SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

Comment by the editor.

A solution to Problem 90 parts (a)–(d) was received from Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

64*. [1993, 132; 1994, 169–170] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

For n a positive integer, let M_n denote the $n\times n$ matrix (a_{ij}), where $a_{ij}=i+j$. Is there a simple formula for $\mathrm{perm}(M_n)$?

Solution by Les Reid and Richard Belshoff, Southwest Missouri State University, Springfield, Missouri.

More generally, let M_n denote the $n \times n$ matrix whose ij entry is $x_i + x_j$. We will show that

$$\mathrm{perm}(M_n) = \sum_{i=0}^{n} i!(n-i)!\sigma_n(i)\sigma_n(n-i)$$

where $\sigma_n(j)$ is the elementary symmetric function of degree j in $x_1, x_2, \dots, x_n$. (i.e. $\sigma_n(0)=1, \sigma_n(1)=x_1+\cdots+x_n, \sigma_n(2)=x_1x_2+\cdots+x_{n-1}x_n, \dots, \sigma_n(n)=x_1x_2\cdots x_n$.) In the original problem, $x_i=i$, and in that case $\sigma_n(i)=S(n+1, n+1-i)$, where the $S(m,j)$ are the Stirling numbers of the first kind, defined recursively by $S(m,0)=0$ $(m\ge 1)$, $S(m,m)=1$ $(m\ge 0)$, and $S(m+1,j)=S(m,j-1)+mS(m,j)$ $(m\ge 0,\ j\ge 1)$.

Now $\mathrm{perm}(M_n)$ is a symmetric function of $x_1, \dots, x_n$. We claim that, in fact, it must be a linear combination of $\sigma_n(0)\sigma_n(n)$, $\sigma_n(1)\sigma_n(n-1)$, $\dots$, $\sigma_n(\lfloor \frac{n}{2}\rfloor)\sigma_n(\lceil \frac{n}{2}\rceil)$.

To see this, first note that $\mathrm{perm}(M_n)=\sum_{\tau\in S_n}\prod_{i=1}^{n}(x_i+x_{\tau(i)})$, so when expanded out, each product is a sum of terms of the form $x_1^{e_1}\cdots x_n^{e_n}$, where $e_i=0,1$, or 2. If we let $f(n,k)$ denote the symmetric polynomial of degree n consisting of the sum of all products with exactly k of the variables squared, then $\mathrm{perm}(M_n)$ is a linear combination of the $f(n,k)$.

To prove our claim, it will suffice to show that each $f(n,k)$ is a linear combination of $\sigma_n(0)\sigma_n(n)$, $\dots$, $\sigma_n(\lfloor \frac{n}{2}\rfloor)\sigma_n(\lceil \frac{n}{2}\rceil)$. Since $\sigma_n(i)$ is a sum of products of i distinct variables and $\sigma_n(n-i)$ is a sum of products of $n-i$ distinct variables, $\sigma_n(i)\sigma_n(n-i)$ is a linear combination of $f(n,0), f(n,1), \dots, f(n,i)$ $(i\le n-i)$. (If the i variables and the $n-i$ variables are distinct, this makes a positive contribution to the $f(n,0)$ term; if they have one variable in common, this contributes to the $f(n,1)$ term, etc.) Since this system is lower triangular with nonzero entries on the diagonal (in fact, they are all equal to 1), it is invertible and the result follows.

Our next claim is that if we can find $\lambda_0, \lambda_1, \ldots, \lambda_{\lfloor \frac{n}{2} \rfloor}$ so that

$$\text{perm}(M_n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \lambda_i \sigma_n(i) \sigma_n(n-i) \tag{1}$$

