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# Explicit Solutions to the $H$-Surface Equation on Tori

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## I. Introduction

A twice continuously differentiable map $x$ from $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^3$ is a solution to the $H$-surface equation if

$$\Delta x = 2H(x_u \wedge x_v) \ \text{ on } \Omega. \tag{1}$$

Here $\Delta x = x_{uu} + x_{vv}$ is the standard Laplacian on $\mathbb{R}^2$ and the wedge symbol denotes the usual cross product. We represent points in $\Omega$ by $(u, v)$ or sometimes $w = u + iv$; points in the target space are represented by $x = (x, y, z) \in \mathbb{R}^3$.

It is important to observe that solutions to (1) remain solutions after a conformal change of coordinates. Thus it makes sense to consider solutions to the $H$-surface equations on a Riemann surface. This may be seen as follows. We set the Dirichlet integral to be

$$D(x) = \iint_\Omega (|x_u|^2 + |x_v|^2)\, du\, dv \tag{2}$$

and the oriented volume functional to be

$$V(x) = \frac{1}{3}\iint_\Omega x \cdot (x_u \wedge x_v)\, du\, dv. \tag{3}$$

Observe that $D(x)$ is invariant under a conformal change of coordinates and that $V(x)$ is a parametric integral invariant under any smooth change of coordinates. Solutions to (1) are extremals of the $H$-functional

$$E_H(x) \equiv D(x) + 4HV(x). \tag{4}$$

To any solution of (1) is attached the Hopf differential

$$(x_w \cdot x_w)dw^2 = F(w)dw^2. \tag{5}$$

This is a holomorphic quadradic differential ($F_{\bar{w}} = 0$) and in local coordinates $F(w) = (|x_u|^2 - |x_v|^2) - 2i(x_u \cdot x_v)$. A solution $x(u, v)$ of (1) represents a surface of constant mean curvature (cmc surface) only when $F(w) \equiv 0$. Solutions of the $H$-surface equation have the same relationship to cmc surfaces as harmonic maps to minimal surfaces.

In an earlier paper [6] and also more recently [7], the author took up the question of solutions to (1) on annular domains that vanish on the boundary. By conformal invariance one may assume that the domain is a standard annulus bounded

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by two concentric circles centered at the origin. We looked for rotationally symmetric solutions. Pulling back to the universal cover by the exponential map, we search for solutions to (1) defined on the vertical strip $\Lambda_a = \{(u, v) \mid -a \leq u \leq a\}$ with $x(u, v + 2\pi) = x(u, v)$ such that $x(u, v)$ vanishes on $\partial\Lambda$. The region $\Lambda_a$ conformally covers the annulus

$$\Omega_A = \{w \mid 1/A < |w| < A\}, \quad A = e^a.$$

Upon trying solutions of the form

$$x(u, v) = \langle f(u)\cos v,\ f(u)\sin v,\ g(u)\rangle \tag{6}$$

we find that the differential equation (1) becomes

$$f'' - f = -2Hfg', \tag{7a}$$

$$g'' = 2Hff'. \tag{7b}$$

We focus on the case $H \neq 0$ and so may rescale by setting $H = -1$. One integration of (7b) gives $g' = Hf^2 + c$ for some constant $c$. Substitute this into (7a) with $H = -1$ and we obtain

$$f'' - (1 + 2c)f + 2f^3 = 0. \tag{8}$$

After one more integration we arrive at

$$f'^2 - (1 + 2c)f^2 + f^4 = \Gamma, \tag{9a}$$

$$g' = -f^2 + c. \tag{9b}$$

Equation (9a) may be thought of as a 1-dimensional dynamical system with potential energy $W(f) = -(1 + 2c)f^2 + f^4$ and can be solved by quadratures (see Figure 1). In earlier work we anayzed the situation when the total energy $\Gamma$ was positive. In this case we chose $L$ positive so that $W(L) = \Gamma$ and produced the solution $f(u)$ to (9a), where $f(u)$ is even in $u$ with $f(0) = L$ and $f'(0) = 0$ and where $f(u)$ is positive on the interval $-a < u < a$ with $f(-a) = f(a) = 0$. Here the period is given by

$$a = P(L, c) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{-(1 + 2c) + L^2(1 + \sin^2\theta)}}. \tag{10}$$

One then obtains $g(u)$ by integrating (9b). We choose $g(0) = 0$ so that $g(u)$ is an odd function of $u$. One wants $g(a) = 0$ as well. Clearly the constant $c$ must be positive, and the condition $g(a) = 0$ becomes

$$\Phi(L, c) = cP(L, c) - I(L, c) = 0, \tag{11a}$$

$$I(L, c) = \int_0^{\pi/2} \frac{L^2 \sin^2\theta \, d\theta}{\sqrt{-(1 + 2c) + L^2(1 + \sin^2\theta)}}. \tag{11b}$$

At this point we observe that the Hopf differential $F(w)$ given by (5) becomes

$$F(w) = f'^2 + g'^2 - f^2 = \Gamma + c^2 \tag{12}$$

(making use of system (9)); hence, if $\Gamma$ is positive then the immersion fails to be conformal.

