1 \to U_1$ be the restriction of π. If $\tilde{U}_1 = \mathrm{Spec}(\tilde{R}_1)$ where $\tilde{R}_1 = \mathbb{C}[t_1]$, then $U_1 = \mathrm{Spec}(R_1)$ where $R_1 = \{f \in \tilde{R}_1 \mid f(x) = f(y)\}$. We choose a trivialization of P over $\tilde{U}_1$ and let $\phi_1\colon G \to G$ be the morphism corresponding to A. As before, we construct $h_1\colon \mathrm{Spec}(B_1) \to \mathrm{Spec}(R_1)$, where B_1 is a R_1-algebra defined by $B_1 = \{s \in S \otimes_{\mathbb{C}} \tilde{R}_1 = S[t_1] \mid s(x) = \phi_1^*(s(y))\} \subset S[t_1]$. We think of $h_1\colon \mathrm{Spec}(B_1) \to \mathrm{Spec}(R_1)$ as the restriction of $T(P, A)$ on $X - \{p_1\}$.

We now glue together h and h_1 into a principal G-bundle on X. The gluing is done over $X - \{p, p_1\} = \mathrm{Spec}(W)$, where $W = \{f \in \tilde{W} = \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 - \{z, z_1\}) \mid f(q_0) = f(q_1)\}$ and $z, z_1 \in \mathbb{P}^1$ are the points over p and p_1, respectively. Note that the restriction of the normalization morphism is $\pi : \mathrm{Spec}(\tilde{W}) \to \mathrm{Spec}(W)$. The restriction of $h\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R)$ to $\mathrm{Spec}(W)$ can be identified with $\bar{h}\colon \mathrm{Spec}(C) \to \mathrm{Spec}(W)$, where $C = \{s \in \tilde{W} \otimes_{\mathbb{C}} S \mid s(q_0) = \phi^*(s(q_1))\} \subset \tilde{W} \otimes_{\mathbb{C}} S$. This identification is induced from the natural W-algebra morphism $B \otimes_R W \to C$, which is an isomorphism. Similarly, the restriction of $h_1\colon \mathrm{Spec}(B_1) \to \mathrm{Spec}(R_1)$ to $\mathrm{Spec}(W)$ can be identified with $\bar{h}_1\colon \mathrm{Spec}(C_1) \to \mathrm{Spec}(W)$, where $C_1 = \{s \in \tilde{W} \otimes_{\mathbb{C}} S \mid s(q_0) = \phi_1^*(s(q_1))\}$. The cocycle of P with respect to the trivializations over $\tilde{U}$ and $\tilde{U}_1$ is a morphism $\lambda\colon \tilde{U} \cap \tilde{U}_1 = \mathrm{Spec}(\tilde{W}) \to G$. The cocycle λ defines an automorphism $\tilde{\lambda}$ of $\tilde{W} \otimes_{\mathbb{C}} S$ corresponding to the action on the right with λ. Since $\phi_1 \circ \lambda(x) = \lambda(y) \circ \phi$, the automorphism $\tilde{\lambda}$ factorizes through a W-algebra isomorphism $\bar{\lambda}$ between C and C_1. We glue $\bar{h}\colon \mathrm{Spec}(C) \to \mathrm{Spec}(W)$ and $\bar{h}_1\colon \mathrm{Spec}(C_1) \to \mathrm{Spec}(W)$ by using the automorphism $\bar{\lambda}\colon \mathrm{Spec}(C) \to \mathrm{Spec}(C)$. The result is a Zariski locally trivial principal G-bundle on X, which we will denote by $T(P, A)$. This construction extends on morphisms. Thus we have a functor T from $\mathcal{D}$ to the category of principal G-bundles on X. If $Q = T(P, A)$, then there is a natural isomorphism $h\colon P \to \pi^*Q$ such that the canonical isomorphism

$j_0^*Q \to j_1^*Q$ is identified with $A\colon j_0^*P \to j_1^*P$. These observations show that T is a two-sided inverse of S. $\square$

If $\lambda \in X_*(T)$ is dominant and $g \in G$, we define a principal G-bundle $P(\lambda, g)$ on X as follows. Consider the bundle P_λ over $\mathbb{P}^1$ constructed in Observation 1.2(3), and recall that P_λ has trivializations over $\mathbb{P}^1 - \{q_0\}$ and $\mathbb{P}^1 - \{q_1\}$ such that the transition function over $\mathbb{P}^1 - \{q_0, q_1\}$ is λ. We consider $A\colon j_0^*P_\lambda \to j_1^*P_\lambda$ as a morphism of G-bundles on Z, which in the given trivializations of P_λ is given by the left multiplication by $g \in G$. The pair (P_λ, A) is an object of the category $\mathcal{D}$. Let $P(\lambda, g)$ be the principal G-bundle on X corresponding, as in Proposition 3.1, to the object (P_λ, A).

