A SIMPLE REDUCTION OF THE POINCARÉ DIFFERENTIAL EQUATION TO CAUCHY MATRIX FORM

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1. Introduction. G. D. Birkhoff was the first who tried to reduce each ordinary homogeneous linear differential equation with analytical coefficients to a certain canonical form [1, 2]. But F. R. Gantmacher [3] and P. Masani [4] gave counterexamples to Birkhoff's theorem [2]. Later, G. D. Birkhoff gave a true result [5] and noticed that his proof [6] is a special case of an important theorem given by D. Hilbert [7] and J. Plemelj [8]. But, H. L. Turrittin [9, 10] considered the reduction of more general systems of homogeneous linear differential equations up to Birkhoff's canonical form.

In the present paper, we prove the reduction of the Poincaré homogeneous differential equation of nth degree [11] with different regular singularities, up to the Cauchy matrix form [3].

2. A Simple Reduction. We will prove the following theorem.

Theorem. The Poincaré differential equation

$$P_n(x)y^{(n)} = \sum_{i=0}^{n-1} P_i(x)y^{(i)}, \tag{1}$$

where $P_n(x) = \prod_{i=1}^{n}(x-d_i) = \varphi_n(x)$, $(d_i \neq d_j;\ i \neq j)$ reduces to the Cauchy matrix form

$$(xI - D)\frac{dY}{dx} = QY, \tag{2}$$

where

$$D = \operatorname{diag}\,(d_1, d_2, \dots, d_n), \tag{3}$$

$$Q = \begin{bmatrix} q_{1,1} & 1 & 0 & \cdots & 0 & 0 \\ q_{2,1} & q_{2,2} & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ q_{n-1,1} & q_{n-1,2} & q_{n-1,3} & \cdots & q_{n-1,n-1} & 1 \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & q_{n,n-1} & q_{n,n} \end{bmatrix}, \tag{4}$$

and

$$Y = (y_1, y_2, \cdots, y_n)^T. \tag{5}$$

<u>Proof</u>. Using the substitution $\varphi_i = \prod_{j=1}^{i}(x - d_j)$, $(1 \le i \le n)$ and the transformation of H. L. Turrittin [9]

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_i \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{2,0}(x) & \varphi_1 & 0 & \cdots & 0 & 0 \\ c_{3,0}(x) & c_{3,1}(x) & \varphi_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_{i,0}(x) & c_{i,1}(x) & c_{i,2}(x) & \cdots & \varphi_{i-1} & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_{n,0}(x) & c_{n,1}(x) & c_{n,2}(x) & \cdots & c_{n,n-2}(x) & \varphi_{n-1} \end{bmatrix} \cdot \begin{bmatrix} y \\ y' \\ y'' \\ \cdot \\ \cdot \\ \cdot \\ y^{(i-1)} \\ \cdot \\ \cdot \\ \cdot \\ y^{(n-1)} \end{bmatrix}, \tag{6}$$

where $\deg c_{i,j}(x) \le j$, for the ith row of the system (2) it follows that,

$$\begin{aligned} (x - d_i)y_i' = \varphi_i y^{(i)} &+ (x - d_i)[\varphi_{i-1}' + c_{i,i-2}(x)]y^{(i-1)} \\ &+ (x - d_i)[c_{i,i-2}'(x) + c_{i,i-3}(x)]y^{(i-2)} \\ &\vdots \\ &+ (x - d_i)[c_{i,1}'(x) + c_{i,0}(x)]y'. \end{aligned} \tag{7}$$

Substituting

$$\varphi_i y^{(i)} = y_{i+1} - c_{i+1,i-1}(x)y^{(i-1)} - \cdots - c_{i+1,0}(x)y,$$

in the right side of the equation (7), we obtain

$$
\begin{aligned}
(x-d_i)y_i' &= y_{i+1} + \Big((x-d_i)[\varphi_{i-1}' + c_{i,i-2}(x)] - c_{i+1,i-1}(x)\Big)y^{(i-1)} \qquad (8)\\
&+ \Big((x-d_i)[c_{i,i-2}'(x) + c_{i,i-3}(x)] - c_{i+1,i-2}(x)\Big)y^{(i-2)}\\
&\vdots\\
&+ \Big((x-d_i)[c_{i,1}'(x) + c_{i,0}(x)] - c_{i+1,i}(x)\Big)y' - c_{i+1,0}(x)y,
\end{aligned}
$$