is satisfied when $\lceil \frac{n}{2} \rceil$ of the x_i are set equal to 1 and the rest to 0; when $\lceil \frac{n}{2} \rceil + 1$ of the x_i are set equal to 1 and the rest to 0; ...; all of the x_i are set equal to 1, then the equation holds for all x_i. To see this, first note that if k of the x_i are set equal to 1 and the rest to 0, then $\sigma_n(i) = \binom{k}{i}$ (which is 0 if $i > k$). Thus, if $\lceil \frac{n}{2} \rceil$ of the x_i are set equal to 1 and the rest to 0, the only nonzero term in (1) is the one containing $\sigma_n(\lfloor \frac{n}{2} \rfloor) \sigma_n(\lceil \frac{n}{2} \rceil)$; if $\lceil \frac{n}{2} \rceil + 1$ of the x_i are set equal to 1 and the rest to 0, then only nonzero terms are the ones containing $\sigma_n(\lfloor \frac{n}{2} \rfloor) \sigma_n(\lceil \frac{n}{2} \rceil)$ and $\sigma_n(\lfloor \frac{n}{2} \rfloor - 1) \sigma_n(\lceil \frac{n}{2} \rceil + 1)$; .... Since this system is invertible (being triangular, with nonzero entries on the bounding antidiagonal), the λ_i are uniquely determined.

We next write (1) in the more symmetric form

$$\text{perm}(M_n) = \sum_{i=0}^{n} \mu_i \sigma_n(i) \sigma_n(n-i). \tag{2}$$

Our claim is that we may take $\mu_i = i!(n-i)!$. To verify this we must compute $\text{perm}(M_n)$ when $x_1, \ldots, x_k$ are set equal to 1 and the rest of the variables to 0, for $k = \lceil \frac{n}{2} \rceil, \ldots, n$. In this case M_n has the form

$$\begin{pmatrix} 2 & \cdots & 2 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

where every entry in $k \times k$ submatrix consisting of the first k rows and first k columns is 2, and every entry in the $(n-k) \times (n-k)$ submatrix consisting of the last $n-k$ rows and columns is 0, and all other entries are 1. Repeatedly expanding along columns from right to left we obtain

$$\operatorname{perm}(M_n) = k(k-1)\cdots(k-(n-k+1))\operatorname{perm}\begin{pmatrix} 2 & 2 & \cdots & 2 & 2 \\ \vdots & & & & \vdots \\ 2 & 2 & \cdots & 2 & 2 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

(The matrix shown is a $k \times k$ matrix with each entry in the first $2k-n$ rows equal to 2, and each entry in the last $n-k$ rows equal to 1.) Therefore,

$$\operatorname{perm}(M_n) = \frac{k!}{(2k-n)!} 2^{2k-n} k!.$$

To finish, we must show that

$$\frac{k!}{(2k-n)!} 2^{2k-n} k! = \sum_{i=0}^{n} i!(n-i)! \binom{k}{i} \binom{n-k}{i} \qquad (k = \lceil \frac{n}{2} \rceil, \ldots, n)$$

But

$$\begin{aligned} \frac{(2k-n)!}{k!k!} \sum_{i=0}^{n} i!(n-i)! \binom{k}{i} \binom{n-k}{i} &= \sum_{i=0}^{n} \binom{2k-n}{k-i} \\ &= \sum_{i=n-k}^{k} \binom{2k-n}{k-i} \\ &= \sum_{j=0}^{2k-n} \binom{2k-n}{j} \\ &= 2^{2k-n} \end{aligned}$$

and the result follows.

97. [1996, 135] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let an (m by 2) determinant

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m-1} & a_{2,m} \end{pmatrix}, \quad m \geq 3$$

be defined by $S(m-1)+\overline{D}$, where

$$S(m-1)=\sum_{r=1}^{m-1}\det\begin{pmatrix} a_{1,r} & a_{1,r+1} \\ a_{2,r} & a_{2,r+1} \end{pmatrix} \quad \text{and} \quad \overline{D}=\det\begin{pmatrix} a_{1,m} & a_{1,1} \\ a_{2,m} & a_{2,1} \end{pmatrix}.$$

Let

$$D=\det\begin{pmatrix} P_{1,k} & P_{2,k} & P_{3,k} & \cdots & P_{m,k} \\ P_{1,k+a} & P_{2,k+a} & P_{3,k+a} & \cdots & P_{m,k+a} \end{pmatrix},$$

where $P_{n,k}$ denotes the nth polygonal number of k "dimensions." For example, $P_{5,3}$ is the 5th triangular number. Evaluate D.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.