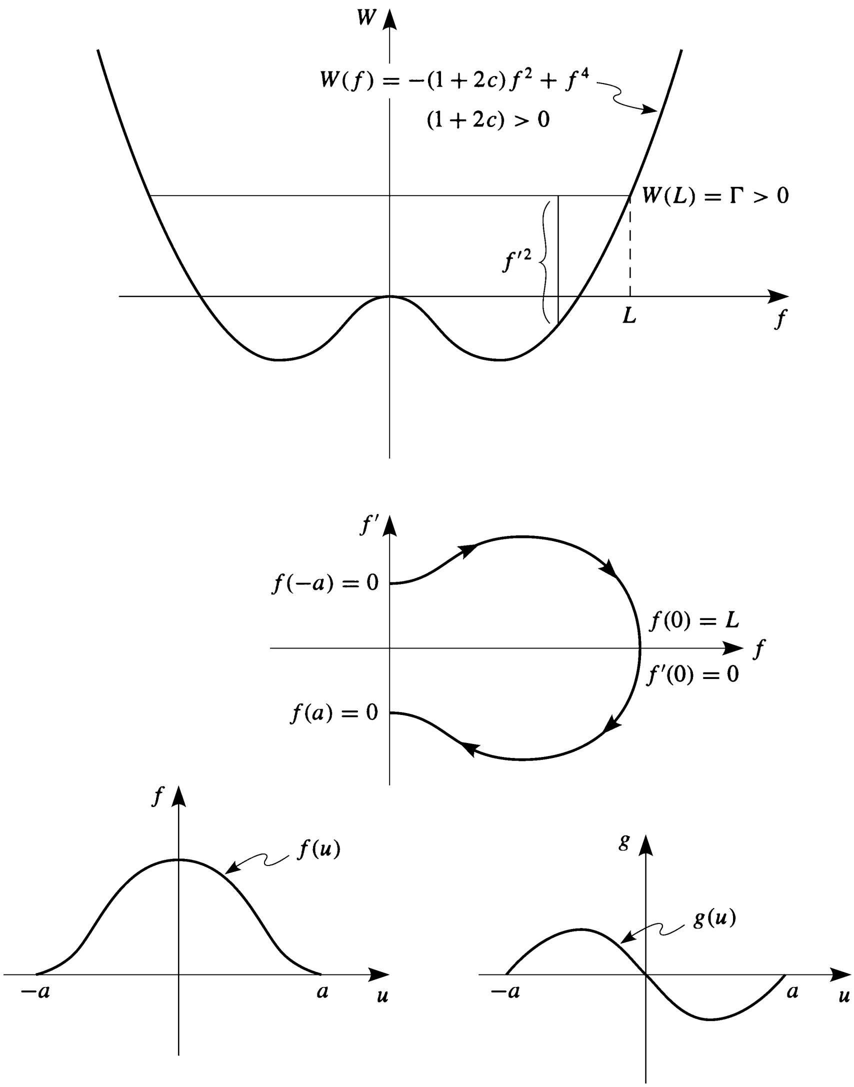


**Figure 1** The Dirichlet Problem

Equations (11) determine a smooth 1-parameter family of solutions $c = c(L)$. The pair $(L, c(L))$ approaches $(1, 0)$, and in this limit the conformal parameter $a = P(L, c)$ approaches $+\infty$ and equation (9a) becomes $f'^2 - f^2 + f^4 = 0$. We end up with the Mercator representation of the round sphere with $\langle f(u), g(u)\rangle = \langle \operatorname{sech} u, -\tanh u\rangle$. At the other extreme we have $P(L, c) \to 0$, so that the conformal type is a thin annulus. See [7] for a detailed discussion.

In Section II we continue our study of the system (9) and observe that, by setting $\Gamma$ to be negative, the possibility of immersed tori of rectangular conformal type appears if one assumes $c > -1/2$. We analyze the situation completely.

In Section III we consider immersed solutions of (1) on tori of nonrectangular conformal type. We give a variational argument to show that a solution of any conformal type must exist and then exhibit the solutions explicitly using the Weierstrass $\mathcal{P}$-function.

At this point we note that those solutions to (1) defined on the annulus $\Omega_A$ that are vanishing on the boundary can be extended to maps on $\mathbb{R}^2 - \{0\}$ by repeated odd reflection. We have at hand already nontrivial solutions to (1) defined on any rectangular torus. Other (trivial) solutions on any torus can be constructed using the Weierstrass $\mathcal{P}$-function, which gives a double cover of $S^2$ from a torus of any conformal type when we identify the complex plane with $S^2$ via stereographic projection.

Recently Yuxin Ge [1] produced solutions to the $H$-surface equation on certain Riemann surfaces by solving an appropriate variational problem. Let $\Omega \subset \mathbb{R}^2$ be a multiply connected domain bounded by circles that contains a "thick enough" annulus. Ge shows the existence of solutions to the $H$-surface equation on $\Omega$ such that on $\partial\Omega$ we have $z = x_\nu = y_\nu = 0$, where $x_\nu, y_\nu$ are the normal derivatives of $x, y$ along $\partial\Omega$. By reflection he obtains solutions to (1) on a Riemann surface. Presumably, if the Riemann surface is a rectangular torus then his solutions would agree with those we construct in Section II. Also, there has been an interesting paper by Koh [3] wherein is suggested an approach to obtaining solutions to the $H$-surface equation on tori using saddle-point techniques. The analysis is still incomplete.

Still open is the existence of nontrivial solutions to the $H$-surface equation with zero boundry data on any multiply connected domain or on any Riemann surface. A discussion of this may be found in Struwe [5].

## II. The Construction for Rectangular Tori

First we state an existence theorem for solutions to the $H$-surface equation on any torus of rectangular conformal type. Such a torus, $T_a$, can be represented in the form $\mathbb{R}^2/G_a$, where $G_a$ is the discrete lattice consisting of all points $(2ma, 2\pi n)$ where $a$ is positive and $(m, n) \in \mathbb{Z}_2$. A fundamental domain for $T_a$ is then

$$\Lambda_a = \{(u, v) \mid -a \le u \le a,\ 0 \le v \le 2\pi\}. \tag{13}$$

We consider vector functions on the Sobolev space $W_1(T_a)$ with square integrable first derivatives and denote by $W_1^s(T_a)$ the subspace of $W_1(T_a)$ consisting of rotationally symmetric functions having the form (6). For such functions we have

$$\iint_{T_a} |x|^2\, du\, dv = 2\pi \int_{-a}^{a} (f^2 + g^2)\, du, \tag{14a}$$

$$D(x) = 2\pi \int_{-a}^{a} (f'^2 + g'^2 + f^2)\, du. \tag{14b}$$

It follows that the space $W_1^s(T_a)$ can be identified with pairs of functions $\langle f, g\rangle$, each in $W_1([-a, a])$, that have continuous periodic extensions to $\mathbb{R}$ with period

$2a$. Note that if $f, g \in W_1([-a,a])$ then $f, g$ are absolutely continuous and so the conditions $f(-a) = f(a)$ and $g(-a) = g(a)$ are meaningful.

Furthermore, for $x(u,v) = \langle f(u)\cos v, f(u)\sin v, g(u)\rangle$ in $W_1^s(T_a)$, the Dirichlet integral $D(x)$ is given by (14b) and the volume functional $V(x)$ becomes

$$V(x) = 2\pi \int_{-a}^{a} ff'g\,du = -\pi \int_{-a}^{a} f^2 g'\,du. \tag{15}$$

THEOREM 2.1. *Consider the set of functions $\langle f, g\rangle \in W_1([-a,a])$ also satisfying $f(-a) = f(a)$, $g(-a) = g(a)$, and $\int_{-a}^{a} g(u)\,du = 0$. In this class of functions that also satisfy the volume constraint $V(x) = 1$, there exists a minimizer $x_0(u,v) = \langle f_0(u)\cos v, f_0(u)\sin v, g_0(u)\rangle$ of the Dirichlet integral $D(x)$. The pair $\langle f_0, g_0\rangle$ will be solutions to the system* (7) *for some constant $H$ and so $x_0(u,v)$ is a solution to the $H$-surface equation on the torus $T_a$.*

*Proof*. Let $\langle f_n, g_n\rangle$ be a minimizing sequence. The boundedness of the Dirichlet integral along with the condition $\int_{-a}^{a} g(u)\,du = 0$ implies that there is a subsequence in $W_1([-a,a])$ converging weakly to a limit function $\langle f_0, g_0\rangle$. The sequence $\langle f_n, g_n\rangle$ is an equicontinuous family and so a subsequence (relabeled) converges uniformly to $\langle f_0, g_0\rangle$, which satisfies the periodicity condition and determines a candidate $x_0(u,v)$ in $W_1^s(T_a)$. We need only check that $V(x_0) = 1$. However, the fact that $x_n(u,v)$ converges weakly and uniformly to $x_0(u,v)$ is sufficient to guarantee this. □

The proof of Theorem 2.1 is similar to a result of Patnaik's, who proved in [4] an existence theorem for rotationally symmetric $H$-surfaces of annular type whose boundary is a pair of coaxial circles. In this case the minimizers are conformal cmc surfaces.

