LINEARIZATION OF TRIGONOMETRIC POLYNOMIALS AND INTEGRALS

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1. The Method. One of the problems encountered in calculus courses is the computation of simple integrals. The existence of a primitive for a continuous function on an interval is a well-known theorem (see [3]), but the actual computation is sometimes quite hard to perform, even impossible. In [2] a geometric approach for computing one particular trigonometric integral is presented and an interesting method for computing trigonometric integrals is given in [1]. We present a way to compute integrals of trigonometric polynomials, via a linearization of these polynomials using complex numbers and Euler's formula (the method is simple, but we did not find any exposition of it in classical textbooks).

Recall that for $x \in \mathbb{R}$, $e^{ix} = \cos x + i \sin x$. By DeMoivre's formula, we have for any $x \in \mathbb{R}$, $e^{-ix} = \cos x - i \sin x$ and for any $n \in \mathbb{Z}$, $(e^{ix})^n = e^{nix}$. Therefore,

$$\begin{cases} \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \\ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}). \end{cases}$$

These are the so-called Euler's formula.

By Newton's binomial development, we can now compute any positive integral power of $\cos x$ and of $\sin x$ and any product of such powers as a linear combination of powers of e^{ix} and e^{-ix}.

1. Let $f(x) = \cos^p x$. Then

$$\begin{aligned} \cos^p x &= \left[\frac{1}{2}(e^{ix} + e^{-ix})\right]^p \\ &= \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{k}(e^{kix}\cdot e^{-(p-k)ix}) \\ &= \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{k}e^{(2k-p)ix} \\ &= \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{p-k}e^{(2k-p)ix}. \end{aligned}$$

Recall that

$$\binom{p}{k} = \binom{p}{p-k}$$

and that $2(p-k)-p = -(2k-p)$. We consider the following two cases.

Case 1. p is odd.

$$\begin{aligned}
\cos^p x &= \frac{1}{2^p}\sum_{k=0}^{(p-1)/2}\binom{p}{k}(e^{kix}+e^{-kix}) \\
&= \frac{1}{2^p}\sum_{k=0}^{(p-1)/2}\binom{p}{k}e^{(2k-p)ix} + \sum_{k=(p+1)/2}^{p}\binom{p}{k}e^{(2k-p)ix} \\
&= \frac{1}{2^p}\sum_{k=0}^{(p-1)/2}\binom{p}{k}e^{(2k-p)ix} + \sum_{k=0}^{(p-1)/2}\binom{p}{p-k}e^{-(2k-p)ix} \\
&= \frac{1}{2^p}\sum_{k=0}^{(p-1)/2}\binom{p}{k}(e^{(2k-p)ix}+e^{-(2k-p)ix}) \\
&= \frac{1}{2^p}\sum_{k=0}^{(p-1)/2}\binom{p}{k}2\cos(2k-p)x \\
&= \frac{1}{2^{p-1}}\sum_{k=0}^{(p-1)/2}\binom{p}{k}\cos(2k-p)x.
\end{aligned}$$

Case 2. p is even.

$$\cos^p x = \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{k}(e^{kix}+e^{-kix})^p$$

$$= \frac{1}{2^p} \sum_{k=0}^{p/2} \binom{p}{k} (e^{(2k-p)ix} + e^{-(2k-p)ix}) + \binom{p}{p/2}$$

$$= \frac{1}{2^p} \left[\sum_{k=0}^{p/2} \binom{p}{k} 2\cos(2k-p)x + \binom{p}{p/2} \right]$$

$$= \frac{1}{2^{p-1}} \sum_{k=0}^{p/2} \binom{p}{k} \cos(2k-p)x + \frac{1}{2^p} \binom{p}{p/2}.$$

Since cos is an even function, it could have been foreseen that for every power we get a linearization with all summands of the form $\cos kx$.

2. Let $f(x) = \sin^q x$. Then,

$$\sin^q x = \left[\frac{1}{2i} (e^{ix} - e^{-ix}) \right]^q$$

$$= \frac{1}{(2i)^q} \sum_{k=0}^{q} \binom{q}{k} (-1)^{-(q-k)} (e^{kix} \cdot e^{-(q-k)ix})$$

$$= \frac{1}{(2i)^q} \sum_{k=0}^{q} \binom{q}{k} (-1)^{-(q-k)} e^{(2k-q)ix}.$$

As before, we have two distinct cases.

Case 1. q is odd.

$$\sin^q x = \frac{1}{(2i)^q} \sum_{k=0}^{(q-1)/2} (-1)^{k-q} \binom{q}{k} (e^{kix} + e^{-kix})$$

$$= \frac{1}{(2i)^q} \sum_{k=0}^{(q-1)/2} (-1)^{k-q} \binom{q}{k} (2i \sin kx).$$

Since q is odd, $(2i)^q$ is a pure imaginary number and we can simplify by i. Thus, we get a linear combination of sines; this was expected since we computed an odd power of an odd function.

