SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

113*. [1998, 46] *Proposed by Kamal Jain, Georgia Institute of Technology, Atlanta, Georgia.*

Find all ordered pairs (a, b) such that

$$\tan(a\pi) = b$$

and a and b are rational numbers.

Solution by Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin.

In his article "Rational Values of Trigonometric Functions" [see pp. 507–508 of *The American Mathematical Monthly*, 52 (1945)], J. M. H. Olmsted proved that the only rational values of $\tan(a\pi)$ (where a is a rational number) are 0 and ± 1.

Thus, if a and b are rational numbers then (a, b) is a solution of $\tan(a\pi) = b$ if and only if (a is an arbitrary integer and $b = 0$) or ($a = \frac{1}{4} + k$ where k is an arbitrary integer and $b = 1$) or ($a = -\frac{1}{4} + k$ where k is an arbitrary integer and $b = -1$).

Also, on the pages leading up to p. 41 of his book *Irrational Numbers*, (Carus Monograph #11) The Mathematical Association of America (distributed by John Wiley and Sons, Inc.), 1963, Ivan Niven proved that if θ is rational in degrees, say $\theta = 2\pi r$ for some rational number r, then the only rational values of the trigonometric functions of θ are as follows: $\sin\theta$, $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$; $\sec\theta$, $\csc\theta = \pm 1, \pm 2$; $\tan\theta$, $\cot\theta = 0, \pm 1$.

114. [1998, 46] *Proposed by Kenneth B. Davenport, 301 Morea Road, Frackville, Pennsylvania.*

(a) Prove that

$$\int_0^\infty \frac{1}{1+x^2} \cdot \frac{4}{4+x^2} \cdot \cdots \cdot \frac{n^2}{n^2+x^2} dx = \frac{\pi}{2} \frac{n}{2n-1}.$$

(b) Prove that

$$\int_0^\infty \frac{1}{1+x^2}\cdot\frac{9}{9+x^2}\cdot\cdots\cdot\frac{(2n+1)^2}{(2n+1)^2+x^2}dx$$

$$=\frac{\pi}{2}\frac{(\Gamma(2n+2))^3}{2^{5n}(2n+1)^3(\Gamma(n+1))^4\prod_{k=1}^n k(2k-1)}.$$

Solution to part (a) by Paul S. Bruckman, 1518 Vanstone Road # 2, Campbell River, British Columbia, Canada.

Let

$$I_n=\int_0^\infty \frac{dx}{x^2+n^2}=\frac{\pi}{2n} \quad \text{and} \quad J_n=\int_0^\infty P_n(x)dx,$$

where

$$P_n(x)=\prod_{k=1}^n (x^2+k^2)^{-1}.$$

Now

$$P_n(x)=\sum_{k=1}^n\left(\frac{A_k}{x-ki}+\frac{\overline{A_k}}{x+ki}\right),$$

where

$$A_k=\lim_{x\to ki}(x-ki)P_n(x)=\lim_{y\to 0} yP_n(y+ki)$$

$$=\lim_{y\to 0}\left(\frac{y}{(y+ki)^2+k^2}\prod_{\substack{j=1\\ j\neq k}}^n[(y+ki)^2+j^2]^{-1}\right)$$

$$=\lim_{y\to 0}\left(\frac{1}{y+2ik}\cdot\prod_{\substack{j=1\\ j\neq k}}^n(-k^2+j^2)^{-1}\right),$$

or

$$A_k = \frac{1}{2ik} \cdot \frac{1}{Q_{k,n}},$$

where

$$Q_{k,n} = \prod_{\substack{j=1 \\ j \neq k}}^{n} (j^2 - k^2).$$

Then

$$P_n(x) = \sum_{k=1}^{n} \left(\frac{1}{2ik} \cdot \frac{1}{x - ki} - \frac{1}{2ik} \cdot \frac{1}{x + ki} \right) \frac{1}{Q_{k,n}} = \sum_{k=1}^{n} \frac{1}{x^2 + k^2} \cdot \frac{1}{Q_{k,n}}.$$

Then,

$$J_n = \pi \sum_{k=1}^{n} \frac{1}{2kQ_{k,n}}.$$

Now,

$$\begin{aligned} Q_{k,n} &= \prod_{j=1}^{k-1} (j^2 - k^2) \prod_{j=k+1}^{n} (j^2 - k^2) \\ &= (-1)^{k-1} \cdot (k-1)! \cdot \frac{(2k-1)!}{k!} (n-k)! \frac{(n+k)!}{(2k)!} \\ &= (-1)^{k-1} \frac{(n-k)!(n+k)!}{2k^2}. \end{aligned}$$