Observe that for $H = 0$ the system (7) gives $f'' - f = 0$ and so $f(u)$ cannot be periodic unless $f(u) \equiv 0$. The minimizer of Theorem 2.1 has $H \neq 0$. By rescaling we may set $H = -1$.

We move on to (9b) and consider the graph of $W(f) = -(1+2c)f^2 + f^4$. We assume that $1+2c$ is positive. The graph has a local maximum at $(0,0)$ and a global minimum at $\sqrt{(1+2c)/2}$ and is increasing for $f > \sqrt{(1+2c)/2}$. It follows that, for any choice of $\Gamma$ with $-(1+2c)^2/4 < \Gamma < 0$, there is a periodic solution $f(u)$ to (9a) of total period $2a$ oscillating between a maximum $f_M$ with $\sqrt{(1+2c)/2} < f_M < \sqrt{1+2c}$ and a minimum $f_m$ lying between 0 and $\sqrt{(1+2c)/2}$ (see Figure 2). Having picked such an $f$ with $f(0) = f_M$ and $f(\pm a) = f_m$, we integrate (9b) to find $g(u)$ where we may set $g(0) = 0$.

We want $g(u)$ to be periodic as well. This will happen if $g(a) = 0$. Note that $g'(0) = c - f_M^2 < c - \left(\frac{1+2c}{2}\right) = -1/2 < 0$ (see Figure 3).

An integration of (9a) gives an inverse representation of $f(u)$,

$$u = \int_F^{f_M} \frac{df}{\sqrt{\Gamma - W(f)}}, \tag{16}$$

so that the period $2a$ is determined by

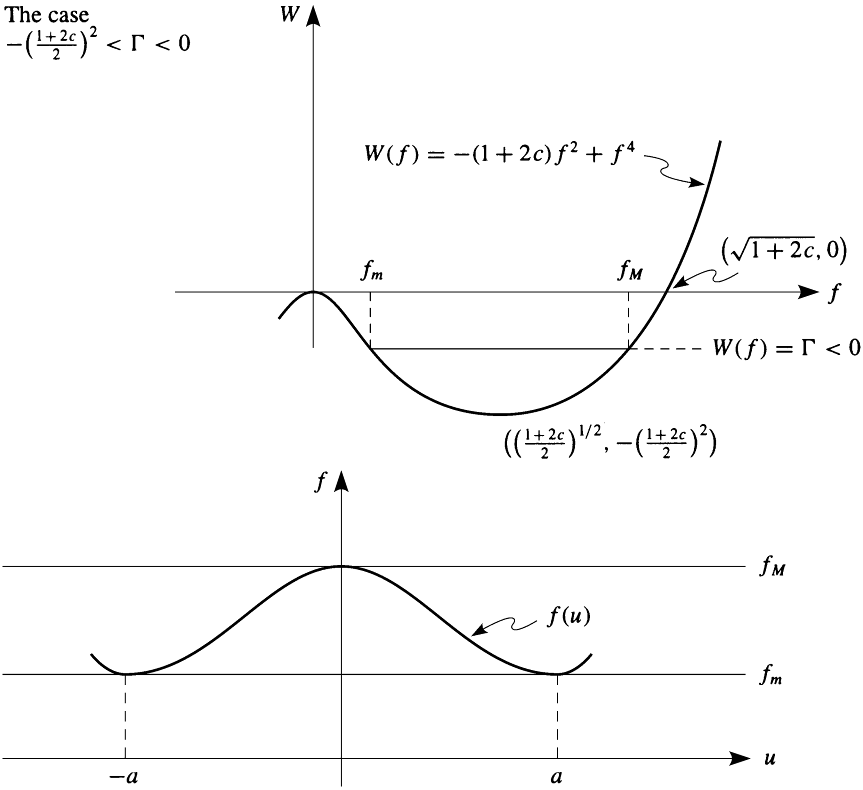


**Figure 2** The Torus Case

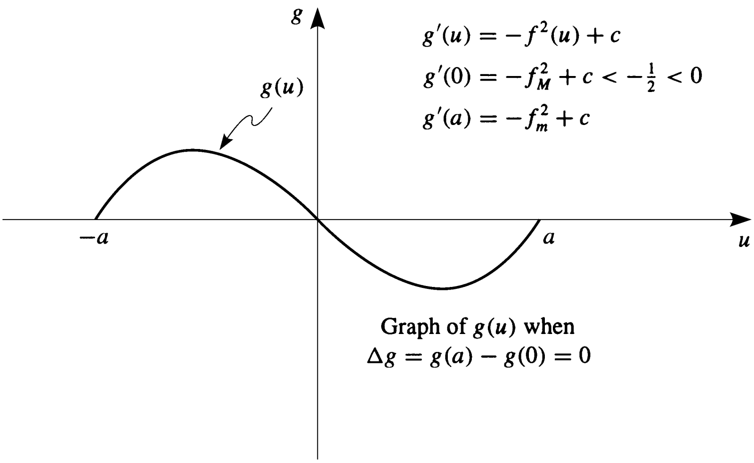


**Figure 3**

$$a = P(c, \Gamma) = \int_{f_m}^{f_M} \frac{df}{\sqrt{\Gamma - W(f)}}; \tag{17}$$

the expression for $\Delta g = g(a) - g(0)$ is

$$\Delta g = \int_0^a (c - f^2)\, du = \int_{f_m}^{f_M} \frac{(c - f^2)\, df}{\sqrt{\Gamma - W(f)}}. \tag{18}$$

Our goals are to determine $(c, \Gamma)$ so that $\Delta g = 0$ and to estimate $P(c, \Gamma)$.

