ASYMPTOTIC BEHAVIOR AND UNIQUENESS OF BLOW-UP SOLUTIONS OF ELLIPTIC EQUATIONS*

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To Neil S. Trudinger on the occasion of his sixty-fifth birthday

Introduction. A number of authors have studied the problem

$$(0.1) \qquad \Delta u = g(u) \text{ in } \Omega, u \equiv \infty \text{ on } \partial\Omega$$

for various choices of the function g, where Ω is a domain in $\mathbb{R}^n$. Following standard practice, we assume initially that

$$(0.2a) \qquad g \text{ is locally Lipschitz on } \mathbb{R}$$

(although this condition can be considerably relaxed (see [30])), and that there is a $t_0 \in \mathbb{R} \cup \{-\infty\}$ such that

$$(0.2b) \qquad g \text{ is increasing and positive on } (t_0, \infty),$$
$$(0.2c) \qquad \lim_{t \to t_0^+} g(t) = 0.$$

(If $t_0 = -\infty$, we also assume that g is integrable at $-\infty$.) In addition, we define the function G by

$$G(s) = \int_{t_0}^{s} g(t)\, dt$$

and we assume that

$$(0.3) \qquad \frac{1}{\sqrt{G}} \text{ is integrable at } \infty.$$

We refer to the survey paper [2] for a detailed discussion of the history of this problem, but we point out specifically that, according to results of Osserman [31] and Keller [17], if (0.2) holds, then there is a solution of (0.1) for Ω a ball if and only if (0.3) holds. For this reason, the function ψ, defined by

$$\psi(s) = \int_s^\infty \frac{1}{\sqrt{2G(t)}}\, dt$$

(with the constant 2 introduced for normalization), plays an important role in the theory. A key element of the theory is to examine the asymptotic behavior of u near $\partial\Omega$ and a number of estimates for this behavior are given in, for example, [1, 21, 28, 29]

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when $\partial\Omega$ is sufficiently smooth. Our goal here is to provide a unified approach to these estimates which can also be applied in less smooth domains. We also provide more detailed estimates, similar to those in [18, 19], but with weaker smoothness for $\partial\Omega$ than in those works. In fact, our main idea is based on the one in [18, 19]; we look at the boundary value problem satisfied by $v = \psi(u)$. Such a function is known to have intrinsic interest, too. If $n = 2$ and $g(u) = 4e^{2u}$ or if $n > 2$ and $g(u) = n(n-2)u^{(n+2)/(n-2)}$, then v is known as the hyperbolic radius of Ω (see [1, Section 3] for a discussion of the hyperbolic radius), but it has rarely been studied by analyzing the boundary value problem it solves. For example, in [18, 19], the author studies the asymptotic behavior of v by looking at the equation solved by $w = (v-d)/d^2$, where d denotes distance to $\partial\Omega$.

As in all the cited works, we make some additional assumptions on g. Specifically, we look at the quantities

$$\gamma = \liminf_{t\to\infty} \frac{\psi(t)\psi''(t)}{\psi'(t)^2},$$

$$\Gamma = \limsup_{t\to\infty} \frac{\psi(t)\psi''(t)}{\psi'(t)^2}.$$

Note that $0 \le \gamma \le \Gamma \le \infty$ because $\psi\psi''/(\psi')^2 = \psi g/\sqrt{2G}$. In addition, we must have $\Gamma \ge 1$ because, otherwise, there are constants $\theta \in [0,1)$ and $t_1 \ge t_0$ such that $\psi^{1-\theta}$ is concave and decreasing on (t_1,∞), which can't happen because this function is positive.

We begin in Section 1 with some representative examples for the function g. We point out here that the different nature of this structure condition from those in the cited works makes a direct comparison difficult, so it's useful to have a full range of examples to work from. (In fact, our main goal here is to illustrate a general method, so we do not always give the best hypotheses for our estimates.) Section 2 lists the properties of a regularized distance function, which we use to construct comparison functions. Our first comparison functions are used in Section 3 to prove first order asymptotic expansions of the solution. To obtain higher order expansions, it will be useful to have some gradient estimates. We derive them in Section 4, and, unlike earlier gradient estimates, we control the direction of the gradient in addition to its length. The higher estimates are obtained in Section 5, using the first order expansion, the gradient estimate and a simple Schauder-type estimate for elliptic equations with unbounded coefficients. Section 6 presents some uniqueness results, and we close in Section 7 by presenting the forms of the estimates for some of our examples and by discussing the uniqueness of solutions in various circumstances.