Solution.

$$D=\frac{am(m-1)(m-2)}{6}.$$

Proof.

$$D=\det\begin{pmatrix} P_{1,k} & P_{2,k} & P_{3,k} & \cdots & P_{m,k} \\ P_{1,k+a} & P_{2,k+a} & P_{3,k+a} & \cdots & P_{m,k+a} \end{pmatrix}.$$

From [1], the formula for the nth polygonal number of k "dimensions" is given by

$$P_{n,k}=\frac{n}{2}\big((k-2)n+(-k+4)\big).$$

Then

$$\det\begin{pmatrix} P_{i,k} & P_{i+1,k} \\ P_{i,k+a} & P_{i+1,k+a} \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{i}{2}\big((k-2)i+(4-k)\big) & \frac{i+1}{2}\big((k-2)(i+1)+(4-k)\big) \\ \frac{i}{2}\big((k+a-2)i+(4-k-a)\big) & \frac{i+1}{2}\big((k+a-2)(i+1)+(4-k-a)\big) \end{pmatrix}$$

$$= \frac{i}{2}\frac{i+1}{2}\det\begin{pmatrix} (k-2)i+(4-k) & (k-2)(i+1)+(4-k) \\ (k+a-2)i+(4-k-a) & (k+a-2)(i+1)+(4-k-a) \end{pmatrix}$$

$$= \frac{i(i+1)}{4}\det\begin{pmatrix} (k(i-1)-2(i-2) & ki-2(i-1) \\ (k+a)(i-1)-2(i-2) & (k+a)i-2(i-1) \end{pmatrix}$$

$$= \frac{i(i+1)}{4}\big(2(k+a)\cdot 1+2k(-1)\big)$$

$$= \frac{i(i+1)}{4}(2a)$$

$$= \frac{ai(i+1)}{2}.$$

Similarly, it can be shown that

$$\det\begin{pmatrix} P_{m,k} & P_{1,k} \\ P_{m,k+a} & P_{1,k+a} \end{pmatrix} = \frac{am(1-m)}{2}.$$

These two derivations give us

$$D = \sum_{i=1}^{m-1}\frac{ai(i+1)}{2} + \frac{am(1-m)}{2}$$

$$= \frac{a}{2}\bigg(\sum_{i=1}^{m-1} i^2 + \sum_{i=1}^{m-1} i + m(1-m)\bigg)$$

$$= \frac{a}{2}\bigg(\frac{m(m-1)(2m-1)}{6} + \frac{m(m-1)}{2} + m(1-m)\bigg)$$

$$= \frac{am(m-1)}{2}\left(\frac{2m-1}{6}+\frac{1}{2}-1\right)$$

$$= \frac{am(m-1)}{2}\left(\frac{2m-1+3-6}{6}\right)$$

$$= \frac{am(m-1)(2m-4)}{12}$$

$$= \frac{am(m-1)(m-2)}{6}.$$

This concludes the evaluation of D.

Reference

1. H. T. Freitag, "From the Legacy of Pythagoras," *Missouri Journal of Mathematical Sciences,* 8 (1996), 55–62.

Comment by the proposer.

It is interesting to note that the result is independent of k as long as the two "dimensions" in question differ by a. In the special case of Problem 96 treated before, $a = 1$; also, $k = 3$. Thus, this is a double generalization.

Also solved by the proposer.

98. [1996, 135–136] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let an (m by 2) determinant

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m-1} & a_{2,m} \end{pmatrix}\ , \quad m \geq 3$$

be defined by $S(m-1) + \overline{D}$, where

$$S(m-1) = \sum_{r=1}^{m-1} \det\begin{pmatrix} a_{1,r} & a_{1,r+1} \\ a_{2,r} & a_{2,r+1} \end{pmatrix} \quad \text{and} \quad \overline{D} = \det\begin{pmatrix} a_{1,m} & a_{1,1} \\ a_{2,m} & a_{2,1} \end{pmatrix}.$$

Let T_n be the nth triangular number and consider an $((m+2)$ by 2) determinant

$$D(m+2) = \det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,m+1} & a_{1,m+2} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,m+1} & a_{2,m+2} \end{pmatrix}$$

such that $a_{i,j} = P_{j,i+2}$, where $P_{n,k}$ is the nth polygonal number of k "dimensions" ($P_{3,5}$ is the third pentagonal number). Prove that

$$D(m+2) = \sum_{i=1}^{m} T_i.$$

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.