These integrals can be simplified. The values $f_m$, $f_M$ are roots of the equations $\Gamma - W(f) = \Gamma + (1 + 2c) f^2 - f^4 = 0$. Upon solving for $f^2$ we find that

$$f^2_{m,M} = B \pm A, \tag{19}$$

where $B = (1+2c)/2$ and $0 < A < B$ is obtained by setting $A^2 = \Gamma + B^2$. Make the change of variable $f^2 = Ax + B$ and find

$$u = \int_F^{f_M} \frac{df}{\sqrt{\Gamma - W(f)}} = \frac{1}{2} \int_X^1 \frac{dx}{\sqrt{Ax + B}\sqrt{1 - x^2}}, \quad F^2 = AX + B. \tag{20}$$

The period then becomes

$$a = P(B, A) = \frac{1}{2} \int_{-1}^1 \frac{dx}{\sqrt{Ax + B}\sqrt{1 - x^2}}. \tag{21}$$

It is convenient to set

$$I(B, A) = \int_{f_m}^{f_M} \frac{f^2\, df}{\sqrt{\Gamma - W(f)}} = \frac{1}{2} \int_{-1}^1 \frac{\sqrt{Ax + B}\, dx}{\sqrt{1 - x^2}}, \tag{22}$$

which leads to

$$\begin{aligned} \Delta g &= \Phi(B, A) = cP(B, A) - I(B, A) \\ &= \left(B - \frac{1}{2}\right) P(B, A) - I(B, A) = -\frac{1}{2} \int_{-1}^1 \frac{(Ax + 1/2)\, dx}{\sqrt{Ax + B}\sqrt{1 - x^2}}. \end{aligned} \tag{23}$$

The equation $\Phi(B, A) = 0$ determines the solutions to the $H$-system equation on rectangular tori. We are led to the following result.

THEOREM 2.2. *The equation* $\Phi(B, A) = 0$ *from* (23) *with* $B = c + 1/2$ *and* $A^2 = \Gamma + B^2$ $(0 < A < B)$ *determines* $A$ *as a function of* $B$, $A = A(B)$, *defined for* $B > 1/2$ *with* $1/2 < A(B) < B$. *As* $B$ *approaches* $1/2$ *the pair* $(B, A(B))$ *approaches* $(1/2, 1/2)$, *corresponding to the case* $c = \Gamma = 0$. *In this limit the toral solutions converge to* $\langle f(u), g(u) \rangle = \langle \operatorname{sech} u, -\tanh u \rangle$, *which is the Mercator representation of the unit sphere with* $x(-\infty, v) = (0, 0, 1) =$ north pole *and* $x(+\infty, v) = (0, 0, -1) =$ south pole. *Also, as* $B$ *approaches* $1/2$ *the period* $a = P(B, A(B))$ *approaches* $+\infty$. *As* $B \to +\infty$, $A(B)$ *also becomes infinite and* $\lim P(B, A(B)) = 0$.

*The family of toral solutions ranges over all conformal types. All of the solutions have* $H = -1$. *Denote by* $x_B$ *the solution corresponding to the pair* $(B, A(B))$. *We have* $\lim D(x_B) = 8\pi$ *as* $B \to 1/2$ *and* $\lim D(x_B) = \infty$ *as* $B$ *becomes infinite. If we rescale by setting* $y_B = t_B x_B$, *where* $t_B$ *is chosen so that* $V(y_B) = 1$, *then* $\lim D(y_B) = 8\pi$ *as* $B \to 1/2$ *and* $\lim D(y_B) = \infty$ *as* $B$ *becomes infinite* (see Figure 4).

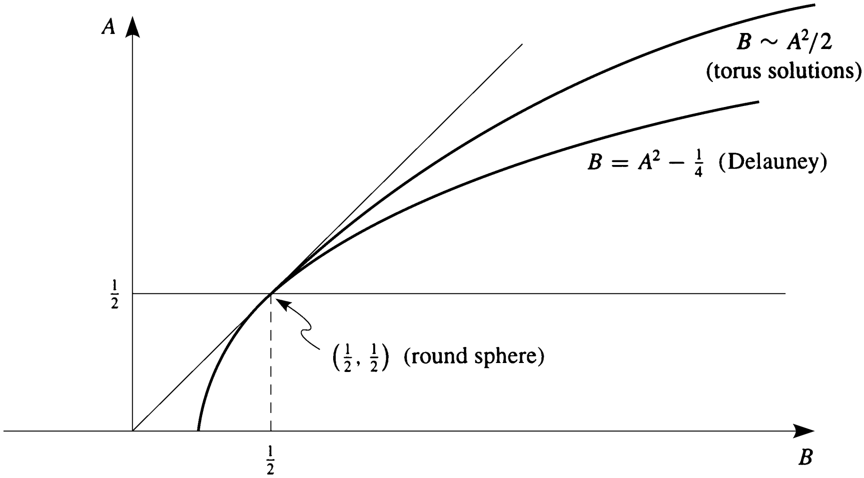


**Figure 4**

*Finally, the generating curve for* $x(u, v)$ *given by* $\langle f(u), g(u)\rangle$ *with* $-a \le u \le a$ *is an embedded curve with* $f(u) > 0$, *so that the corresponding immersed surface is embedded. These surfaces are never conformal.*

*Proof.* The proof follows from a sequence of assertions involving the integrals (21)–(23).

1. For $A < 1/2$ we have $Ax + 1/2 > 0$ for $x \in [-1, 1]$ and so $\Phi(B, A) < 0$ using (23).

2. For $A = 1/2$, $\Phi(B, A)$ is still negative and for $A = B = 1/2$ we have $\Phi(1/2, 1/2) = -1$. Otherwise, $\Phi(B, B) = +\infty$ when $B > 1/2$ and equals $-\infty$ for $0 < B < 1/2$.

3. $\Phi(B, 0) = -\pi/2\sqrt{B} < 0$ and so, for every $B > 1/2$, there exists $A(B)$ with $\Phi(B, A(B)) = 0$.

4. We have

$$\Phi_A(B, A) = \frac{-1}{4}\int_{-1}^{1} \frac{x}{\sqrt{Ax + B}} \frac{dx}{\sqrt{1 - x^2}} - \frac{1}{4}\int_{-1}^{1} \frac{(B - 1/2)x\,dx}{(Ax + B)^{3/2}\sqrt{1 - x^2}}. \tag{24}$$

Consequently, if $B > 1/2$ and $0 < A < B$ then we see that $\Phi_A(B, A)$ is positive. Thus the equation $\Phi(B, A) = 0$ determines $A = A(B)$ defined for $B > 1/2$ with $1/2 < A(B) < B$, and $\lim A(B) = 1/2$ as $B \to 1/2$.

5. $\lim_{B\to+\infty} \sqrt{B}\Phi(B, A) = -\pi/4$ for any fixed $A$ because

$$\sqrt{B}\Phi(B, A) = -\frac{\sqrt{B}}{2}\int_{-1}^{1}\frac{(Ax+1/2))\,dx}{\sqrt{Ax+B}\sqrt{1-x^2}}$$
$$\to -\frac{1}{4}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = -\frac{\pi}{4}. \quad (25)$$

It follows that $\lim_{B\to+\infty} A(B) = +\infty$.