Then,

$$\begin{aligned} J_n &= \frac{\pi}{(2n)!}\sum_{k=1}^{n}(-1)^{k-1}\cdot k\binom{2n}{n-k} \\ &= \frac{\pi}{(2n)!}\sum_{k=0}^{n-1}(-1)^{n-1-k}(n-k)\binom{2n}{k} \\ &= \frac{\pi}{(2n-1)!}\left(\frac{1}{2}\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2n}{k} - \sum_{k=0}^{n-2}(-1)^{n-k}\binom{2n-1}{k}\right). \end{aligned}$$

Now, if

$$R_n = \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2n}{k} \quad \text{and} \quad S_n = \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2n+1}{k},$$

then

$$J_n = \frac{\pi}{(2n-1)!}\left(\frac{1}{2}R_n - S_{n-1}\right).$$

Note that

$$R_n = \sum_{k=n+1}^{2n}(-1)^{n-1-k}\binom{2n}{k},$$

so

$$2R_n = \sum_{k=0}^{2n}(-1)^{n-1-k}\binom{2n}{k} + \binom{2n}{n}.$$

But

$$\sum_{k=0}^{2n}(-1)^{n-1-k}\binom{2n}{k} = (-1)^{n-1}(1-1)^{2n} = 0 \quad (\text{if } n \geq 1),$$

and so

$$R_n = \frac{1}{2}\binom{2n}{n}.$$

Thus,

$$J_n = \frac{\pi}{(2n-1)!}\left(\frac{1}{4}\binom{2n}{n} - S_{n-1}\right).$$

It remains to show that $S_n = nC_n$, where

$$C_n = \frac{\binom{2n}{n}}{n+1}$$

is the nth Catalan number. For then

$$\begin{aligned} J_n &= \frac{\pi}{(2n-1)!}\left(\frac{1}{4}\binom{2n}{n} - \frac{n-1}{n}\binom{2n-2}{n-1}\right) \\ &= \frac{\pi}{(2n-1)!}\left(\frac{2n(2n-1)}{4n^2} - \frac{n-1}{n}\right)\binom{2n-2}{n-1} \\ &= \frac{\pi}{(2n)!}\binom{2n-2}{n-1} = \frac{\pi}{2n(2n-1)[(n-1)!]^2}. \end{aligned}$$

Now,

$$\begin{aligned} S_n &= \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2n+1}{k} = (-1)^{n-1} + \sum_{k=1}^{n-1}(-1)^{n-1-k}\left(\binom{2n}{k} + \binom{2n}{k-1}\right) \\ &= (-1)^{n-1} + \sum_{k=1}^{n-1}(-1)^{n-1-k}\binom{2n}{k} - \sum_{k=0}^{n-2}(-1)^{n-1-k}\binom{2n}{k} \\ &= \binom{2n}{n-1} = \frac{(2n)!}{(n-1)!(n+1)!} = \frac{n}{n+1}\binom{2n}{n} = nC_n. \end{aligned}$$

Therefore,

$$J_n = \frac{\pi}{2n(2n-1)[(n-1)!]^2}.$$

Hence, the result follows.

Solution to part (b) by the proposer. The product is initialized at $n = 1$, so begin with

$$\int_0^\infty \frac{1}{x^2+1}\frac{9}{x^2+9}dx.$$

For simplicity, we could ignore the product in the numerator and treat just the partial fractions arising from the product,

$$\frac{1}{x^2+1}\cdot\frac{1}{x^2+9}\cdot\frac{1}{x^2+25}\cdots.$$

Observe that

$$\frac{1}{x^2+1}\cdot\frac{1}{x^2+9} = \frac{1/8}{x^2+1} - \frac{1/8}{x^2+9},$$

where the numerators of the two partial fractions are given by $1/(9-1)$ and $1/(1-9)$ and

$$\frac{1}{x^2+1}\cdot\frac{1}{x^2+9}\cdot\frac{1}{x^2+25} = \frac{1/192}{x^2+1} - \frac{1/128}{x^2+9} + \frac{1/384}{x^2+25},$$