and hence, after the substitution

$$(x-d_i)[\varphi_{i-1}' + c_{i,i-2}(x)] - c_{i+1,i-1}(x) = q_{i,i}\varphi_{i-1}, \qquad (9)$$

the coefficients $q_{i,i}$ and the polynomials $c_{i,i-2}(x)$ from $c_{i+1,i-1}(x)$ can be determined, i.e.

$$
\begin{aligned}
q_{i,i} &= -c_{i+1,i-1}(d_i)/\varphi_{i-1}(d_i),\\
c_{i,i-2}(x) &= -\varphi_{i-1}' + [q_{i,i}\varphi_{i-1}(x) + c_{i+1,i-1}(x)]/(x-d_i).
\end{aligned}
$$

Further, by using

$$\varphi_{i-1}y^{(i-1)} = y_i - c_{i,i-2}(x)y^{(i-2)} - \cdots - c_{i,0}(x)y,$$

the right side of equation (8) becomes

$$
\begin{aligned}
(x-d_i)y_i' &= y_{i+1} + q_{i,i}y_i\\
&+ \Big((x-d_i)[c_{i,i-2}'(x) + c_{i,i-3}(x)] - c_{i+1,i-2}(x) - q_{i,i}c_{i,i-2}(x)\Big)y^{(i-2)}\\
&\vdots\\
&+ \Big((x-d_i)[c_{i,1}'(x) + c_{i,0}(x)] - c_{i+1,i}(x) - q_{i,i}c_{i,1}(x)\Big)y'\\
&- [c_{i+1,0}(x) + q_{i,i}c_{i,0}(x)]y,
\end{aligned}
$$

and hence, using

$$(x-d_i)[c'_{i,i-2}(x)+c_{i,i-3}(x)]-c_{i+1,i-2}(x)-q_{i,i}c_{i,i-2}(x)=q_{i,i-1}\varphi_{i-2}, \qquad (10)$$

the constants $q_{i,i-1}$ and the polynomials $c_{i,i-3}(x)$ of $c_{i+1,i-2}(x)$, $c_{i,i-2}(x)$ and $q_{i,i}$ can be determined, i.e.

$$\begin{aligned} &q_{i,i-1}=-[c_{i+1,i-2}(d_i)+q_{i,i}c_{i,i-2}(d_i)]/\varphi_{i-2}(d_i),\\ &c_{i,i-3}(x)=-c'_{i,i-2}(x)+[q_{i,i-1}\varphi_{i-2}+q_{i,i}c_{i,i-2}(x)+c_{i+1,i-2}(x)]/(x-d_i). \end{aligned}$$

Continuing this procedure, it follows that

$$(x-d_i)[c'_{i,i-j}(x)+c_{i,i-j+1}(x)]=c_{i+1,i-j}(x)+\sum_{r=0}^{j-2}q_{i,i-r}c_{i-r,i-j}(x)+q_{i,i-j+1}\varphi_{i-j}, \qquad (11)$$

can successively determine the constants $q_{i,i-j+1}$ and the polynomials $c_{i,i-j+1}(x)$ for $j=2,3,\dots,i-1$, where finally,

$$-[c_{i+1,0}(x)+\sum_{r=0}^{i-2}q_{i,i-r}c_{i-r,0}(x)]=q_{i,1}. \qquad (12)$$

In particular, for the nth element y_n, using equation (1) in the first part of these calculations, we obtain

$$\begin{aligned} (x-d_i)y'_n &= \varphi_n y^{(n)}+(x-d_i)[\varphi'_{n-1}+c_{n,n-2}(x)]y^{(n-1)}+\cdots\\ &=\Big((x-d_n)[\varphi'_{n-1}+c_{n,n-2}(x)]+P_{n-1}(x)\Big)y^{(n-1)}\\ &+\Big((x-d_n)[c'_{n,n-2}(x)+c_{n,n-3}(x)]+P_{n-2}(x)\Big)y^{(n-2)}\\ &\vdots\\ &+\Big((x-d_n)[c'_{n,1}+c_{n,0}(x)]+P_1(x)\Big)y'+P_0(x)y. \end{aligned}$$

Hence, we obtain the required formulas for $i = n$ by replacing $c_{n+1,n-j}(x)$ by $-P_{n-j}(x)$, $(1 \le j \le n)$ in (9), (11), and (12), completing the proof of the theorem.