We note that there has been a lot of work on a modification of (0.1) in which the Laplace operator is replaced by a more general, possibly nonlinear, elliptic operator. For example, [16] is concerned with the blow-up problem for the equation $\mathrm{div}(|Du|^{p-2}Du) = g(u)$, and [8, 9, 7] look at the equation $\Delta u + au = b(x)g(u)$ with a a suitable constant and b a nonnegative function satisfying some additional technical conditions relating b to g and a. We defer a study of such problems to a future work, but point out here that we are able to study g from a larger class of functions than in those works (when specialized to $p = 2$ in [16] and to $b \equiv 1$ and $a = 0$ in [8, 9, 7]).

1. Examples. We start with two standard examples.

EXAMPLE 1.1. Let

$$g(s) = s^q$$

with $q > 1$. Then $t_0 = 0$,

$$G(s) = \frac{s^{q+1}}{q+1}, \quad \psi(s) = \frac{\sqrt{2(q+1)}}{q-1} s^{(1-q)/2},$$

and

$$\frac{\psi(s)g(s)}{\sqrt{2G(s)}} \equiv \frac{q+1}{q-1},$$

so $\gamma = \Gamma = (q+1)/(q-1)$.

EXAMPLE 1.2. Now, let

$$g(s) = e^s.$$

So $t_0 = -\infty$,

$$G(s) = e^s, \quad \psi(s) = \sqrt{2}e^{-s/2},$$

and

$$\frac{\psi(s)g(s)}{\sqrt{2G(s)}} \equiv 1,$$

so $\gamma = \Gamma = 1$.

Next, we look at a generalization of Example 1.1.

EXAMPLE 1.3. Let's suppose g satisfies

(1.1) $$2 + \eta \le \frac{sg(s)}{G(s)} \le 2 + \theta \text{ if } s \ge t_1$$

for constants t_1, η, and θ satisfying $0 < \eta \le \theta$ and $t_1 \ge t_0$. Then simple integration (see, for example, [23, Lemma 1.1(b)]) shows that

$$\frac{2}{\theta} \le \frac{\psi(s)\sqrt{2G(s)}}{s} \le \frac{2}{\eta}.$$

Hence

$$\frac{\eta+2}{\theta} \le \frac{\psi(s)g(s)}{\sqrt{2G(s)}} \le \frac{\theta+2}{\eta},$$

and therefore $\gamma \ge (\eta+2)/\theta$ and $\Gamma \le (\theta+2)/\eta$. (Note that, if we weaken (1.1) to

$$2 \le \frac{sg(s)}{G(s)} \le 2 + \theta \text{ if } s \ge t_1,$$

then we still obtain $\gamma \ge 2/\theta$ but we don't obtain a finite upper bound for Γ.)

This example includes functions like those studied in [8]. Specifically, that work examined blow-up solutions for the differential equation

$$\Delta u + au = b(x)g(u)$$

for a a suitable constant, b a nonnegative function, and g a regularly varying function (at infinity) with index $q > 1$. Regularly varying functions were first introduced by Karamata in 1930 and are defined by the condition

$$\lim_{s \to \infty} \frac{g(\lambda s)}{g(s)} = \lambda^q$$

for all $\lambda > 0$. We refer to [6] for a thorough discussion of such functions, but we note that [6, Theorem 1.5.11] (also called Karamata's Theorem) says that $\lim_{s \to \infty} sg(s)/G(s) = q + 1$ in this case, so $\gamma = \Gamma = (q+1)/(q-1)$. The important element in [8] that we do not deal with here is that b is allowed to vanish in part of Ω and it need not be bounded away from zero near $\partial\Omega$.

We can also look at functions g with faster growth at infinity.

EXAMPLE 1.4. Let's suppose that $g(s) = \exp(h(s))$ for $s \geq t_1$ for some $t_1 \geq \max\{t_0, 0\}$ (the behavior of g on (t_0, t_1) is not important here), where h is a function satisfying

$$\eta \leq \frac{sh'(s)}{h(s)} \leq \theta$$

for positive constants $\eta \leq \theta$. In this case, explicit formulae for G and ψ are not available, but we can still compute the asymptotic behavior of $\psi g/\sqrt{2G}$. For this computation, we recall that the usual proof of l'Hôpital's rule (see, for example, [32, Theorem 5.13]) says that, if k and k_1 are differentiable functions defined on $[a, \infty)$ for some real number a with k_1' never zero, and if $\lim_{s \to \infty} k_1(s) = \infty$, then

$$\liminf_{s \to \infty} \frac{k'(s)}{k_1'(s)} \leq \liminf_{s \to \infty} \frac{k(s)}{k_1(s)} \leq \limsup_{s \to \infty} \frac{k(s)}{k_1(s)} \leq \limsup_{s \to \infty} \frac{k'(s)}{k_1'(s)}.$$