6. Let $B = kA$ where $k > 1$. We have

$$\lim_{A\to+\infty}\frac{\Phi(kA, A)}{\sqrt{A}} = -\frac{1}{2}\int_{-1}^{1}\frac{x\,dx}{\sqrt{x+k}\sqrt{1-x^2}} > 0. \quad (26)$$

It follows that, for any $\lambda$ $(0 < \lambda < 1)$, we have $0 < A(B) < \lambda B$ for $B$ sufficiently large. Observe that $A(B)$ has sublinear growth.

7. We now estimate the period $P(B, A(B))$. Since $\Phi(B, A(B)) = 0$ we may use (23) to write $P(B, A(B)) = I(B, A(B))/(B - 1/2)$. However,

$$I(B, A(B)) < \frac{1}{2}\int_{-1}^{1}\frac{\sqrt{A+B}\,dx}{\sqrt{1-x^2}} < \frac{\sqrt{B}}{\sqrt{2}}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \frac{\pi\sqrt{B}}{\sqrt{2}}. \quad (27)$$

Thus,

$$P(B, A(B)) < \frac{\pi}{\sqrt{2}}\frac{\sqrt{B}}{(B-1/2)}, \quad (28)$$

which shows that the limit of $P(B, A(B)) = 0$ as $B \to +\infty$. We already know $\lim P(B, A(B)) = P(1/2, 1/2) = +\infty$ as $B \to 1/2$. Our immersed tori range over all conformal types.

8. Let $x_B$ be the immersion corresponding to the pair $(B, A(B))$. We claim that $\lim D(x_B) = +\infty$ as $B$ becomes infinite.

Start now with the formula for $D(x)$ given by (14b); making use of the identities in system (9), one finds $f'^2 + g'^2 + f^2 = (\Gamma + c^2) + 2f^2$. This leads to

$$D(x_B) = 4\pi(\Gamma + c^2)P + 8\pi I, \quad (29)$$

where $P$ and $I$ are the integrals (21) and (22). Now use $\Phi = cP - I = 0$ to obtain

$$D(x_B) = 4\pi[\Gamma + c^2 + 2c]P(B, A(B)); \quad (30)$$

since $-((1+2c)/2)^2 < \Gamma < 0$ it follows that $\Gamma + c^2 + 2c > -(c+1/2)^2 + c^2 + 2c = c - 1/4 = B - 3/4$. We have

$$D(x_B) > 4\pi(B - 3/4)P(B, A(B)) = \frac{4\pi(B-3/4)I(B,(A(B))}{(B-1/2)}. \quad (31)$$

However,

$$I(B, A(B)) = \frac{1}{2}\int_{-1}^{1}\frac{\sqrt{Ax+B}\,dx}{\sqrt{1-x^2}} > \frac{1}{2}\int_{0}^{1}\frac{\sqrt{B}\,dx}{\sqrt{1-x^2}} = \frac{\pi\sqrt{B}}{4}. \quad (32)$$

We finally get

$$D(x_B) > \frac{4\pi(B-3/4)}{(B-1/2)}\cdot\frac{\pi\sqrt{B}}{4}, \quad (33)$$

showing that $D(x_B) \to +\infty$ as $B$ becomes infinite.

Since $H = -1$ we have $D(x_B) + 6HV(x_B) = D(x_B) - 6V(x_B) = 0$. We now set $y_B = t_B x_B$, where $t_B > 0$ is chosen so that $V(y_B) = t_B^3 V(x_B) = 1$; we find $t_B^3 = 6/D(x_B)$. This leads to the identity $D(y_B) = 6^{2/3} D(x_B)^{1/3}$ and so $\lim D(y_B) = +\infty$ as $B \to +\infty$ as well.

Observe next that, as $B$ approaches $1/2$, the pair $(B, A(B))$ must approach $(1/2, 1/2)$. Since $B = (1+2c)/2$ and $\Gamma = A^2 - B^2$, we see that $\Gamma < 0$ and the pair $(c, \Gamma)$ approaches $(0,0)$. It follows from (21) that the period $a = P(B, A)$ becomes infinite while the generating pair $\langle f(u), g(u)\rangle$ converges to the solution of system (9) with $c = \Gamma = 0$. That is, $f'^2 - f^2 + f^4 = 0$ and $g' = -f^2$ with initial conditions $f(0) = 1$, $f'(0) = 0$, $g(0) = 0$, and $g'(0) = -1$. We find that $\langle f(u), g(u)\rangle = \langle \operatorname{sech} u, -\tanh u\rangle$, which gives the Mercator representation of the unit sphere.

Finally, we claim that the generating curve $\langle f(u), g(u)\rangle$ for any one of these tori is a simple closed curve. For $0 \le u \le a$, we have that $f(u)$ is strictly decreasing from $f_M$ to $f_m > 0$ while $g(u)$ is negative for $0 < u < a$ with $g(0) = g(a) = 0$. This assertion is clear (see Figure 5). The immersed surface $x_B(u, v)$ is also embedded. □

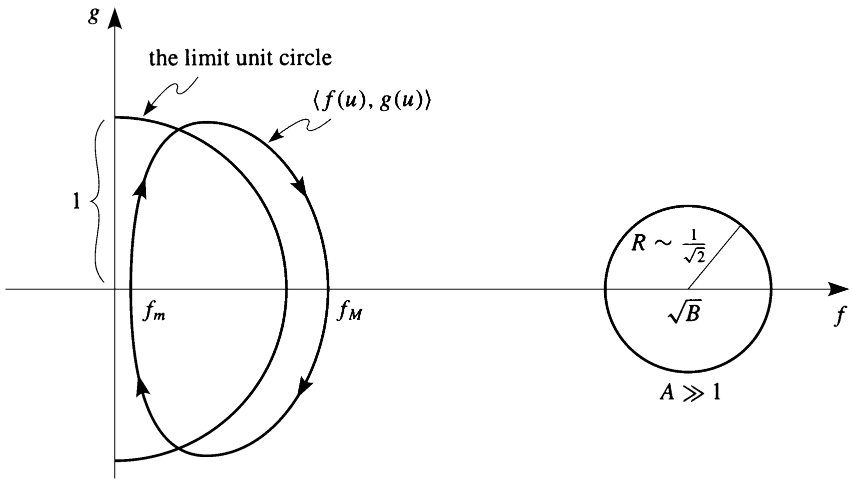


**Figure 5** The Generating Curves for Tori

Our family of toroidal $H$-surfaces converges to a round sphere as $B \to 1/2$ with the north pole at $(0,0,1)$ and south pole at $(0,0,-1)$. The toral surfaces resemble a sphere from which two small disks near the poles have been removed and a narrow tube inserted, connecting the two holes. In this limit the conformal type has $a = P(B, A) \to +\infty$. We now investigate the shape of the toral $H$-surfaces when $B$ becomes infinite. We have the following lemma.