Proof. From the solution to Problem 96 [2],

$$D(m+2) = \frac{m(m+1)(m+2)}{6},$$

substituting $m+2$ for m. Also, for all i,

$$T_i = \frac{i(i+1)}{2}.$$

(See [1], p. 58.) Then

$$\begin{aligned}
\sum_{i=1}^{m} T_i &= \sum_{i=1}^{m} \frac{i(i+1)}{2} \\
&= \frac{1}{2}\left(\frac{m(m+1)(2m+1)}{6} + \frac{m(m+1)}{2}\right) \\
&= \frac{m(m+1)}{2}\left(\frac{2m+1+3}{6}\right) \\
&= \frac{m(m+1)(m+2)}{6} \\
&= D(m+2).
\end{aligned}$$

References

1. H. T. Freitag, "From the Legacy of Pythagoras," *Missouri Journal of Mathematical Sciences,* 8 (1996), 55–62.

2. J. T. Bruening, "Solution to Problem 96," *Missouri Journal of Mathematical Sciences,* 9 (1997), 121–122.

Also solved by the proposer.

99. [1996, 136] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

For what positive integer values of k is $4k+1$ never a prime?

Solution I by Dale Woods, Reeds Spring, Missouri and the proposer.

$4k+1$ is always composite if $k \equiv 2 \pmod 3$.

Solution II by Dale Woods, Reeds Spring, Missouri.

If $4n+1=p$, where p is prime, then $4k+1$ is composite for $k \equiv n \pmod p$, $k>1$.

Solution III by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Any odd number greater than 1 can be written in the form $4n+1$ or $4n-1$ for some positive integer n.

The product of two odd numbers fits one of the following three cases:

i) $(4n+1)(4m+1) = 4(4mn+m+n)+1$,

ii) $(4n-1)(4m-1) = 4(4mn-m-n)+1$, and

iii) $(4n+1)(4m-1) = 4(4mn+m-n)-1$.

Let

$$\begin{aligned} A &= \{4mn+m+n \mid m \text{ and } n \text{ are positive integers}\}, \\ B &= \{4mn-m-n \mid m \text{ and } n \text{ are positive integers}\}, \text{ and} \\ C &= \{4mn+m-n \mid m \text{ and } n \text{ are positive integers}\}. \end{aligned}$$

Suppose $4k+1$ is composite. Then $4k+1$ must be the product of two odd numbers both greater than 1. It follows from the three cases listed above that

$$\begin{aligned} 4k+1 &= (4n+1)(4m+1) = 4(4mn+m+n)+1 \text{ or} \\ 4k+1 &= (4n-1)(4m-1) = 4(4mn-m-n)+1. \end{aligned}$$

Hence,

$$A \cup B = \{k \mid 4k + 1 \text{ is composite, } k \text{ a positive integer}\}.$$

That is, $4k+1$ is composite if and only if $k = 4mn+m+n$ or $k = 4mn-m-n$ for some positive integers m and n.

Similarly one can establish that

$$C = \{k \mid 4k - 1 \text{ is composite}, k \text{ a positive integer}\}.$$

That is, $4k - 1$ is composite if and only if $k = 4mn + m - n$ for some positive integers m and n.

Solution IV by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas.

First note that $4k + 1$ must be an odd number. Let $p \geq 3$ be a prime number. Let k_p be the smallest positive integer such that $4k_p + 1 = mp$ (a multiple of p) where $m \geq 3$. Let $k = k_p + pd$, where $p \geq 3$ and $d \geq 0$. Then

$$4k + 1 = 4(k_p + pd) + 1 = 4k_p + 1 + 4pd = mp + 4pd = p(m + 4d).$$

Hence, $4k + 1 = p(m + 4d)$ which is not a prime number. Thus, for each $p \geq 3$, k_p must be determined since $k = k_p + pd$.

Since $p \geq 3$ is a prime number, then $p \equiv 3$ or $1 \pmod 4$. If $p \equiv 3 \pmod 4$, then $k_p = (3p - 1)/4$ and if $p \equiv 1 \pmod 4$, then $k_p = (5p - 1)/4$.

