# CONSTRUCTING PYTHAGOREAN TRIPLE PRESERVING MATRICES

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**1. Introduction.** In the first paper we found that a 3×3 matrix which is a PTPM, i.e., converts a Pythagorean triple into a Pythagorean triple, has to be of the type H, where

$$H = \begin{pmatrix} ((r^2-t^2)-(s^2-u^2))/2 & rs-tu & ((r^2-t^2)+(s^2-u^2))/2 \\ rt-su & ru+st & rt+su \\ ((r^2+t^2)-(s^2+u^2))/2 & rs+tu & ((r^2+t^2)+(s^2+u^2))/2 \end{pmatrix}.$$

We also recall that every Pythagorean triple has the form $(m^2-n^2, 2mn, m^2+n^2)$ and furthermore, is a PPT if the pair $(m,n)$ satisfies the four conditions listed below.

I-1. $m, n$ are positive integers
I-2. $m > n$
I-3. $\gcd(m,n) = 1$
I-4. $m + n \equiv 1 \pmod 2$.

We have seen that $(m^2-n^2, 2mn, m^2+n^2)H = (M^2-N^2, 2MN, M^2+N^2)$, where the pairs $(m,n)$ and $(M,N)$ are related by the matrix equation

$$(m,n)\begin{pmatrix} r & s \\ t & u \end{pmatrix} = (M,N).$$

If we start with a pair $(m,n)$ which satisfies I-1 to I-4 and multiply it with the matrix

$$R = \begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

the new pair $(M,N)$ may not satisfy I-1 to I-4. To assure that $(M,N)$ also satisfies I-1 to I-4, suitable restrictions must be imposed on $r$, $s$, $t$, and $u$. These are listed as R-1 to R-4 below.

R-1. $r$, $s$, $t$, and $u$ are integers, $r$ and $s$ are positive, but $t$ and $u$ can be negative, so long as $r + t \geq 0$ and $s + u \geq 0$.
R-2. $r + t \geq s + u \geq 0$
R-3. $\Delta = ru - st = \pm 1$
R-4. $r + s \equiv 1 \pmod 2$ and $t + u \equiv 1 \pmod 2$.

Since each PPT originates from a pair $(m, n)$, the task of transforming one PPT into another reduces to one of converting one pair $(m, n)$ into another pair $(M, N)$. The relation between the two pairs $(m, n)$ and $(M, N)$ is given by the matrix equation

$$(m, n)\begin{pmatrix} r & s \\ t & u \end{pmatrix} = (M, N).$$

In this paper we will see some examples of matrices which are PTPM. We will design a matrix which converts a given PPT into another given PPT. We will analyze the difficulties involved in this process. We will see why it is problematic to come up with an algorithm for designing a matrix which does a prescribed conversion of one given PPT into another.

**2. Example of a PTPM.** First we will see examples of PTPM's. To construct a PTPM (Pythagorean Triple Preserving Matrix), we must start with a matrix

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

where $r$, $s$, $t$, and $u$ obey the restrictions R-1 to R-4. Some examples are

$$R_1 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 5 & 2 \\ -2 & -1 \end{pmatrix}, R_3 = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Using the values of $r$, $s$, $t$, and $u$ from $R_1$ in the $3 \times 3$ matrix

$$A = \begin{pmatrix} ((r^2 - t^2) - (s^2 - u^2))/2 & rs - tu & ((r^2 - t^2) + (s^2 - u^2))/2 \\ rt - su & ru + st & rt + su \\ ((r^2 + t^2) - (s^2 + u^2))/2 & rs + tu & ((r^2 + t^2) + (s^2 + u^2))/2 \end{pmatrix},$$

we get

$$A_1 = \begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix}.$$

Similarly one can construct other $3 \times 3$ matrices using the values of $r$, $s$, $t$, and $u$ as given in the matrices $R_2$, $R_3$, and $R_4$. Let us verify that the matrix $A_1$ is a PTPM. Let $(m, n)$ be any pair which satisfies I-1 to I-4. On multiplying with $R_1$, we get

$$(m, n) \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = (3m + 2n, 2m + n) = (M, N).$$

Let us verify that the new pair $(M, N) = (3m + 2n, 2m + n)$ satisfies I-1 to I-4.

Verification.

I-1. $3m + 2n$ and $2m + n$ are both positive integers
I-2. $3m + 2n > 2m + n$
I-3. $\gcd(3m + 2n, 2m + n) = \gcd(3m + 2n - (2m + n), 2m + n) = \gcd(m + n, 2m + n) = \gcd(m + n, m) = \gcd(m, n) = 1$
I-4. $[(3m + 2n) + (2m + n)] \pmod 2 = (5m + 3n) \pmod 2 = (5m + 5n) \pmod 2 \equiv (m + n) \pmod 2 \equiv 1 \pmod 2$.