THEOREM 3.2. *Any principal G-bundle on X is isomorphic with $P(\lambda, g)$ as defined previously, where $\lambda \in X_*(T)$ is dominant and $g \in G$. If $\lambda, \lambda' \in X_*(T)$ are dominant and if $g, g' \in G$, then the principal G-bundles $P(\lambda, g)$ and $P(\lambda', g')$ are isomorphic if and only if $\lambda' = \lambda$ and $g' = (ab')^{-1}g(ab)$, where $a \in Z(\lambda)$ and $b, b' \in U(\lambda)$.*

Proof. Let Q be a principal G-bundle on X. Using Proposition 3.1, we can think of the principal G-bundle Q as being obtained from a principal G-bundle P on $\mathbb{P}^1$ by gluing the fibers over q_0 and q_1 using a morphism $A\colon j_0^*P \to j_1^*P$ of G-bundles on Z. From Observation 1.2(2), the bundle P has trivializations over $\mathbb{P}^1 - \{q_0\}$ and $\mathbb{P}^1 - \{q_1\}$ such that the transition function over $\mathbb{P}^1 - \{q_0, q_1\}$ is a dominant element $\lambda \in X_*(T)$. In these trivializations of P, the morphism A is given by the left multiplication with an element $g \in G$. It follows from all this that the bundle Q is isomorphic with $P(\lambda, g)$ as defined previously. This ends the proof of the first part of the theorem.

As in the proof of Theorem 2.2, an automorphism $\phi \in \mathrm{Aut}(P)$ replaces the gluing data $g \in G$ with $\phi_{q_1} \cdot g \cdot \phi_{q_0}^{-1}$, where $\phi_{q_i} \in G$ represents the automorphism of the fiber $P(q_i)$ over q_i induced by ϕ in the corresponding trivializations of P. By Theorem 1.4, there exist $a \in Z(\lambda)$ and $b, b' \in U(\lambda)$ such that $\phi_{q_1} = ab'$ and $\phi_{q_0} = ab$. It follows that the automorphism ϕ replaces the gluing data g with $(ab')g(ab)^{-1}$. Since λ is uniquely determined from Q, we have proved the claim of the theorem. $\square$

Motivated by the preceding theorem, we define an equivalence relation $\cong$ on G by $g \cong g'$ if and only if $g' = (ab')g(ab)^{-1}$ for some $a \in Z(\lambda)$ and $b, b' \in U(\lambda)$. The data for the construction of principal G-bundles on X is a pair $(\lambda, \hat{g})$, where λ is a dominant element of $X_*(T)$ and $\hat{g}$ is an equivalence class of the equivalence relation $\cong$ on G. For a complete description of the principal G-bundles on X, one needs a description of all the equivalence classes of the relation $\cong$ on G.

Part IV

In this part we study Zariski locally trivial principal G-bundles over a cycle of rational curves. A cycle of rational curves is a connected reduced complex curve

$X = X_1 \cup \cdots \cup X_k$ with $k \geq 2$ irreducible components X_i $(1 \leq i \leq k)$, all of them smooth rational curves, such that (a) X_i intersects X_{i+1} in a nodal singular point q_i for all $1 \leq i \leq k-1$ and (b) X_k intersects X_1 in a nodal singular point q_k. We identify X_i with X_{i+k} and q_i with q_{i+k}. The curve X is obtained from k copies $X_1, \ldots, X_k$ of a smooth rational curve by gluing $X_i - \{q_{i-1}\}$ to $X_{i+1} - \{q_{i+1}\}$ at the point q_i for $1 \leq i \leq k$, as in Part II. The starting point of the study of principal bundles over X is Proposition 4.1, which states that any principal G-bundle over X is obtained from principal G-bundles P_i over X_i by gluing the fibers over the common points q_i, $1 \leq i \leq k$. The proof is similar to the proof of the Proposition 2.1.