We now apply this inequality with $k(s) = g(s)s/h(s)$ and $k_1(s) = G(s)$. Then

$$\frac{k'(s)}{k_1'(s)} = \frac{\left(\frac{1}{h(s)} - \frac{sh'(s)}{h(s)^2}\right)g(s) - \frac{s}{h(s)}h'(s)g(s)}{g(s)} = \frac{h(s) - sh'(s)}{h(s)^2} + \frac{sh'(s)}{h(s)}.$$

Since $\lim_{s \to \infty} h(s) = \infty$, it follows that

$$\eta \leq \liminf_{s \to \infty} \frac{s}{h(s)} \frac{g(s)}{G(s)} \leq \limsup_{s \to \infty} \frac{s}{h(s)} \frac{g(s)}{G(s)} \leq \theta.$$

If we take $k(s) = \psi(s)$ and $k_1(s) = (s/h(s))/\sqrt{2G(s)}$, we find that

$$\frac{k'(s)}{k_1'(s)} = \frac{1}{\left(\frac{sh'(s)}{h(s)^2} - \frac{1}{h(s)}\right) + \frac{s}{h(s)}\frac{g(s)}{2G(s)}},$$

so

$$\frac{2}{\theta} \leq \liminf_{s \to \infty} \frac{h(s)}{s} \frac{\psi(s)}{1/\sqrt{2G(s)}} \leq \limsup_{s \to \infty} \frac{h(s)}{s} \frac{\psi(s)}{1/\sqrt{2G(s)}} \leq \frac{2}{\eta}.$$

Therefore

$$\gamma \geq \frac{\eta}{\theta}, \quad \Gamma \leq \frac{\theta}{\eta}.$$

This example was inspired by the structure conditions in [9]. There are several important differences to note, however. In [9], the equation has the form $\Delta u + au = b(x)g(u)$, and the function g has the form $\exp_m(h)$ with $\exp_m$ being the m-th iterated exponential ($\exp_1 = \exp$ and $\exp_m = \exp(\exp_{m-1})$) and h being a function of regular variation. Unlike the situation for our previous example, this class of gs is not contained in ours even if we make the straightforward modifications needed to replace the exponential by an iterated exponential. On the other hand, we allow functions that cannot be written in the form used in [9].

Other examples can be generated from these via the following lemma.

LEMMA 1.1. *Let g satisfy (0.2) and (0.3). Suppose that g_0 is locally Lipschitz on $\mathbb{R}$ and that there is a constant $t_1 \in \mathbb{R} \cup \{-\infty\}$ such that g_0 is positive and increasing on (t_1, ∞) with*

$$\lim_{t \to t_1^+} g_0(t) = 0.$$

Define G_0 by

$$G_0(s) = \int_{t_1}^{s} g_0(t)\, dt.$$

Suppose also that there are real constants $A > 0$ and $t_2 \geq \max\{t_0, t_1\}$ such that $g(s) \leq A g_0(s)$ for all $s \geq t_2$. Then, for any $\varepsilon > 0$, there is a $t_3 \geq t_2$ such that

$$(1.2) \qquad G(s) \leq (A + \varepsilon) G_0(s)$$

for all $s \geq t_3$. Moreover, if g satisfies (0.3), then $1/\sqrt{G_0}$ is integrable at infinity. If ψ_0 is defined by

$$\psi_0(s) = \int_s^{\infty} \frac{1}{\sqrt{2 G_0(t)}}\, dt,$$

then $\psi_0(s) \leq \sqrt{A + \varepsilon}\, \psi(s)$ for all $s \geq t_3$.

Proof. If $s \geq t_2$, then

$$G(s) = G(t_2) + \int_{t_2}^{s} g(t)\, dt$$
$$\leq G(t_2) + A(G_0(s) - G(t_2))$$
$$\leq G(t_2) + A G_0(s)$$

Then, because $G_0(s) \to \infty$ as $s \to \infty$, there is a constant t_3 such that

$$G(t_2) \leq \varepsilon G_0(t_3),$$

so $s \geq t_3$ implies (1.2) because G_0 is increasing.