where the numerators of the partial fractions are given by the products $1/[(9-1)(25-1)]$, $1/[(1-9)(25-9)]$, and $1/[(1-25)(9-25)]$. Furthermore,

$$\frac{1}{x^2+1}\cdot\frac{1}{x^2+9}\cdot\frac{1}{x^2+25}\cdot\frac{1}{x^2+49} = \frac{1/9216}{x^2+1} - \frac{1/5120}{x^2+9} + \frac{1/9216}{x^2+25} - \frac{1/46080}{x^2+49},$$

where the numerators of the partial fractions are given by the products $1/[(9-1)(25-1)(49-1)]$, $1/[(1-9)(25-9)(49-9)]$, $1/[(1-25)(9-25)(49-25)]$, and $1/[(1-49)(9-49)(25-49)]$.

Multiply the numerators of the partial fractions through by the last term in the series, in this case by 8, 384, 46080, ... The terms in this series are given by

$$1 \cdot 8, 6 \cdot 8^2, 90 \cdot 8^3, 2520 \cdot 8^4, 113400 \cdot 8^5, \ldots$$

where the terms 1, 6, 90, 2520, 113400 are given by the product of the consecutive hexagonal numbers

$$\prod_{k=1}^{n} k(2k-1).$$

So now, following the integration of

$$\int_0^\infty \frac{1}{x^2+1}\frac{1}{x^2+9}dx,$$

we have

$$\frac{\pi}{2} \cdot \frac{1}{8} \cdot \left(1 - \frac{1}{3}\right) = \frac{\pi}{2} \cdot \frac{1}{8} \cdot \frac{2}{3}$$

and for the next 3 fractions in the product, we obtain for

$$\int_0^\infty \frac{1}{x^2+1} \cdot \frac{1}{x^2+9} \cdot \frac{1}{x^2+25}dx,$$

that

$$\frac{\pi}{2} \cdot \frac{1}{384} \cdot \left(2 - \frac{3}{3} + \frac{1}{5}\right) = \frac{\pi}{2} \cdot \frac{1}{384} \cdot \frac{6}{5}$$

and for four fractions in the product, we obtain

$$\int_0^\infty \frac{1}{x^2+1} \cdot \frac{1}{x^2+9} \cdot \frac{1}{x^2+25} \cdot \frac{1}{x^2+49}$$

$$= \frac{\pi}{2} \cdot \frac{1}{46080} \cdot \left(5 - \frac{9}{3} + \frac{5}{5} - \frac{1}{7}\right)$$

$$= \frac{\pi}{2} \cdot \frac{1}{46080} \cdot \frac{20}{7}.$$

The next two results for a product of 5 and 6 fractions will be

$$\frac{\pi}{2}\cdot\frac{1}{2580\cdot 8^4}\left(14-\frac{28}{3}+\frac{20}{5}-\frac{7}{7}+\frac{1}{9}\right)=\frac{\pi}{2}\cdot\frac{1}{2520\cdot 8^4}\cdot\frac{70}{9}$$

and

$$\frac{\pi}{2}\cdot\frac{1}{113400\cdot 8^5}\left(42-\frac{90}{3}+\frac{75}{5}-\frac{35}{7}+\frac{9}{9}-\frac{1}{11}\right)=\frac{\pi}{2}\cdot\frac{1}{113400\cdot 8^5}\cdot\frac{252}{11}.$$

So now the series of fractions of the right-most term, namely

$$\frac{2}{3},\frac{6}{5},\frac{20}{7},\frac{70}{9},\frac{252}{11},\ldots$$

is given by

$$\frac{(2n)!}{(n!)^2(2n+1)}.$$

Therefore,

$$\int_0^\infty \frac{1}{x^2+1}\cdot\frac{1}{x^2+9}\cdot\frac{1}{x^2+(2n+1)^2}dx=\frac{\pi}{2}\cdot\frac{1}{8^n}\cdot\frac{(2n)!}{(2n+1)(n!)^2}\cdot\frac{1}{\prod_{k=1}^n k(2k-1)}.$$

Now multiplying both sides through by the product $1\times 9\times 25\cdots$ which is

$$\frac{[(2n+2)!]^2}{2^{2n+2}[(n+1)!]^2}$$

and recalling that we want the product to begin with 1×9, $1\times 9\times 25$, etc., it follows that

$$\frac{\pi}{2}\cdot\frac{1}{8^n}\cdot\frac{(2n)!}{(2n+1)(n!)^2}\cdot\frac{[(2n+2)!]^2}{2^{2n+2}[(n+1)!]^2}\frac{1}{\prod_{k=1}^n k(2k-1)}.$$