This shows that the pair $(M, N) = (3m + 2n, 2m + n)$ satisfies I-1 to I-4.
Let us prove that the matrix

$$A_1 = \begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix}$$

constructed above converts a PPT into a PPT. Let $(m^2 - n^2, 2mn, m^2 + n^2)$ be a given PPT. This supposes that the pair $(m, n)$ satisfies the conditions I-1 to I-4. On multiplying with the matrix $A_1$, we get

$$(m^2 - n^2, 2mn, m^2 + n^2) \begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix}$$

$$= (5m^2 + 8mn + 3n^2, 12m^2 + 14mn + 4n^2, 13m^2 + 16mn + 5n^2).$$

We note that

$$\begin{aligned}
5m^2 + 8mn + 3n^2 &= (3m+2n)^2 - (2m+n)^2 \\
12m^2 + 14mn + 4n^2 &= 2(3m+2n)(2m+n), \\
13m^2 + 16mn + 5n^2 &= (3m+2n)^2 + (2m+n)^2.
\end{aligned}$$

In other words the triple $(5m^2+8mn+3n^2, 12m^2+14mn+4n^2, 13m^2+16mn+5n^2)$ is of the type $(M^2-N^2, 2MN, M^2+N^2)$ and hence, is a PPT. This verifies that the matrix

$$\begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix}$$

is a PTPM. If we choose $(m,n)$ to be $(2,1)$, it means that we have chosen the triple $(m^2-n^2, 2mn, m^2+n^2) = (3,4,5)$ as our PPT, and on multiplying with the matrix $A_1$, we get

$$(3,4,5)\begin{pmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix} = (39, 80, 89).$$

We verify that $39^2 + 80^2 = 89^2$.

**3. Construction of a PTPM Which Converts a Given PPT into Another Given PPT.** Let us now construct a PTPM (it will convert every PPT into a PPT) that will convert a given PPT into another given PPT. For example, let us construct a PTPM which will convert $(3,4,5)$ into $(5,12,13)$. Now $(3,4,5) = (m^2-n^2, 2mn, m^2+n^2)$, where the pair $(m,n) = (2,1)$ and $(5,12,13) = (M^2 - N^2, 2MN, M^2+N^2)$, where the pair $(M,N) = (3,2)$. We need a matrix

$$R = \begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

such that

$$(2,1)\begin{pmatrix} r & s \\ t & u \end{pmatrix} = (3,2).$$

This requires solving the system of equations $2r + t = 3$ and $2s + u = 2$. We notice that we have two equations with four variables, and hence there are infinitely many solutions. So our matrix $R$ could be

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and in fact any matrix of the type

$$\begin{pmatrix} 1+\alpha & 1+\beta \\ 1-2\alpha & -2\beta \end{pmatrix}$$

where $\alpha$ and $\beta$ are integers, will convert $(2, 1)$ into $(3, 2)$. However the values of $r$, $s$, $t$, and $u$ must obey the conditions R-1 to R-4 listed on page 2. We see that the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfies R-1 to R-4. Choosing the values of $r = 2$, $s = 1$, $t = -1$, and $u = 0$, the $3 \times 3$ matrix is

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}.$$

We can verify that

$$(3, 4, 5) \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix} = (5, 12, 13).$$

**4. Discussion of Constructing a $3 \times 3$ Matrix for a Desired Purpose.** Mathematicians love algorithms. We did not give an algorithm to design a $3 \times 3$

PTPM which converts a given PPT into another given PPT. In this section we will list a number of difficulties one encounters in trying to find such an algorithm.

1. Suppose we want to convert $(5,12,13)$ into $(3,4,5)$. The triples $(3,4,5)$ and $(5,12,13)$ come from the pairs $(m,n)=(2,1)$ and $(3,2)$, respectively. Hence, we need a $2\times 2$ matrix to convert $(3,2)$ into $(2,1)$. Could we not use the inverse of the matrix just found? Let

$$R=\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Its inverse

$$R^{-1}=S=\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},$$

but the matrix $S$ does not satisfy the conditions R-1 to R-4 and is therefore not acceptable. Next, instead of $S$ we will try another approach. We want to find a matrix satisfying R-1 to R-4. We must have

$$(3,2)\begin{pmatrix} r & s \\ t & u \end{pmatrix}=(2,1).$$

Clearly the matrices must be of the type

$$\begin{pmatrix} 2+2\alpha & 1+2\beta \\ -2-3\alpha & -1-3\beta \end{pmatrix},$$

where $\alpha$ and $\beta$ are integers. Some examples are

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix},\begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 3 \\ -2 & -4 \end{pmatrix},\begin{pmatrix} 4 & 3 \\ -5 & -4 \end{pmatrix},\text{ and }\begin{pmatrix} 0 & 3 \\ 1 & -4 \end{pmatrix}.$$

We note that all the matrices shown above are unacceptable because each one violates at least one of the conditions listed in R-1 to R-4. Thus, it is not possible to find any matrix satisfying R-1 to R-4.