To prove that $1/\sqrt{2 G_0}$ is integrable at infinity, we observe that

$$\frac{1}{\sqrt{2 G_0(s)}} \leq \frac{\sqrt{A + \varepsilon}}{\sqrt{2 G(s)}}$$

if $s \geq t_3$. This inequality immediately implies the inequality for ψ and ψ_0. $\square$

Note that, if there are constants $A \geq B > 0$ such that $Bg_0(s) \leq g(s) \leq Ag_0(s)$ for $s \geq t_2$, then

$$\gamma \geq \frac{B}{A} \liminf_{s \to \infty} \frac{\psi_0(s)g_0(s)}{\sqrt{2G_0(s)}},$$

and

$$\Gamma \leq \frac{A}{B} \limsup_{s \to \infty} \frac{\psi_0(s)g_0(s)}{\sqrt{2G_0(s)}}.$$

In particular, if $\lim_{t \to \infty} g(s)/g_0(s) = 1$, then

$$\gamma = \liminf_{s \to \infty} \frac{\psi_0(s)g_0(s)}{\sqrt{2G_0(s)}},$$

and

$$\Gamma = \limsup_{s \to \infty} \frac{\psi_0(s)g_0(s)}{\sqrt{2G_0(s)}}.$$

Further examples can be generated by starting with a suitable function ψ.

EXAMPLE 1.5. Suppose that there is a decreasing C^3 function h such that

$$\psi(s) = \exp(h(\ln s))$$

for s sufficiently large, say $s \geq s_0$. Then

$$\frac{\psi(s)\psi''(s)}{(\psi'(s))^2} = 1 + \frac{h''(\ln s)}{(h'(\ln s))^2} - \frac{1}{h'(\ln s)}$$

and

$$g(s) = \exp(-2h(\ln s)) \left(-\frac{s}{h'(\ln s)} - \frac{sh''(\ln s)}{(h'(\ln s))^3} + \frac{s}{(h'(\ln s))^2} \right).$$

Hence

$$g'(s) = \exp(-2h(\ln s))h_0(\ln s),$$

where

$$h_0 = 2 + \frac{1}{h'} + \frac{1}{(h')^2} - \frac{h'''}{(h')^3} + \frac{3h''}{(h')^2} \left(1 - \frac{1}{h'} + \frac{h''}{(h')^2} \right).$$

It follows that g is increasing if $h_0(t) > 0$ for $t > \ln(t_0)$, with t_0 suitably chosen so that $g(t_0) = 0$. (This can be arranged easily for a given $t_0 < s_0$ if $h'(t) < 0$ and

$$1 + \frac{h''(t)}{(h'(t))^2} - \frac{1}{h'(t)} > 0$$

for $t \geq \ln(s_0)$. Just define g to be linear from t_0 to s_0 and continuous at s_0.)
    In particular, if

$$h(t) = t(\eta + \theta \sin(\ln(\ln t)))$$

for constants $\eta < 0$ and $\theta \in (0, -\eta)$, then

$$h'(t) = \eta + \theta \sin(\ln(\ln t)) + \frac{\theta}{t \ln t} \cos(\ln(\ln t)),$$

so, for given negative numbers $A < B$ we can make $A \leq h'(t) \leq B$ for suitably large t by choosing $\eta < 0$ and $\beta \in (0, -\eta)$ appropriately. On the other hand $|h''(t)| + |h'''(t)| = o(|h'(t)|)$ and

$$2 + \frac{1}{h'} + \frac{1}{(h')^2} \geq \frac{7}{4}.$$

Therefore we can make $h_0(s)$ positive for sufficiently large s and we can make γ and Γ be arbitrary numbers satisfying $1 < \gamma \leq \Gamma < \infty$ by choosing $\eta < 0$ and $\theta \in (0, -\alpha)$ appropriately. Hence the distinction between γ and Γ is important.

On the other hand, we can also arrange to have $\gamma \in (0, 1)$. (Note that in all the examples so far, either we have $\gamma \geq 1$ by explicit calculation of γ or we only have a lower bound on γ.) This time, we take

$$h(t) = \eta t + \theta \sin t$$

for suitable $\eta < 0$ and $\theta \in (0, -\eta)$. The calculations are more delicate, and we do not attempt to show that we can choose γ and Γ completely arbitrarily. First, we have

$$h'(t) = \eta + \theta \cos t,$$
$$h''(t) = -\theta \sin t,$$
$$h'''(t) = -\theta \cos t,$$

so

$$1 + \frac{h''(t)}{(h'(t))^2} - \frac{1}{h'(t)} < 1$$

is equivalent to $h''(t) < h'(t)$. This inequality is just $\eta + \theta \sin t > -\theta \cos t$, which is satisfied for $t = \pi/4$ if $\theta > -\eta/\sqrt{2}$. For convenience, we now choose $\theta = -\frac{3}{4}\eta$. Then