Here are two examples. If $p = 3$, then $k_p = 2$ so $k = 2 + 3d$ where $d \geq 0$. For these values of k, $4k+1$ is never a prime. If $p = 5$, then $k_p = 6$ so $k = 6 + 5d$ where $d \geq 0$. For these values of k, $4k + 1$ is never a prime.

Solution V by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We begin by showing that $4k+1$ is never a prime when $k = 2^{3n-2}$ where n is an arbitrary positive integer. When $k = 2^{3n-2}$ where n is an arbitrary positive integer, $4k + 1 = 8^n + 1$. Since $2^n \equiv -1 \pmod{(2^n + 1)}$, $(2^n)^3 \equiv (-1)^3 \pmod{(2^n + 1)}$. Thus, $8^n+1 \equiv 0 \pmod{(2^n+1)}$, so 2^n+1 divides 8^n+1. Clearly $1 < 2^n+1 < 8^n+1$ where n is an arbitrary positive integer. It follows that $8^n + 1$ is never a prime for each positive integer n.

Next we establish that $4k + 1$ is never a prime when $k = 5 \cdot 2^{2n-2}$ where n is an arbitrary positive integer. When $k = 5 \cdot 2^{2n-2}$ where n is an arbitrary positive integer, $4k + 1 = 5 \cdot 2^{2n} + 1$. Because $2 \equiv -1 \pmod 3$, $2^{2n} \equiv (-1)^{2n} \pmod 3$. It follows that $5 \cdot 2^{2n} \equiv 5 \pmod 3$, making $5 \cdot 2^{2n} + 1 \equiv 0 \pmod 3$. Clearly $1 < 3 < 5 \cdot 2^{2n} + 1$ where n is an arbitrary positive integer. Thus $5 \cdot 2^{2n} + 1$ is never a prime for each positive integer n.

We now demonstrate that $4k+1$ is never a prime when $k = 2m^{3n}$ where m and n are arbitrary positive integers. When $k = 2m^{3n}$ where m and n are arbitrary positive integers, $4k+1 = 8m^{3n}+1 = (2m^n)^3+1 = (2m^n)^3+1^3$ which is divisible by $2m^n+1$. Clearly $1 < 2m^n+1 < (2m^n)^3+1$ where m and n are arbitrary positive integers. It follows that $(2m^n)^3+1$ is never a prime for each positive integer n and for each positive integer m.

Solution VI by James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri.

<u>Solution</u>. For $k \geq 5$, $4k+1$ is not a prime if and only if k is of the form $mp+r$, where m is a positive integer, p is an odd prime, and r is the unique solution of $4x \equiv -1 \pmod{p}$ such that $0 < r < p$.

<u>Proof</u>. Before proceeding with the actual proof, consider the following comments.

1. Examples of this form for k are $3m+2$, $5m+1$, $7m+5$, $11m+8$, $13m+3$, $17m+4$, and there would be a form for each odd prime. Since $k = 5 = 3 \cdot 1 + 2$ would be the smallest value of k represented here, and since for $k < 5$, $9 = 4 \cdot 2 + 1$ is the only integer of the form $4k+1$ that is not prime, assume $k \geq 5$.
2. $4x \equiv -1 \pmod{p}$ has a unique solution r modulo p, since $\gcd(4, p) = 1$; and since $r = 0$ is obviously not a solution, there is a unique solution to the congruence in the interval $0 < r < p$. (See [1], pp. 97, 98, Theorem 4–7 and its Corollary.)

For the proof of this solution, let $4k+1$, $k \geq 5$, be composite. Since $k \geq 5$, there exists an odd prime $p \leq \sqrt{4k+1} < k$ such that $p|(4k+1)$ and p does not divide k. By the Division Algorithm, there exist unique integers m and r such that $k = mp + r$, where $0 < r < p$. $r \neq 0$ since p does not divide k and m is positive since k and p are positive. $4k+1 = 4(mp+r)+1 = 4mp+4r+1$, so that $p|(4k+1)$ implies $p|(4r+1)$. Thus, r is the unique solution of $4x \equiv -1 \pmod{p}$ such that $0 < r < p$, and k is of the form $mp+r$.

