STIRLING'S FORMULA: AN APPLICATION OF THE CENTRAL LIMIT THEOREM

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The large factorials are approximated through the use of Stirling's formula:

$$n! \simeq \sqrt{2\pi}\, n^{n+1/2} e^{-n}.$$

The proof of the Stirling's formula can be found in many texts, such as [1], [2], and [3].

In this short note Stirling's formula is derived as an application of the Central Limit Theorem. Thus, this proof can be introduced in a mathematical statistics course.

Let $X_1, X_2, \dots, X_n$ be a random sample from an exponential distribution with mean 1.

By the Central Limit Theorem the limiting distribution of

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}}$$

is standard normal.

That is,

$$Z_n \xrightarrow{d} Z \sim N(0,1) \quad \text{as} \quad n \to \infty.$$

Thus, for every x,

$$P(Z_n \leq x) \to P(Z \leq x) \quad \text{as} \quad n \to \infty.$$

Since $X_i \sim \exp(1), i = 1, 2, \dots, n$, and all independent, $\displaystyle\sum_{i=1}^{n} X_i$ has a Gamma distribution with probability density function

$$f(t) = \frac{t^{n-1} e^{-t}}{(n-1)!} \quad t \geq 0, \quad \text{and zero otherwise.}$$

Hence,

$$P(Z_n \le x) = P\left(\sum_{i=1}^{n} X_i \le n + x\sqrt{n}\right) = \int_0^{n+x\sqrt{n}} \frac{t^{n-1}e^{-t}}{(n-1)!}\,dt.$$

Thus,

$$\int_0^{n+x\sqrt{n}} \frac{t^{n-1}e^{-t}}{(n-1)!}\,dt \simeq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt. \tag{1}$$

Differentiating both sides of (1) with respect to x, we have

$$\sqrt{n}\frac{(n+x\sqrt{n})^{n-1}e^{-(n+x\sqrt{n})}}{(n-1)!} \simeq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{2}$$

Since (2) is true for all x, taking $x = 0$ in (2) we get

$$\frac{\sqrt{n}\,n^{n-1}e^{-n}}{(n-1)!} \simeq \frac{1}{\sqrt{2\pi}}$$

which gives the desired result.

References

1. T. M. Apostol, *Calculus*, Vol. II, Blaisdell Publishing Co., 1962.
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., John Wiley & Sons, 1968.
3. K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth Inc., 1981.