$$1 + \frac{h''(t)}{(h'(t))^2} - \frac{1}{h'(t)} > 0$$

for all t if and only if $(h')^2 \geq h' - h''$. From our choice of θ, we infer that

$$(h'(t))^2 = \eta^2 (1 - \frac{3}{4} \cos t)^2 \geq \frac{\eta^2}{16}$$

and

$$h'(t) - h''(t) = \eta[1 - \frac{3}{4} \cos t + \frac{3}{4} \sin t] \leq |\eta|(\frac{3}{4}\sqrt{2} - 1).$$

Since $(\frac{3}{4}\sqrt{2} - 1) < \frac{1}{16}$, it follows that $0 < \gamma < 1$ if $|\eta| \geq 1$. In addition,

$$h_0 \geq \frac{7}{4} - \frac{C}{\eta^2}$$

for some absolute constant C. Hence $h_0 \geq 0$ if η is sufficiently large (and nega-tive).With this choice for η and $\theta = -\frac{3}{4}\eta$, we conclude that $\gamma \in (0,1)$ (and hence $\gamma < \Gamma$ because $\Gamma \geq 1$) and that g satisfies (0.2) and (0.3).

Our final example has very slow growth at infinity. Since these functions satisfy the limit condition

$$\lim_{t \to \infty} \frac{\psi(\beta t)}{\psi(t)} = 1$$

for all $\beta \in (0,1)$, they do not fall under the scope of, for example, [1].

EXAMPLE 1.6. Now, we suppose that

(1.3) $g(s) = 2s(\ln s)^{2+\theta} + (2+\theta)s(\ln s)^{1+\theta}$

for some positive constant θ. Then

$$G(s) = s^2 (\ln s)^{2+\theta}, \quad \psi(s) = \frac{\sqrt{2}}{\theta}(\ln s)^{-\theta/2}.$$

Since

$$\frac{\psi \psi''}{(\psi')^2} = \frac{2\ln s + 2 + \theta}{\theta},$$

it follows that $\gamma = \Gamma = \infty$. As we shall see, we can obtain estimates in this case as well. In addition (at least for θ sufficiently large), we can get uniqueness results. These results have been proved also by Cîrstea and Du [7] (for all $\theta > 0$), but their proof takes strong account of the logarithmic growth of g.

2. Regularized distance. We recall from [22] the following definitions. First, we write d for distance to the boundary of Ω, that is,

$$d(x) = \inf\{|x - y| : y \in \partial\Omega\}.$$

We say that $\rho : \Omega \to \mathbb{R}$ is a *regularized distance* if $\rho \in C^2(\Omega)$ and the ratio ρ/d is bounded above and below in Ω by positive constants. If, in addition, there is a $\delta_0 > 0$ such that $|D\rho|$ is bounded away from zero on the set where $d(x) < \delta_0$, then we say that ρ is *proper*.

To proceed, it will be convenient to introduce standard weighted Hölder spaces. We write $\operatorname{diam}\Omega$ for the diameter of Ω. If $\delta \in (0,1]$ and $\beta \geq -\delta$, we define the weighted Hölder seminorm:

$$[u]_\delta^{(\beta)} = \sup\left\{d^{\delta+\beta}\frac{|u(x) - u(y)|}{|x - y|^\delta} : x \in \Omega, 0 < |x - y| < \frac{1}{2}d(x)\right\}.$$

Finally, for k a nonnegative integer, $\delta \in (0,1]$, and $\beta \geq -k - \delta$, we also define

$$|u|_{k+\delta}^{(\beta)} = \sum_{j=0}^{k} |D^j u|_0^{(j+\beta)} + [D^k u]_\delta^{(k+\delta+\beta)},$$

and we write $H_{k+\delta}^{(\beta)}$ for the set of all $u \in C^{k,\delta}(\Omega)$ such that $|u|_{k+\delta}^{(\beta)}$ is finite. (These are equivalent to the norms in [15, (6.10)] if $\beta \geq 0$ and to the norms in [14, (2.1)] if

$\beta \geq -k - \delta$.) Note that, if β is a positive non-integer less than or equal to $2 + \delta$, then $u \in H_\delta^{(-\beta)}$ implies that $u \in C^\beta(\overline{\Omega})$.

We also have a simple existence theorem for regularized distances.