Simplifying, we get

$$\frac{\pi}{2}\cdot\frac{1}{2^{3n}}\cdot\frac{(2n)!}{(2n+1)(n!)^2}\cdot\frac{[(2n+1)!]^2(2n+2)^2}{2^{2n}2^2(n!)^2(n+1)^2}\frac{1}{\prod_{k=1}^{n}k(2k-1)}$$

and

$$\frac{\pi}{2}\cdot\frac{1}{2^{5n}}\cdot\frac{(2n+1)!}{(2n+1)^2(n!)^2}\cdot\frac{[(2n+1)!]^2}{(n!)^2}\frac{1}{\prod_{k=1}^{n}k(2k-1)}$$

resulting finally in

$$\frac{\pi}{2}\cdot\frac{1}{2^{5n}}\cdot\frac{[(2n+1)!]^3}{(2n+1)^2(n!)^4}\frac{1}{\prod_{k=1}^{n}k(2k-1)}.$$

And from here we obtain, as stated, the result in terms of the gamma function,

$$\frac{\pi}{2}\frac{[\Gamma(2n+2)]^3}{2^{5n}(2n+1)^2[\Gamma(n+1)]^4\prod_{k=1}^{n}k(2k-1)}$$

for $n=1,2,\dots$.

Remark by the proposer. The integration product formulas are very similar to theorems discovered by Ramanujan (see e.g. *The Man Who Knew Infinity* by Robert Kanigel, Washington Sq. Press, 1991, p. 165). Also, the proposer would like to express appreciation to George Andrews, Ph.D. Penn State University for his helpful assistance and encouragement.

115. [1998, 47] *Proposed by Kenneth B. Davenport, 301 Morea Road, Frackville, Pennsylvania.*

(a) Prove that

The number of ways of expressing every number of the form $3(2n-1)$, $n \geq 1$, as the sum of three numbers is equal to the sum of an nth ranked hexagonal number and an $(n-1)$th square number.

(b) Prove that

The number of ways of expressing every number of the form $4m$, $m \geq 1$, as the sum of four numbers is equal to the sum of the first m tetrahedral numbers, then subtract the sum of the first $m-3$ pentagonal numbers, the first $m-6$ pentagonal numbers, and so on until you reach 0, 1, or 2.

Solution by Paul S. Bruckman, 1518 Vanstone Road # 2, Campbell River, British Columbia, Canada. The generating function for the number of partitions of n into at most m parts is the coefficient of x^n in

$$P_m(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

Since

$$P_m(x) = \prod_{j=1}^{m} \sum_{k=0}^{\infty} x^{jk} \quad \text{for } |x| < 1,$$

this implies

$$P_m(x) = \sum_{k_1,k_2,\ldots,k_m \geq 0} x^{k_1+2k_2+\cdots+mk_m}.$$

We seek the coefficient of x^n, where

$$n = \sum_{j=1}^{m} jk_j \quad \text{for each } k_j \geq 0.$$

The coefficient of x^n in

$$Q_m(x) = P_m(x) - P_{m-1}(x) = \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)} \tag{1}$$

is the number of partitions of n into exactly m parts, which we denote as $p(n,m)$.

To solve part (a), for $m = 3$ we have

$$Q_3(x) = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}. \tag{2}$$

The trick is to decompose this into partial fractions. For $m = 3$, it's not too bad. Thus we find that

$$Q_3(x) = \frac{1}{6}(1-x)^{-3} - \frac{1}{4}(1-x)^{-2} - \frac{1}{72}(1-x)^{-1} - \frac{1}{8}(1+x)^{-1} + \frac{1}{9}(1-\omega^2 x)^{-1} + \frac{1}{9}(1-\omega x)^{-1}$$

where $\omega = \exp(2i\pi/3)$. Then

$$p(n,3) = \frac{1}{6}\binom{n+2}{2} - \frac{1}{4}(n+1) - \frac{1}{72} - \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3}.$$

After some simplification

$$p(n,3) = \frac{n^2 - 1 + A_n}{12} \tag{3}$$

where

$$A_n = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 6 \\ 0, & \text{if } n \equiv \pm 1 \pmod 6 \\ -3, & \text{if } n \equiv \pm 2 \pmod 6 \\ 4, & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

This is the exact expression, but may be more elegantly expressed as

$$p(n,3) = \left\langle \frac{n^2}{12} \right\rangle$$

where $\langle\cdot\rangle$ is the "nearest integer" function. [cf. Vol. 2, p. 160 of Dickson's *History of the Theory of Numbers*.]