LEMMA 2.1. *Let $(B_n, A_n)$ be a sequence with $A_n \to +\infty$ and such that $B_n/A_n^2 \to \lambda > 0$ (i.e., $B_n = \lambda A_n^2 + \Delta_n$, where $\Delta_n/A_n^2 \to 0$). Consider the corresponding* *pair of functions* $\langle f_n, g_n\rangle$ *solving system* (9) *when restricted to the interval* $-a \le u \le a$, *so that the curve will not close up unless* $\Phi(B_n, A_n) = 0$. *Asymptotically, the pair of functions will satisfy the identity*

$$\left(f_n - \sqrt{\lambda}A_n\right)^2 + g_n^2 = 1/4\lambda. \tag{34}$$

*The generating curves describe a round circle with center* $\left(\sqrt{\lambda}A_n, 0\right)$ *and radius* $1/2\sqrt{\lambda}$.

*Proof*. We refer to equations (20)–(23), where we made the change of variable $f^2 = Ax + B$ or $F^2 = AX + B$ with the function $X(u)$ determined by (20). From (23) one may express $g$ in terms of $X$ by

$$g = \frac{-1}{2}\int_X^1 \frac{(Ax + 1/2)\,dx}{\sqrt{Ax+B}\sqrt{1-x^2}}, \quad -1 < X < 1. \tag{35}$$

Now let $A = A_n$ with $B_n = \lambda A_n^2 + \Delta_n$, so that in the limit we find

$$4\lambda g^2 = 1 - X^2. \tag{36}$$

Solving for $f$ in terms of $X$ leads to

$$X = \frac{1}{A}(f^2 - B) = \frac{\left(f - \sqrt{\lambda A^2 + \Delta}\right)\left(f + \sqrt{\lambda A^2 + \Delta}\right)}{A}. \tag{37}$$

Since $B_n = \lambda A_n^2 + \Delta_n$ and since by (20) we have $B - A < f^2 < B + A$, it follows that $\lim\left(f_n + \sqrt{\lambda A_n^2 + \Delta_n}\right)/A_n = 2\sqrt{\lambda}$; this shows that

$$X^2/4\lambda \cong \left(f_n(X) - \sqrt{\lambda}A_n\right)^2. \tag{38}$$

This, along with (36), leads to

$$\left(f_n - \sqrt{\lambda}A_n\right)^2 + g_n^2 = 1/4\lambda. \tag{39}$$

□

We now have the following result.

THEOREM 2.3. *Consider the family of toroidal H-surfaces* ($H = -1$) *determined by the condition* $\Phi(B, A) = 0$. *We claim that* $\lim B/A^2(B) = \lambda = 1/2$ *as* $B$ *becomes infinite. It follows that the generating curve for the immersed torus is asymptotically the round circle*

$$\left(f - \sqrt{B}\right)^2 + g^2 = 1/2 \tag{40}$$

*with center* $\left(\sqrt{B}, 0\right)$ *and radius* $1/\sqrt{2}$.

*Proof*. The proof is based on the following estimate. Let $B = \lambda A^2$ with $\Phi(B, A)$ given by (23). We claim

$$\lim_{A\to\infty} A\Phi(\lambda A^2, A) = \frac{(1-2\lambda)\pi}{8\lambda^{3/2}}. \tag{41}$$

From this it follows that if $A = A(B)$ is determined by $\Phi(B, A) = 0$ then $B/A^2 \to \lambda = 1/2$ and so the theorem follows from Lemma 2.1.

We can write $\Phi(\lambda A^2, A)$ as the sum

$$\begin{aligned}\Phi(\lambda A^2, A) &\\ &= I_1 + I_2\\ &= \frac{1}{2}\int_{-1}^{1}\frac{x\,dx}{\sqrt{\lambda+(x/A)}\sqrt{1-x^2}} - \frac{1}{4A}\int_{-1}^{1}\frac{dx}{\sqrt{\lambda+(x/A)}\sqrt{1-x^2}}.\end{aligned} \tag{42}$$

From the binomial expansion we have

$$(\lambda + (x/A))^{-1/2} = \frac{1}{\sqrt{\lambda}}\left[1 - \frac{x}{2\lambda A} + O(1/A^2)\right]. \tag{43}$$

This leads to

$$I_1 = \frac{1}{4\lambda^{3/2}A}\int_{-1}^{1}\frac{x^2\,dx}{\sqrt{1-x^2}} + O(1/A^2) \tag{44a}$$

$$I_2 = -\frac{1}{4\sqrt{\lambda}A}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} + O(1/A^2), \tag{44b}$$

giving us

$$\Phi(\lambda A^2, A) = \frac{\pi}{8A\lambda^{3/2}}(1-2\lambda) + O(1/A^2). \tag{45}$$

□

REMARKS. Another special class of surfaces contained in the family are the Delauney surfaces. These occur exactly when the immersion is conformal and is determined by the condition $\Gamma + c^2 = 0$ from (12). From the identities $A^2 = \Gamma + B^2$ and $B = c + 1/2$ this condition can be rewritten as $A^2 = B - 1/4$, so the factor $\lambda = 1$. From Lemma 2.1, the limit curves are circles $\left(f - \sqrt{B}\right)^2 + g^2 = 1/4$ with radius $r = 1/2$ as it should be.

Since $\Gamma + c^2 = 0$ we must have $\Gamma < 0$. Now $-(1+2c)^2/4 = -(c+1/2)^2 = -c^2 - c - 1/4 < \Gamma = -c^2 < 0$, which can only happen if $c > -1/4$. We find:

$$\begin{aligned} c &= -1/4, && \text{the cylinder } f(u) \equiv 1/2;\\ -1/4 < c &< 0, && \text{unduloids;}\\ c = \Gamma &= 0, && \text{the unit sphere;}\\ c &> 0, && \text{nodoids.}\end{aligned}$$

Finally, our family of surfaces also includes surfaces of revolution with Gauss curvature $K = -1$. For some members of the family this will occur precisely when $f'(a) = g'(a) = 0$, so that the generating curve will contain a cusp. This will occur when $f_m = \sqrt{c}$, leading to $B - A = f_m^2 = c = B - 1/2$ and hence $A = 1/2$. This is equivalent to $\Gamma + c(c+1) = 0$.