Example 1. Let us consider the following Poincaré differential equation of third degree

$$\varphi_3(x)y''' = P_2(x)y'' + P_1(x)y' + P_0(x)y,$$

where

$$\begin{aligned}
&d_1 = 0,\ d_2 = 1,\ d_3 = 2,\\
&P_2(x) = (x-3)(x-4), \quad \varphi_1(x) = x,\\
&P_1(x) = x - 5, \quad \varphi_2(x) = x(x-1),\\
&P_0(x) = 6, \quad \varphi_3(x) = x(x-1)(x-2).
\end{aligned}$$

From the Cauchy matrix system, the coefficients of the matrix Q, given by (4), have values

$$q_{1,1} = 8, \quad q_{2,1} = -36, \quad q_{2,2} = -5, \quad q_{3,1} = -14, \quad q_{3,2} = -3, \quad q_{3,3} = 1,$$

and the coefficients of the matrix of the transformation (6) are

$$c_{3,1}(x) = -2x + 7, \quad c_{3,0}(x) = -4, \quad c_{2,0}(x) = -8.$$

Example 2. Now we consider the reduction of Gauss' hypergeometric equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0,$$

where

$$\begin{gathered}
\varphi_1(x) = x, \quad \varphi_2(x) = x(x-1), \quad d_1 = 0, \quad d_2 = 1,\\
P_1(x) = c - (a+b+1)x, \quad P_0(x) = -ab.
\end{gathered}$$

The coefficients of the matrix Q, given by (4), have values

$$q_{1,1} = 1 - c, \quad q_{2,1} = -ab - [c - (a+b+1)](c-1), \quad q_{2,2} = c - (a+b+1),$$

and the coefficient of the matrix of the transformation (6) is $c_{2,0}(x) = c - 1$.

References

1. G. D. Birkhoff, "Singular Points of Ordinary Linear Differential Equations," *Trans. Amer. Math. Soc.*, 10 (1909), 436–470.

2. G. D. Birkhoff, "Equivalent Singular Points of Ordinary Linear Differential Equations," *Math. Ann.*, 74 (1913), 134–139.

3. F. R. Gantmacher, *The Theory of Matrices*, Nauka, Moscow, 1988, 420–421, (in Russian).

4. P. Masani, "On Result of G. D. Birkhoff on Linear Differential Systems," *Proc. Amer. Math. Soc.*, 10 (1959), 696–698.

5. G. D. Birkhoff, "A Theorem on Matrices of Analytical Functions," *Math. Ann.*, 74 (1913), 122–133.

6. G. D. Birkhoff, "On a Simple Type of Irregular Singular Point," *Trans. Amer. Math. Soc.*, 14 (1913), 463.

7. D. Hilbert, "Grundzüge einer Allgemeine Theorie der Linearen Integralgleichungen," *Göttinger Nachr.*, 1905, 307–338.

8. J. Plemelj, "Riemannsche Funktionenscharen mit Gegebner Monodromiegruppe," *Monatsh. Math. Phys.*, 19 (1908), 211–246.

9. H. L. Turrittin, "Reduction of Ordinary Differential Equations to Birkhoff Canonical Form," *Trans. Amer. Math. Soc.*, 107 (1963), 485–507.

10. H. L. Turrittin, "Convergent Solutions of Ordinary Linear Homogeneous Differential Equations in the Neighborhood of an Irregular Singular Point," *Acta. Math.*, 93 (1955), 27–66.

11. H. Poincaré, "Sur les Équations Linéaires aux Differentieles Ordinaries et aux Différences Finies," *Amer. J. Math.*, 7 (1885), 1–56.

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