2. The discussion above suggests that the conditions R-1 to R-4 are perhaps too restrictive. These conditions are sufficient but are not necessary. Perhaps they could be relaxed. The condition R-3 is not necessary if we use the fact that the pair $(m, n)$ satisfies I-1 to I-4. For example, the matrix

$$E = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

has $\det E = 2$. Yet,

$$(m, n) \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = (M, N)$$

is such that $\gcd(M, N) = \gcd(2m, m+n)$. Simple analysis shows that the $\gcd(M, N)$ divides $2m$ as well as $m + n$. Since $m + n = 1 \pmod 2$, 2 is not a factor of $\gcd(M, N)$. Also, a prime $p > 2$ cannot be a factor of $\gcd(M, N)$ because it will have to divide both $m$ and $n$. This is not possible because $m$ and $n$ are relatively prime. Summarizing, the matrix

$$E = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

has determinant 2 and yet it converts a pair $(m, n)$ with $\gcd = 1$ into a pair $(M, N)$ with $\gcd = 1$. This proves that R-3 is sufficient but is not necessary.

3. Throughout our discussion we required that both the $2 \times 2$ matrix $R$, as well as the $3 \times 3$ matrix $A$, have entries which are integers. It is a matter of taste whether the entries should be integers or whether we should relax the condition to allow the entries to be rational numbers.

4. If we do allow the elements of $R$ and $A$ to be rational numbers, there is indeed an algorithm to find a $2 \times 2$ matrix $R$ and hence, a $3 \times 3$ matrix $A$ which converts a given PPT into another given PPT.

As an example of a matrix with rational elements which converts a given PPT into another given PPT, suppose we want to convert $(5,12,13)$ into $(3,4,5)$. Again, we need to convert $(3,2)$ into $(2,1)$. The rotational matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

will rotate any ordered pair through an angle $\theta$. Using this matrix and suitable scalar multiplication, we obtain a matrix

$$F = \frac{1}{13}\begin{pmatrix} 8 & -1 \\ 1 & 8 \end{pmatrix}$$

which converts $(3,2)$ into $(2,1)$. We see that

$$(3,2)\frac{1}{13}\begin{pmatrix} 8 & -1 \\ 1 & 8 \end{pmatrix} = \frac{1}{13}(3,2)\begin{pmatrix} 8 & -1 \\ 1 & 8 \end{pmatrix} = \frac{1}{13}(26,13) = (2,1).$$

We note that the matrix

$$F = \begin{pmatrix} 8/13 & -1/13 \\ 1/13 & 8/13 \end{pmatrix}$$

has rational entries and has determinant $\frac{5}{13}$. Using $F$ as our $2 \times 2$ matrix

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

the corresponding $3 \times 3$ matrix is given by

$$B = \frac{1}{169}\begin{pmatrix} 63 & -16 & 0 \\ 16 & 63 & 0 \\ 0 & 0 & 65 \end{pmatrix}.$$

We can verify that $(5, 12, 13)B = (3, 4, 5)$.

5. Lastly, by our definition $(3, 4, 5)$ is a PPT (of type $A$). We know that $(\pm 3, \pm 4, \pm 5)$ are all Pythagorean triples. So is $(4, 3, 5)$ (PPT of type $B$) and its variations with negative signs. Our analysis and, more specifically, our conditions R-1 to R-4 would be different and surely much easier to satisfy if we relax our definition of PPT to allow other Pythagorean triples.

6. By our definition, $(3, 4, 5)$ is a PPT of type $A$ and $(4, 3, 5)$ is a PPT of type $B$. Our matrix $A$ converts type PPTA into PPTA. If we want to convert PPTA into PPTB, we only need to interchange column 1 and column 2. For example, $(3, 4, 5)$ when multiplied by

$$\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}$$

gives the result $(5, 12, 13)$, but when multiplied by

$$\begin{pmatrix} 2 & 1 & 2 \\ -1 & -2 & -2 \\ 2 & 2 & 3 \end{pmatrix}$$

gives the result $(12, 5, 13)$.

**5. Conclusion.** We have seen examples of matrices which convert Pythagorean triples into Pythagorean triples and in addition convert an assigned Pythagorean triple into another assigned one. We have seen that the task of constructing a matrix converting $(m^2 - n^2, 2mn, m^2 + n^2)$ into $(M^2 - N^2, 2MN, M^2 + N^2)$ reduces to finding a suitable $2 \times 2$ matrix $R$ which converts $(m, n)$ into $(M, N)$. We have seen the difficulties caused by the set of conditions R-1 to R-4 on the elements of $R$. Perhaps a different set of conditions would ease this situation and perhaps a suitable set of conditions would lead to an algorithm to design a PTPM. Because we have focused on designing a matrix, we have purposely left out other results concerning $A$ and $R$. For example we found that

$$\det\begin{pmatrix} ((r^2-t^2)-(s^2-u^2))/2 & rs-tu & ((r^2-t^2)+(s^2-u^2))/2 \\ rt-su & ru+st & rt+su \\ ((r^2+t^2)-(s^2+u^2))/2 & rs+tu & ((r^2+t^2)+(s^2+u^2))/2 \end{pmatrix}$$

$$= \left(\det\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right)^3.$$

This area of research is very rich and extensive and we have had to leave many unanswered questions. The story of Pythagorean triples has always been, and will always remain, an exciting area of mathematical investigation.

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