A RELATIONSHIP BETWEEN QUADRATIC EQUATIONS, RIGHT TRIANGLES AND FIBONACCI TYPE SEQUENCES

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Although no isosceles right triangle can be a Pythagorean triangle, if

$a = 21669693148613788330547979729286307164015202768699465346081691992338845992696,$

$b = a+1$, and $c = \sqrt{a^2+b^2}$, then the triple (a, b, c) gives the sides of a Pythagorean triangle which is nearly isosceles [1]. This example prompts the question of whether Pythagorean triangles can always be found that are nearly similar to right triangles given by the triple $(a, b, \sqrt{a^2+b^2})$ where a and b are positive integers.

A question that needs to be asked here is what does "nearly similar to" mean? One possibility would be to find a right triangle given by the triple (d, e, f) such that the ratio d/e is approximately equal to a/b. Better still, can we find a sequence of Pythagorean triangles given by $\{(a_n, b_n, c_n)\}$ such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}?$$

If so, we would say that each of the triangles in this sequence are approximately similar to the triangle (a, b, c).

To develop a general method for generating Pythagorean triangles which are nearly similar to a given right triangle with integral sides a and b, consider the quadratic equation

$$x^2 - \frac{2a}{b}x - 1 = 0.$$

This equation has roots

$$\alpha = \frac{a+\sqrt{a^2+b^2}}{b} \quad \text{and} \quad \beta = \frac{a-\sqrt{a^2+b^2}}{b},$$

where $0 < |\beta| < |\alpha|$. It follows that

$$\lim_{n\to\infty} \left(\frac{\beta}{\alpha}\right)^n = 0.$$

Define the sequence $\{x_n\}$ by $x_n = \alpha^n + \beta^n$. Since

$$\alpha^{n-1}\left(\alpha^2 - \frac{2a}{b}\alpha - 1\right) + \beta^{n-1}\left(\beta^2 - \frac{2a}{b}\beta - 1\right) = 0,$$

then

$$\alpha^{n+1} + \beta^{n+1} = \frac{2a}{b}(\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1}).$$

This shows for $n \geq 1$,

$$x_{n+1} = \frac{2a}{b}x_n + x_{n-1}.$$

Because the sequence begins with $x_0 = 2$ and $x_1 = 2a/b$, each term will be a rational number with b^n in the denominator of x_n. Further,

$$\begin{aligned} \lim_{n\to\infty} \frac{x_{n+1}}{x_n} &= \lim_{n\to\infty} \frac{\alpha^{n+1} + \beta^{n+1}}{\alpha^n + \beta^n} \\ &= \lim_{n\to\infty} \frac{\alpha + \left(\frac{\beta}{\alpha}\right)^n \beta}{1 + \left(\frac{\beta}{\alpha}\right)^n} \\ &= \alpha. \end{aligned}$$

Define a sequence $\{(a_n, b_n, c_n)\}$ of triples by $a_n = (x_{n+1}^2 - x_n^2) \cdot b^{2(n+1)}$, $b_n = 2x_{n+1}x_n \cdot b^{2(n+1)}$ and $c_n = (x_{n+1}^2 + x_n^2) \cdot b^{2(n+1)}$. Then a_n, b_n, and c_n are positive integers satisfying $a_n^2 + b_n^2 = c_n^2$. Hence, the triple (a_n, b_n, c_n) is a Pythagorean triple for each integer n. The right triangles given by this sequence of triples become nearer to being similar to the right triangle given by the triple $(a, b, \sqrt{a^2+b^2})$ as n gets larger. To verify this note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{x_{n+1}^2 - x_n^2}{2x_{n+1}x_n} \\
&= \frac{1}{2} \lim_{n\to\infty} \left(\frac{x_{n+1}-x_n}{x_n}\right)\left(\frac{x_{n+1}+x_n}{x_{n+1}}\right) \\
&= \frac{1}{2} \lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n} - 1\right)\left(\frac{x_n}{x_{n+1}} + 1\right) \\
&= \frac{1}{2}(\alpha - 1)\left(\frac{1}{\alpha} + 1\right) \\
&= \frac{1}{2}\frac{\alpha^2 - 1}{\alpha} \\
&= \frac{1}{2}\left(\frac{2a}{b} \cdot \frac{\alpha}{\alpha}\right) \\
&= \frac{a}{b}.
\end{aligned}$$

The preceding procedure gives a method for generating infinitely many integral right triangles such that the ratio of the two legs converge to a/b, where a and b are integral legs of a right triangle, even though the hypotenuse may be irrational.

It is worth noting that the sequence $\{x_n\}$ defined by $x_n = \alpha^n + \beta^n$ could also be defined by $x_n = \alpha^n - \beta^n$ or

$$x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If

$$x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$a = 1$ and $b = 2$ then the sequence is the Fibonacci sequence and α is the golden ratio. In this case the sequence of triangles converge to the right triangle given by $(1, 2, \sqrt{5})$.

References

1. A. H. Beiler, *Recreations in the Theory of Numbers*, Dover, 1964.
2. E. C. Kennedy, "Concerning Sequences of Integral Right Triangles," *American Mathematical Monthly*, 45 (1935), 557–558.