LEMMA 2.1. *Let $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$. Then there is a proper regularized distance $\rho \in H_3^{(-1-\alpha)}$; if $\alpha = 1$, then $\rho \in C^{1,1}$ as well. Moreover, we can take this ρ so that there is a positive δ_0 so that $|D\rho| = 1$ if $\rho < \delta_0$.*

Proof. The existence of a proper regularized distance is just [22, Theorem 1.3]. To see that we can make $|D\rho| = 1$ near the boundary, let ρ_0 be any proper regularized distance. Then there is a positive constant δ_0 such that $|D\rho| > 0$ for $\rho < 2\delta_0$. If g is a smooth, increasing, nonnegative function which is zero on $[0, \delta_0]$ and positive on (δ_0, ∞), then ρ defined by

$$\rho = \frac{\rho_0}{(|D\rho_0|^2 + g(\rho_0))^{1/2}}$$

is the desired function. $\square$

We also point out that, for this regularized distance ρ, we have $|d - \rho| = O(d^{1+\alpha})$. The proof of this fact is easy. Let $x \in \Omega$, set $r = d(x)$ and let y be any point in $\partial\Omega$ such that $|x - y| = d(x)$. Then

$$\rho(x) = \int_0^1 D\rho(tx + (1-t)y) \cdot (x - y) \, dt,$$

and $D\rho(y) = (x - y)/r = D(tx + (1-t)y)$ for $t \in [0, 1]$. Hence

$$|\rho(x) - d(x)| \leq \int_0^1 |D\rho(tx + (1-t)y) - D\rho(y)| \, dt |x - y| \leq Cd(x)^{1+\alpha}$$

because $|(tx + (1-t)y) - y| \leq |x - y|$.

3. Basic expansions. In this section, we prove some simple pointwise estimates for $v = \psi(u)$. An easy calculation shows that

(3.1) $$\Delta v = \tilde{G}(v)(|Dv|^2 - 1) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

where

$$\tilde{G} = \frac{g \circ \psi^{-1}}{\sqrt{2G \circ \psi^{-1}}}.$$

For our estimates, it is very useful to introduce the operator Q defined by $Qw(x) = \Delta w - \tilde{G}(v)(|Dw|^2 - 1)$.

Our first step is a general estimate on how fast v approaches zero at the boundary.

LEMMA 3.1. *Suppose $\partial\Omega \in C^{1,\alpha}$. Then there is a positive constant H, determined only by g and Ω, such that $v \leq Hd$ in Ω.*

Proof. Let R be a positive number such that Ω is contained in a strip of the form $\xi \cdot x \in [a, a + 2R]$ for some unit vector ξ and some $a \in \mathbb{R}$. For $\varepsilon > 0$, set $w_\varepsilon^+ = (1 + \varepsilon)(\xi \cdot x - a)$ and $w_\varepsilon^- = (1 + \varepsilon)(a + 2R - \xi \cdot x)$. Since $|Dw_\varepsilon^\pm| = 1 + \varepsilon > 1$ and $\Delta w_\varepsilon^\pm = 0$, it follows that $Qw_\varepsilon^\pm < 0$ in Ω, so [15, Theorem 10.1] implies that $w_\varepsilon^\pm \geq v$ in Ω. Sending $\varepsilon \to 0$ gives $v \leq \min\{\xi \cdot x - a, a + 2R - \xi \cdot x\} \leq R$.

To proceed, we first note that, from Lemma 2.1, there is a constant H_1 such that $|\Delta\rho| \leq H_1\rho^{\alpha-1}$ in Ω. Now, we set

$$\delta_1 = \min\{\delta_0, [\alpha/(4H_1 + 1)]^{1/\alpha}\},$$

$$H = 2 + \frac{2R}{\delta_1}, \quad B = 2H_1H/[\alpha(1 + \alpha)],$$

and $\Omega' = \{x \in \Omega : \rho(x) < \delta_1\}$. It follows that $w = H\rho - B\rho^{1+\alpha}$ satisfies $Qw < 0$ in Ω' because $|Dw| > 1$ and $\Delta w \leq 0$ there. In addition, $w \geq v$ on $\partial\Omega'$, so [15, Theorem 10.1] implies that $w \geq v$ in Ω'. Therefore $v \leq H\rho$ if $\rho < \delta_1$, and the definition of H along with the previously proved inequality $v \leq R$ implies that $v \leq H\rho$ if $\rho \geq \delta_1$. $\square$

A similar (but more complicated) argument shows the corresponding lower bound for v. Because we don't need this result in the sequel, we omit the proof.