## III. Immersions of Nonrectangular Type

Suppose the $H$-surface of revolution about the $z$-axis has a generating function $\langle x(u), y(u), z(u)\rangle$, not necessarily a planar curve. The surface representation then takes the form

$$x(u, v) = \langle x(u)\cos v - y(u)\sin v,\ x(u)\sin v + y(u)\cos v,\ z(u)\rangle. \tag{46}$$

The immersion will have periods $\langle 2a, 2b\rangle$ and $\langle 0, 2\pi\rangle$ when

$$x(u + 2a)\cos 2b - y(u + 2a)\sin 2b = x(u), \tag{47a}$$

$$y(u + 2a)\sin 2b + y(u + 2a)\cos 2b = y(u). \tag{47b}$$

The vector function $x(u, v)$ in (44) will be a solution to the $H$-surface equation (1) if

$$x'' - x = 2H(-xz'), \tag{48a}$$

$$y'' - y = 2H(-yz'), \tag{48b}$$

$$z'' = 2H(xx' + yy'). \tag{48c}$$

An integration of (48c) gives $z' = c + H(x^2 + y^2)$, leading to the system

$$x'' = (1 - 2cH)x - 2H^2(x^2 + y^2)x, \tag{49a}$$

$$y'' = (1 - 2cH)y - 2H^2(x^2 + y^2)y, \tag{49b}$$

$$z' = c + H(x^2 + y^2). \tag{49c}$$

We now show that, for every choice of periods $\langle 2a, 2b\rangle$ and $\langle 0, 2\pi\rangle$ with $a > 0$, there is a solution $x(u, v)$ to the $H$-surface equation with these periods and having the form (46).

As in Section II we denote the flat torus $T(a, b)$ as $\mathbb{R}^2/G(a, b)$, where $G(a, b)$ is the discrete translation group with fundamental periods $(2a, 2b)$ and $(0, 2\pi)$. We form the Sobolev spaces $W_1(T(a, b))$ and $W_1^s(T(a, b))$, where the latter space are those periodic functions of the form (46) satisfying the periodicity conditions (47). A fundamental domain $\Lambda(a, b)$ is the parallelogram in $\mathbb{R}^2$ with vertices $(-a, -b)$, $(a, b)$, $(-a, -b + 2\pi)$, and $(a, b + 2\pi)$. A direct calculation gives

$$D(x) = 2\pi \int_{-a}^{a} [(x'^2 + y'^2 + z'^2] + (x^2 + y^2)]\, du, \tag{50a}$$

$$V(x) = 2\pi \int_{-a}^{a} z(xx' + yy')\, du = -\pi \int_{-a}^{a} z'(x^2 + y^2)\, du. \tag{50b}$$

THEOREM 3.1. *For each* $(a, b)$ *where* $a$ *is positive, there exists a member* $x_0$ *of the space* $W_1^s(T(a, b))$ *that is a minimizer of the Dirichlet integral* $D(x)$, *subject to the volume constraint* $V(x) = 1$. *Such a minimizer is obtained from a generating curve* $\langle x_0(u), y_0(u), z_0(u)\rangle$, *which is a solution to the system* (48) *for some* $H \neq 0$; *hence,* $x_0(u, v)$ *solves the* $H$*-surface equation.*

*Proof.* In the class $W_1^2(T(a, b))$, the Dirichlet integral and volume functional are given by the single variable integrals (48). The argument now proceeds exactly as in the proof of Theorem 2.1 of Section II. Observe that, for $H = 0$, the only periodic solution is when $x(u) = y(u) = 0$. □

Since the solution in Theorem 3.1 has $H \neq 0$, we can rescale so that $H = -1$, which we now do. Our goal is to exhibit the solution to the system (48) when $H = -1$. In this case the system (48) is

$$x'' = (1 + 2c)x - 2(x^2 + y^2)x, \tag{51a}$$

$$y'' = (1 + 2c)y - 2(x^2 + y^2)y, \tag{51b}$$

$$z' = c - (x^2 + y^2). \tag{51c}$$

We seek solutions to this system satisfying the periodicity conditions (47). The equations (51a,b) can be interpreted as the equations of planar motion of a particle acted upon by a central force. Upon introduction of polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (51) becomes

$$\langle x'', y''\rangle = ((1 + 2c)r - 2r^3)u_r, \tag{52a}$$

$$z' = c - r^2, \tag{52b}$$

where $u_r = \langle x/r, y/r\rangle$ is the outward-pointing unit radial vector. We search for solutions $r = r(u)$ and $\theta = \theta(u)$. In polar coordinates one finds

$$D(x) = 2\pi \int_{-a}^{a} (r'^2 + r^2\theta'^2 + r^2 + z'^2)\, du, \tag{53a}$$

$$V(x) = 2\pi \int_{-a}^{a} zrr'\, du = -\pi \int_{-a}^{a} z'r^2\, du. \tag{53b}$$

We have conservation of angular momentum, and in polar coordinates the system (52) becomes

$$r'^2 + h^2/r^2 - (1 + 2c)r^2 + r^4 = \Gamma, \tag{54a}$$

$$r^2\theta' = h, \tag{54b}$$

$$z' = c - r^2. \tag{54c}$$

The solution to the system depends on three constants $(c, h, \Gamma)$. For $h = 0$ there is no angular momentum and we are back in the framework of Section II. Otherwise we can solve (54a) by quadratures to find $r(u)$. Integrations of (54b,c) then give us $\theta(u)$ and $z(u)$. As in Section II, we may rewrite (54a) in the form

$$r'^2 + W(r) = \Gamma, \tag{55a}$$

$$W(r) = h^2/r^2 - (1 + 2c)r^2 + r^4 \tag{55b}$$

(see Figure 6).

We now exhibit the solution $r(u)$ to system (55) explicitly using the Weierstrass $\mathcal{P}$-function. We use the conventions for this function as expressed in [2].

Let $\mathcal{P}(\omega)$ be the Weierstrass $\mathcal{P}$-function solving the differential equation

$$\mathcal{P}'^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3, \tag{56}$$

where the cubic equation $4z^3 - g_2z - g_3 = 0$ with $g_2, g_3$ real and the discriminant $\Delta = g_2^3 - 27g_3^2$ positive, so that the cubic equation has three distinct real roots $e_1 > e_2 > e_3$ and also $4z^3 - g_2z - g_3 = 4(z - e_1)(z - e_2)(z - e_3)$. Then