We wish to show $p(3(2n-1),3) = H_n + S_{n-1}$ where H_n is the nth ranked Hexagonal number and S_{n-1} is the $(n-1)$th ranked Square number. $A_{3(2n-1)} = 4$ since all odd multiples of 3 are congruent to 3 (mod 6). Thus, using (3)

$$\begin{aligned} p(3(2n-1),3) &= \frac{[3(2n-1)]^2 - 1 + 4}{12} \\ &= \frac{(36n^2 - 36n + 9) + 3}{12} \\ &= 3n^2 - 3n + 1. \end{aligned}$$

Now,

$$\begin{aligned} H_n + S_{n-1} &= n(2n-1) + (n-1)^2 \\ &= 2n^2 - n + n^2 - 2n + 1 \\ &= 3n^2 - 3n + 1 \end{aligned}$$

and we are done with part (a).

With part (b), Dickson's, supra, is in error. That is,

$$p(n,4) \neq \left\langle \frac{n^3 + 3n^2 - 4}{144} \right\rangle.$$

The general formula based on (1) is

$$p(n,4) = \frac{n^3 + 3n^2 - 9n \cdot o_n + B_n}{144} \tag{4}$$

where $o_n = \frac{1}{2}(1-(-1)^n)$ and

$$B_n = \begin{cases} 0, & \text{if } n \equiv 0 \pmod{12} \\ 5, & \text{if } n \equiv 1 \pmod{6} \\ -20, & \text{if } n \equiv 2 \pmod{12} \\ -27, & \text{if } n \equiv 3 \pmod{6} \\ 32, & \text{if } n \equiv 4 \pmod{12} \\ -11, & \text{if } n \equiv 5 \pmod{6} \\ -36, & \text{if } n \equiv 6 \pmod{12} \\ 16, & \text{if } n \equiv 8 \pmod{12} \\ -4, & \text{if } n \equiv 10 \pmod{12}. \end{cases}$$

So now using (4) it can be shown that the following identity holds.

$$p(4n,4) = \sum_{k=1}^{n} T_k - \sum_{j=1}^{[\frac{1}{3}(n-1)]} \sum_{k=1}^{n-3j} P_k, \quad n = 1, 2, \ldots$$

where T_n is the nth "tetrahedral". $T_n = \binom{n+2}{3}$ and P_n is the nth "pentagonal", with $P_n = \frac{3n^2-n}{2}$. Let

$$f(x) = \sum_{n=1}^{\infty} p(4n,4)x^n, \tag{5}$$

$$g(x) = \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} T_k, \tag{6}$$

$$h(x) = \sum_{n=1}^{\infty} x^n \sum_{j=1}^{[\frac{1}{3}(n-1)]} \sum_{k=1}^{n-3j} P_k. \tag{7}$$

These expressions are assumed valid for all $|x| < 1$.

We're required to prove $f(x) = g(x) - h(x)$ and so substituting $4n$ for n into (4) we obtain

$$p(4n,4) = \frac{64n^3 + 48n^2 + B_{4n}}{144}$$

where

$$B_{4n} = \begin{pmatrix} 0 \\ 32 \\ 16 \end{pmatrix},$$

depending on whether

$$4n \equiv \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix} \pmod{12}.$$

And so,

$$p(4n,4) = \frac{4n^3 + 3n^2 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}{9}, \quad n \equiv \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \pmod{3}.$$

Then

$$\begin{aligned} f(x) &= \frac{1}{9}\sum_{n=1}^{\infty}\left(4n^3 + 3n^2 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right)x^n \\ &= \frac{1}{9}\sum_{n=1}^{\infty}(4n^3 + 3n^2)x^n + \frac{1}{9}\sum_{n=0}^{\infty}(2x + x^2)x^{3n}. \end{aligned}$$

Now,

$$\begin{aligned} 4n^3 + 3n^2 &= 4n^{(3)} + 15n^{(2)} + 7n^{(1)} \\ &= 24\binom{n}{3} + 30\binom{n}{2} + 7\binom{n}{1}. \end{aligned}$$