LEMMA 3.2. *Suppose $\partial\Omega \in C^{1,\alpha}$. Then there is a positive constant h, determined only by g and Ω such that $v \geq hd$ in Ω.*

We are now ready to prove a sharper upper estimate on v provided $\psi g/\sqrt{2G}$ is bounded away from 0.

THEOREM 3.3. *Suppose $\partial\Omega \in C^{1,\alpha}$ and suppose that $\gamma > 0$. Then there are constants $\delta \leq \alpha$ and K such that*

$$v \leq \rho + K\rho^{1+\delta}.$$

Proof. Let H be the constant from Lemma 3.1. With $\delta_1 \in (0, \min\{1, \delta_0\}]$ and $\delta \in (0, \alpha]$ to be chosen, we set $w = \rho + H\delta_1^{-\delta}\rho^{1+\delta}$. We also set

$$\Omega_1 = \{x \in \Omega : \rho(x) < \delta_1\}, \quad \Sigma = \{x \in \Omega : \rho(x) = \delta_1\}.$$

Then $w = (1 + H)\delta_1 \geq v$ on Σ and $w = 0 = v$ on $\partial\Omega$, so $w \geq v$ on $\partial\Omega_1$.

In addition, with H_1 as in the proof of Lemma 3.1 (so $|\Delta\rho| \leq H_1\rho^{\alpha-1}$), we have

$$\Delta w = [1 + (1 + \delta)H\delta_1^{-\delta}\rho^\delta]\Delta\rho + \delta(1 + \delta)H\delta_1^{-\delta}\rho^{\delta-1}$$
$$\leq [1 + (1 + \delta)H]H_1\rho^{\alpha-1} + \delta(1 + \delta)H\delta_1^{-\delta}\rho^{\delta-1}.$$

while

$$|Dw|^2 - 1 = H\delta_1^{-\delta}(1 + \delta)\rho^\delta[2 + H\delta_1^{-\delta}(1 + \delta)\rho^\delta] \geq 2H\delta_1^{-\delta}(1 + \delta)\rho^\delta.$$

Let us now assume that δ_1 is so small that $\psi(s)g(s) \geq \frac{1}{2}\gamma\sqrt{2G(s)}$ if $s > \psi^{-1}(H\delta_1)$. Then $\tilde{G}(v) \geq \gamma/v \geq \gamma/(H\rho)$ in Ω_1, so

$$\tilde{G}(v)(|Dw|^2 - 1) \geq \frac{1}{2}\gamma\delta_1^{-\delta}(1 + \delta)2\rho^{\delta-1},$$

and hence

$$Qw \leq \rho^{\delta-1}[(1 + (1 + \delta)H)H_1 + \delta(1 + \delta)H\delta_1^{-\delta} - \gamma\delta_1^{-\delta}(1 + \delta)].$$

By choosing $\delta = \min\{\alpha, \gamma/(2H)\}$ and taking δ_1 sufficiently small, we see that $Qw < 0$ in Ω_1 and hence $w \geq v$ there. This gives our inequality for $\rho < \delta_1$ and it's clear for $\rho \geq \delta_1$. $\square$

We now observe that the constant δ can be taken independent of H.

COROLLARY 3.4. *Theorem 3.3 is true for any δ satisfying $\delta \leq \alpha$ and $\delta < 2\gamma$.*

Proof. Choose ε so that

$$\delta(1+\varepsilon)^2 + \varepsilon^2 + 3\varepsilon < 2\gamma.$$

We then follow the proof of Theorem 3.3. By choosing δ_1 sufficiently small, we have from Theorem 3.3 that $v \leq (1+\varepsilon)\rho$ and $\tilde{G}(v) \geq (\gamma - \varepsilon)/((1+\varepsilon)\rho)$ in Ω_1, so we proceed with $1 + \varepsilon$ in place of H. It then follows that

$$Qw \leq \rho^{\delta-1}[(1 + (1+\delta)(1+\varepsilon))H_1 - \varepsilon\delta_1^{-\delta}(1+\delta)]$$

because

$$\delta(1+\varepsilon) + \varepsilon < \frac{2(\gamma - \varepsilon)}{1+\varepsilon}.$$

Choosing δ_1 even smaller guarantees that $Qw < 0$ in this case. □

In particular, if $\gamma > \alpha/2$ (which is the case for any $\alpha < 1$ for Examples 1.1, 1.2, and 1.4), then we can take $\delta = \alpha$.

Lower bounds for v are proved by similar methods but with some subtle variation.