Case 2. q is even.

$$\begin{aligned}\sin^q x &= \frac{1}{(2i)^q}\sum_{k=0}^{q/2}\binom{q}{k}(-1)^{k-q}(e^{kix}+e^{-kix})+\binom{p}{p/2}\\ &= \frac{1}{(2i)^q}\sum_{k=0}^{q/2}\binom{q}{k}(-1)^{k-q}2\cos kx+\binom{p}{p/2}\\ &= \frac{1}{2^{q-1}i^q}\sum_{k=0}^{q/2}\binom{q}{k}(-1)^{k-q}2\cos kx+\frac{1}{2^q}\binom{p}{p/2}.\end{aligned}$$

Here q is even, thus i^q is a real number (it equals ± 1). We get a linear combination of cosines; this was expected as we computed an even power of an odd function, getting an even function.

3. Now consider the general case, where $f(x) = \cos^p x \cdot \sin^q x$. As above, $\cos^p x$ and $\sin^q x$ are linear combinations of sines and/or cosines; thus the product $\cos^p x \cdot \sin^q x$ is a linear combination of terms either of the form $\cos ax \cdot \cos bx$ or of the form $\cos ax \cdot \sin bx$. Using the well-known formula:

$$\begin{cases}\cos\alpha\cdot\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))\\ \cos\alpha\cdot\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))\end{cases},$$

we get a linearization of $\cos^p x \cdot \sin^q x$ for any natural numbers p and q.

2. Examples and Applications. Here are some examples.

Example 1. $f(x) = \cos^2 x$.

$$\begin{aligned}\cos^2 x &= \left[\frac{1}{2}(e^{ix}+e^{-ix})\right]^2 = \frac{1}{4}(e^{2ix}+2+e^{-2ix})\\ &= \frac{1}{4}(2\cos 2x+2) = \frac{1}{2}\cos 2x+\frac{1}{2}.\end{aligned}$$

Example 2. $f(x) = \cos^3 x$.

$$\cos^3 x = \left[\frac{1}{2}(e^{ix} + e^{-ix})\right]^3 = \frac{1}{8}(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix})$$

$$= \frac{1}{8}(e^{3ix} + e^{-3ix} + 3(e^{ix} + e^{-ix})) = \frac{1}{8}(2\cos 3x + 6\cos x)$$

$$= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.$$

Case 3. $f(x) = \sin^3 x \cdot \cos^2 x$.

$$\sin^3 x \cdot \cos^2 x = \left[\frac{1}{2i}(e^{ix} - e^{-ix})\right]^3 \left[\frac{1}{2}(e^{ix} + e^{-ix})\right]^2$$

$$= \frac{1}{-32i}(e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix})(e^{2ix} + 2 + e^{-2ix})$$

$$= \frac{-1}{32i}(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix})(e^{2ix} + 2 + e^{-2ix})$$

$$= \frac{-1}{32i}(e^{5ix} - e^{3ix} - 2e^{ix} + 2e^{-ix} + e^{-3ix} - e^{-5ix})$$

$$= \frac{-1}{32i}(2i\sin 5x - 2i\sin 3x - 4i\sin x)$$

$$= \frac{-1}{16}\sin 5x + \frac{1}{16}\sin 3x + \frac{1}{8}\sin x.$$

The main application of this linearization is in computing integrals of trigonometric polynomials. We will end this note with an example of the calculation of an integral.

Example 4. Define the integral

$$I = \int_0^{\pi/2} \sin^3 x \cdot \cos^2 x dx.$$

We linearize the trigonometric polynomial $\sin^3 x \cdot \cos^2 x$ by the method described previously; computing the integral is straightforward.

$$\begin{aligned}
I &= \int_0^{\pi/2} \sin^3 x \cdot \cos^2 x dx \\
&= \int_0^{\pi/2} \left(\frac{-1}{16} \sin 5x + \frac{1}{16} \sin 3x + \frac{1}{8} \sin x \right) dx \\
&= \frac{-1}{16} \int_0^{\pi/2} \sin 5x dx + \frac{1}{16} \int_0^{\pi/2} \sin 3x dx + \frac{1}{8} \int_0^{\pi/2} \sin x dx \\
&= \left[\frac{1}{80} \cos 5x - \frac{1}{48} \cos 3x - \frac{1}{8} \cos x \right]_0^{\pi/2} \\
&= \frac{2}{15}.
\end{aligned}$$

References

1. E. Deeba and A. Gunawardena, "Trigonometric Integration," *Int. J. Math. Educ. Sci. Technol.*, 26, (1995), 763–767.

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