Now assume k is of the form $mp+r$, where m is a positive integer, p is an odd prime, and r is the unique solution of $4x \equiv -1 \pmod{p}$. $4r \equiv -1 \pmod{p}$ implies $p|(4r+1)$, so $4k+1 = 4(mp+r)+1 = 4mp+4r+1$ has p as a factor, and the other factors of $4k+1$ are greater than 1 since m is positive. So $4k+1$ is never a prime when k is of the form described here.

<u>References</u>

1. D. M. Burton, *Elementary Number Theory*, 2nd ed., Wm. C. Brown Publishers, Dubuque, Iowa, 1989.

Solution VII by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, Missouri.

If $k = n^2 + n$, then $4k + 1 = 4(n^2 + n) + 1 = 4n^2 + 4n + 1$ is a perfect square and hence, not a prime. The problem is more interesting if we try to find necessary conditions on k for which $4k + 1$ is not a prime.

Suppose $k = an^2 + bn$. Then $4k + 1 = 4(an^2 + bn) + 1 = 4an^2 + 4bn + 1$. If $4k+1$ is not a prime then this quadratic in n should have rational roots. Hence, its discriminant $16b^2 - 16a$ should be a perfect square. Let $16j^2 = 16b^2 - 16a$. Then $j^2 = b^2 - a$ or $a = b^2 - j^2$ and $k = (b^2 - j^2)n^2 + bn$, where $0 \le j < b$. To reduce the number of variables, let $bn = c$ and $jn = d$. Then $0 \le d < c$ and $k = c^2 - d^2 + c$. Now $4k+1 = 4(c^2 - d^2 + c) + 1 = 4c^2 + 4c + 1 - 4d^2$ factors as $(2c+1+2d)(2c+1-2d)$ and hence, is not a prime.

We have seen that when $k = c^2 - d^2 + c$, $4k + 1$ is composite. We now prove the converse. If $4k + 1$ factors into two factors, say x and y with $x \ge y > 1$, we will show that there exist integers c and d such that $k = c^2 - d^2 + c$, $0 \le d < c$ with $x = (2c + 1 + 2d)$ and $y = (2c + 1 - 2d)$.

Since $4k + 1$ is odd so are x and y. Let $x = 2r + 1$ and let $y = 2s + 1$, where r and s are positive integers and $r \ge s$. We would like to find suitable c and d such that $2r + 1 = 2c + 1 + 2d$ and $2s + 1 = 2c + 1 - 2d$. This implies that $r + s = 2c$ and $r - s = 2d$. We see that both x and y must be congruent to $1 \pmod 4$ or both must be congruent to $3 \pmod 4$. If this is not so, then the product xy would not be of the form $4k + 1$ which is congruent to $1 \pmod 4$. This implies that r and s are either both odd or both even and the quantities $r + s$ and $r - s$ are both even. Now we can choose integers c and d such that $(r+s)/2 = c$ and $(r-s)/2 = d$. Then $4k + 1 = xy = (2r + 1)(2s + 1) = (2c + 1 + 2d)(2c + 1 - 2d) = 4c^2 + 4c - 4d^2 + 1$ and k is of the form $c^2 - d^2 + c$. Thus, we have shown the following.

Result. For positive integers k, the quantity $4k+1$ is never a prime if and only if k is of the form $c^2 - d^2 + c$, where $0 \le d < c$.

100*. [1996, 136] *Proposed by Bryan Dawson, Emporia State University, Emporia, Kansas.*

Let C be the set of constructible numbers. Let $f\colon C \to \mathbb{R}$ be given by $f(x) = n$ where n is the minimum number of arcs necessary to construct a segment of length x under the following rules:

1) Only compass and straightedge may be used for the construction.
2) The construction starts with only a segment of unit length and this segment may not be used for any other purposes than measurement (i.e., the construction cannot be built using the segment; $f(1) = 1$).
3) The number of uses of the straightedge must be finite.

Prove or disprove that f has a point of continuity.

Comments by the editor and the proposer.

No solution was received. Therefore, the problem remains open. The proposer notes that $f(2)=1$ and

$$f^{-1}(2) \subseteq \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}, 3, 4\right\}.$$

Also, if k is the minimum number of arcs necessary to construct a regular pentagon, then there are infinitely many x such that $f(x) \leq k$.