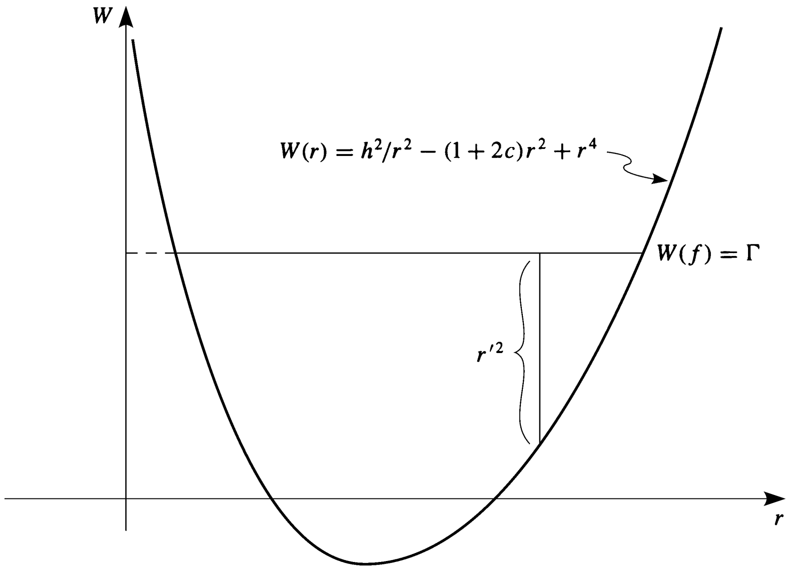


**Figure 6** Potential Function for Tori with Nonrectangular Conformal Type

$\mathcal{P}(\omega)$ will have two fundamental periods $2\omega_1, 2\omega_2$, where $\omega_1 > 0$ and $\omega_2/i > 0$. In this case $\mathcal{P}(\omega_1) = e_1$, $\mathcal{P}(\omega_2) = e_3$, and $\mathcal{P}(\omega_1 + \omega_2) = e_2$. We have

$$\omega_1 = \int_{e_1}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}, \qquad \omega_2 = \int_{-\infty}^{e_3} \frac{dx}{\sqrt{g_3 + g_2 x - x^3}}. \tag{57}$$

Consider the function $\mathcal{P}(\omega_2 + u)$. This is a real-valued function of $u$ with $\mathcal{P}(\omega_2) = e_3$ and $\mathcal{P}(\omega_2 + \omega_1) = e_2$ with $e_3 < e_2$, so that $\mathcal{P}(\omega_2 + u)$ is a periodic function of $u$ oscillating between $e_2$ and $e_3$ with a half-period $a = \omega_1$.

Let $d$ be any value greater than $e_2$ and set

$$r^2(u) = d - \mathcal{P}(\omega_2 + u). \tag{58}$$

Substitute this into (56) to find

$$r'^2 - \frac{(4d^3 - g_2 d - g_3)}{4r^2} - 3dr^2 + r^4 = \frac{g_2 - 12d^2}{4}. \tag{59}$$

THEOREM 3.2. *The function* $r(u)$ *defined* (*see* (59)) *by* $r^2(u) = d - \mathcal{P}(\omega_2 + u)$ *represents a solution to equations* (54) *precisely when*

$$4h^2 = -(4d^3 - g_2 d - g_3), \tag{60a}$$

$$3d = 1 + 2c, \tag{60b}$$

$$4\Gamma = g_2 - 12d^2; \tag{60c}$$

*here we assume that* $g_2, g_3$ *are real and* $\Delta = g_2^3 - 27g_3^2 > 0$. *Setting*

$$4z^3 - g_2 z - g_3 = 4(z-e_1)(z-e_2)(z-e_z)$$

*with* $e_1 > e_2 > e_3$, *we must choose* $e_2 < d < e_1$.

*Given the triple* $(d, g_2, g_3)$ *with* $e_2 < d < e_1$ *and* $g_2, g_3$ *as before, we can determine constants* $(c, h, \Gamma)$ *from system* (60). *Also, starting with the triple* $(c, h, \Gamma)$ *we can find the triple* $(d, g_2, g_3)$. *In fact,*

$$3d = 1 + 2c, \tag{61a}$$

$$g_2 = 4(\Gamma + 3d^2), \tag{61b}$$

$$g_3 = 4(h^2 - 4\Gamma d - 2d^3). \tag{61c}$$

*Proof.* The relations (60) are found by comparing (54) and (59). From (58a) we see that $4d^3 - g_2 d - g_3 < 0$. Since $d > e_2$, we must have $e_2 < d < e_1$. □

NOTE. The condition $h \neq 0$ restricts $d$ to lie in the interval $e_2 < d < e_1$. By choosing $h = 0$ we have either $d = e_2$ or $d = e_1$. In either case, $d$ is a root of the cubic $4d^3 - g_2 d - g_3 = 0$ (see equation (60a)).

If $d = e_2 = \mathcal{P}(\omega_2 + \omega_1)$ we obtain $r^2(u) = \mathcal{P}(\omega_2 + \omega_1) - \mathcal{P}(\omega_2 + u)$ so that $r^2(\omega_1) = 0$. We obtain those solutions to the $H$-surface equation that are defined on annular domains and vanish on the boundary. If $d = e_1 = \mathcal{P}(\omega_1)$ then $r^2(u) = \mathcal{P}(\omega_1) - \mathcal{P}(\omega_2 + u)$ and so $r^2(u)$ ranges in the interval $e_1 - e_2 \leq r^2(u) < e_1 - e_3$. Since $h = 0$, we are back to the solutions discussed in Section II.

We derive the complete representation of $x(u, v)$ by integrating (54b,c), which yields our final result as follows.

THEOREM 3.3. *Let* $r^2(u) = d - \mathcal{P}(\omega_2 + u)$, *where* $\mathcal{P}(\omega)$ *is the Weierstrass* $\mathcal{P}$-*function satisfying the conditions of Theorem 3.2. We then have*

$$\theta(u) = h \int_0^u \frac{dt}{d - \mathcal{P}(\omega_2 + t)}, \tag{62a}$$

$$z(u) = \frac{(d-1)u}{2} + \int_0^u \mathcal{P}(\omega_2 + t)\, dt. \tag{62b}$$

*The solution* $x(u, v)$ *given by* (46) *has basic periods* $\langle a, b\rangle$ *and* $\langle 0, 2\pi\rangle$, *where* $a = \omega_1$ *and*

$$b = \theta(\omega_1) = h \int_0^{\omega_1} \frac{du}{d - \mathcal{P}(\omega_2 + u)}; \tag{63}$$

*the closing condition for* $z(u)$ *is*

$$0 = \frac{(d-1)}{2}\omega_1 + \int_0^{\omega_1} \mathcal{P}(\omega_2 + u)\, du. \tag{64}$$

*Proof.* This is a direct consequence of taking the expression for $r^2(u)$ in (58) and using the formulas (54). □

NOTE. The anti-derivative of $\mathcal{P}(\omega)$ is the $\zeta$-function $\zeta(u)$ with $\zeta'(u) = -\mathcal{P}(u)$. This allows (62b) to be rewritten as

$$z(u) = \frac{(d-1)}{2}u + \zeta(\omega_2) - \zeta(\omega_2 + u); \tag{65}$$

the closing condition (65) becomes

$$0 = \frac{(d-1)}{2}\omega_1 + \zeta(\omega_2) - \zeta(\omega_2 + \omega_1). \tag{66}$$

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