PROPOSITION 4.1. *Let $Y = Y_1 \cup \cdots \cup Y_k$ be a reduced curve with $k \geq 2$ irreducible components Y_i such that* (a) *Y_i intersects Y_{i+1} in a point q_i for $1 \leq i \leq k$ and* (b) *the structure sheaf $\mathcal{O}_Y$ is given by $(f_1, \ldots, f_k) \in \mathcal{O}_{Y_1} \oplus \cdots \oplus \mathcal{O}_{Y_k}$ with $f_i(q_i) = f_{i+1}(q_i)$ for $1 \leq i \leq k$. Here we identify Y_i with Y_{i+k} and q_i with q_{i+k}. Let j_i be the inclusion of q_i in Y_i and let j_i' be the inclusion of q_i in Y_{i+1}. Let $\mathcal{D}_k$ be the category of tuples $(P_1, A_1, P_2, A_2, \ldots, A_{k-1}, P_k, A_k)$, where P_i is a principal G-bundle over Y_i and $A_i \colon j_i^* P_i \to j_i'^* P_{i+1}$ is a morphism of G-bundles over q_i. A morphism in $\mathcal{D}_k$ between $(P_1, A_1, P_2, A_2, \ldots, A_{k-1}, P_k, A_k)$ and $(Q_1, B_1, Q_2, B_2, \ldots, B_{k-1}, Q_k, B_k)$ is given by a tuple $(\phi_1, \ldots, \phi_k)$, where $\phi_i \colon P_i \to Q_i$ is a G-morphism over Y_i $(1 \leq i \leq k)$ such that $B_i \circ j_i^* \phi_i = j_i'^* \phi_{i+1} \circ A_i$. There is an equivalence of categories between the category of principal G-bundles on Y and the category $\mathcal{D}_k$. Under this correspondence, the bundles $P_1, \ldots, P_k$ are the pull-backs of the bundle P.*

The classification of the principal G-bundles over X is presented in Theorem 4.2. We describe first the parameters of the classification. The data for the construction of a principal G-bundle on X is a triple $\mathbf{d} = (\Lambda, N, \hat{g})$, where $\Lambda = (\lambda_1, \ldots, \lambda_k)$ is a tuple of dominant elements of $X_*(T)$, $N = (n_1, \ldots, n_{k-1})$ is a tuple of elements of $N(T)$, and $\hat{g}$ is an equivalence class of an equivalence relation $\sim$ described momentarily. If $a \in Z(\lambda_1)$, define the sets $S_{i+1}(a) = Z(\lambda_{i+1}) \cap n_i S_i(a) n_i{}^{-1} U(\lambda_{i+1})$ $(1 \leq i \leq k-1)$ and $S_1(a) = aU(\lambda_1)$. We define an *equivalence relation* on G by $g \sim (ab) g (\tilde{a} c)^{-1}$, where $b \in U(\lambda_1)$, $c \in U(\lambda_k)$, $a \in Z(\lambda_1)$, and $\tilde{a} \in S_k(a)$; here $\hat{g}$ is an equivalence class with respect to the equivalence relation $\sim$. If $\mathbf{d}$ is as just described, let $P(\mathbf{d})$ be the principal G-bundle associated to $(P_{\lambda_1}, n_1, P_{\lambda_2}, \ldots, n_{k-1}, P_{\lambda_k}, g)$, considered as an object in $\mathcal{D}_k$. Here $g \in G$ is a representative of the equivalence class $\hat{g}$. Any other representative will define an isomorphic principal G-bundle.

THEOREM 4.2. *Any principal G-bundle on X is isomorphic with $P(\mathbf{d})$, for some data $\mathbf{d}$ as described previously.*

Proof. The proof is similar to that for Theorem 2.2. We start with a principal G-bundle on X. Using Proposition 4.1, we can think of the principal G-bundle P as being obtained from principal G-bundles over the components by gluing the fibers over q_i, $1 \leq i \leq k$. Let P_i be the pull-back of P to the component X_i, $1 \leq i \leq k$.

From Observation 1.2(2), the bundle P_i has trivializations over $X_i - \{q_{i-1}\}$ and over $X_i - \{q_i\}$ such that the transition function over $X_i - \{q_{i-1}, q_i\} = \mathbb{C}^*$ is λ_i, a dominant element of $X_*(T)$. We fix the two trivializations of P_{λ_i}. Using Proposition 4.1, the principal G-bundle P is obtained from the principal G-bundles P_{λ_i} over X_i by gluing data $(A_1, \ldots, A_{k-1}, A_k)$, where $A_i\colon j_i^* P_{\lambda_i} \to j_i'^* P_{\lambda_{i+1}}$ is an isomorphism of principal G-bundles over q_i. Using the fixed trivializations of P_i and P_{i+1} over q_i, the morphism A_i corresponds to an element $g_i \in G$ and so the gluing data is the tuple $(g_1, \ldots, g_k)$. We consider automorphisms $(\phi_1, \ldots, \phi_k)$ of P, where $\phi_i \in \mathrm{Aut}(P_{\lambda_i})$, and we consider how such an automorphism affects a gluing tuple $(g_1, \ldots, g_k)$. As in the proof of Theorem 2.2, we can construct automorphisms of P such that the tuple $(g_1, \ldots, g_k)$ is replaced with $(n_1, \ldots, n_{k-1}, g)$, where $n_1, \ldots, n_{k-1} \in N(T)$ and $g \in G$. A similar argument shows that an automorphism of P replaces the tuple $(n_1, \ldots, n_{k-1}, g)$ with the tuple $(n_1, \ldots, n_{k-1}, g')$ if and only if $g' \sim g$, where $\sim$ is the equivalence relation previously defined. This completes the proof. $\square$

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