Then,

$$f(x) = \frac{1}{9}\sum_{n=0}^{\infty}\left(24\binom{n+3}{3}x^{n+3} + 30\binom{n+2}{2}x^{n+2} + 7\binom{n+1}{1}x^{n+1}\right)$$

$$+\frac{1}{9}(2x+x^2)(1-x^3)^{-1}$$

$$= \frac{1}{9}\left(24x^3(1-x)^{-4} + 30x^2(1-x)^{-3} + 7x(1-x)^{-2} + (2x+x^2)(1-x^3)^{-1}\right).$$

After some simplification,

$$f(x) = \frac{x(1+x)(1+x+2x^2)}{(1-x)^4(1+x+x^2)}, \quad |x| < 1. \tag{8}$$

Next,

$$g(x) = \sum_{k=1}^{\infty} T_k \sum_{n=k}^{\infty} x^n = \sum_{k=1}^{\infty} T_k \sum_{n=0}^{\infty} x^{n+k}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{k=1}^{\infty} T_k x^k = (1-x)^{-1}\sum_{k=0}^{\infty}\binom{k+3}{3}x^{k+1}$$

$$= x(1-x)^{-1}(1-x)^{-4} = x(1-x)^{-5}$$

so

$$g(x) = x(1-x)^{-5}, \quad |x| < 1. \tag{9}$$

Finally,

$$h(x) = \sum_{j=1}^{\infty}\sum_{n=3j+1}^{\infty} x^n \sum_{k=1}^{n-3j} P_k = \sum_{j=1}^{\infty} x^{3j}\sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} P_k$$

$$= x^3(1-x^3)^{-1}\sum_{k=1}^{\infty} P_k \sum_{n=0}^{\infty} x^{n+k} = x^3(1-x)^{-1}(1-x^3)^{-1}\sum_{k=1}^{\infty} P_k x^k.$$

Now,

$$\sum_{k=1}^{\infty} P_k x^k = \sum_{k=1}^{\infty}\left(3\binom{k}{2}+\binom{k}{1}\right)x^k = \sum_{k=0}^{\infty}\left(3\binom{k+2}{2}x^{k+2}+\binom{k+1}{1}x^{k+1}\right)$$

$$= 3x^2(1-x)^{-3} + x(1-x)^{-2} = x(1+2x)(1-x)^{-3}.$$

Then,

$$h(x) = \frac{x^4(1+2x)}{(1-x)^5(1+x+x^2)}, \quad |x|<1. \tag{10}$$

From here, it is a fairly straight-forward exercise to show

$$f(x) = g(x) - h(x)$$

which establishes the given identity.

Also, it might be noted that the expression in Dickson's, supra, is valid for n even, in which case a more concise expression is

$$p(n,4) = \left\langle \frac{n^3+3n^2}{144} \right\rangle$$

but for odd n

$$p(n,4) = \left\langle \frac{n^3+3n^2-9n}{144} \right\rangle.$$

116. [1998, 47] *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.*

Let n be a fixed positive real number and let

$$I_n(t) = \int_0^1 \left[\log\left(\frac{1-rt}{1-r}\right)\right]^n \frac{1-r}{(1-rt)^2}dr,$$

where $0 < t < 1$.

Find an upper bound for $I_n(t)$ as a function of n in terms of the gamma and zeta functions.

Solution by the proposers. Let

$$u = \log\left(\frac{1-tr}{1-r}\right).$$

Then

$$du = \frac{1-t}{(1-tr)(1-r)}dr$$

and we get

$$I_n(t) = \int_0^\infty \frac{u^n e^{-u}}{e^u - t}du.$$

Since $0 < t < 1$, $e^u - t > e^u - 1$ and so

$$I_n(t) < \int_0^\infty \frac{u^n}{e^u(e^u-1)}du = \int_0^\infty \left(\frac{-u^n}{e^u} + \frac{u^n}{e^u-1}\right)du$$

$$= -\Gamma(n+1) + \int_0^\infty \frac{u^n}{e^u-1}du = -\Gamma(n+1) + \int_0^\infty \sum_{k=0}^\infty u^n e^{-(k-1)u}du$$

$$= -\Gamma(n+1) + \Gamma(n+1)\sum_{k=0}^\infty \frac{1}{(k+1)^{n-1}} = -\Gamma(n+1) + \Gamma(n+1)\zeta(n+1).$$