THEOREM 3.5. *Suppose $\partial\Omega \in C^{1,\alpha}$ and suppose $\gamma > 0$. If $\delta \leq \alpha$ and $\delta < \gamma/(\gamma+2)$, then there is a constant K such that*

$$v \geq \rho - K\rho^{1+\delta}.$$

Proof. Let H_1 be as in Lemma 3.1. With $\delta_1 \in (0, \min\{1, \delta_0\}]$ and $\delta \in (0, \alpha]$ to be chosen, we set $w = \rho - \delta_1^{-\delta}\rho^{1+\delta}$. We also set

$$\Omega_1 = \{x \in \Omega : \rho(x) < \delta_1\}, \quad \Sigma = \{x \in \Omega : \rho(x) = \delta_1\}.$$

Since $\delta_1^{-\delta}\rho^\delta \in (0,1)$, it follows that $-\delta \leq 1 - \delta_1^{-\delta}(1+\delta)\rho^\delta$, and hence

$$\Delta w \geq \rho^{\delta-1}[-H_1 - \delta(1+\delta)\delta_1^{-\delta}].$$

In addition,

$$|Dw|^2 - 1 = \delta_1^{-\delta}(1+\delta)\rho^\delta[-2 + \delta_1^{-\delta}\rho^\delta(1+\delta)] \leq \delta_1^{-\delta}(1+\delta)\rho^\delta(\delta-1).$$

From Theorem 3.3, we can arrange $\tilde{G}(v) \geq \gamma/(2\rho)$ if δ_1 is sufficiently small. It follows that

$$w \geq \rho^{\delta-1}\left[-H_1 + (1+\delta)\delta_1^{-\delta}\left(-\delta + \frac{\gamma}{2}(1-\delta)\right)\right],$$

Since $\delta < \gamma/(\gamma+2)$, we can take δ_1 small enough to obtain $w > 0$. Since $v = \psi(u)$, it follows that $v \geq 0$ in Ω, so $v \geq w$ on $\partial\Omega_1$. Again [15, Theorem 10.1] implies that $v \geq w$ in Ω_1, which leads to the desired estimate. □

Again, we can improve the constant δ in this theorem.

COROLLARY 3.6. *Theorem 3.5 is true for δ satisfying $\delta \leq \alpha$ and $\delta < 2\gamma$.*

Proof. Let δ_1 be the constant from the proof of Theorem 3.5 and fix $\varepsilon \in (0,1)$ such that

$$\frac{\gamma - \varepsilon}{1 + \varepsilon}(2 - \varepsilon(1 + \delta)) > \delta.$$

With $\delta_2 \in (0, \delta_1)$ to be further specified, we now set

$$w = \rho - \varepsilon \delta_2^{-\delta} \rho^{1+\delta}.$$

Again, by choosing δ_2 sufficiently small, we can arrange $w > 0$ in

$$\Omega_2 = \{x \in \Omega : \rho(x) < \delta_2\}.$$

Furthermore, from Theorem 3.5, we have $v \geq (1 - \varepsilon)\rho$ in Ω_2 for δ_2 sufficiently small, so $v \geq w$ on $\partial\Omega_2$. The proof is completed as before. $\square$

Note that Theorems 3.3 and 3.5 imply that

$$\lim_{d(x) \to 0} \frac{v(x)}{d(x)} = 1.$$

For $\partial\Omega \in C^{1,1}$, this limit behavior is well-known even without the assumption $\gamma > 0$. See, for example, [13, Theorem 6.8]. (The assertion there is for C^2 boundary but the proof there applies directly to $C^{1,1}$ boundaries.) On the other hand, the argument in [13] seem to be tied very closely to the exact structure of $C^{1,1}$ domains and the Laplace operator because it is based on estimates of radial symmetric solutions of the partial differential equation. In addition, there is no easy way to determine the rate of convergence of the ratio v/d to 1 from that proof.

4. Gradient estimates. From our pointwise bounds, we can derive estimates on the gradient of v as well.

THEOREM 4.1. *Suppose $\partial\Omega \in C^{1,\alpha}$. If $\gamma > 0$ and $\Gamma < \infty$, then, for any δ satisfying $\delta \leq \alpha$ and $\delta < 2\gamma$, there is a constant K_2 such that*

$$|Dv - D\rho| \leq K_2\rho^\delta.$$

In addition, if $\delta < 1$, then $v \in C^{1,\delta}(\overline{\Omega})$.

Proof. Our first step is to obtain a gradient bound for v. So fix $x_0 \in \Omega$ and set $R = d(x_0)/2$. If R is sufficiently small (say $R < R_0$), then we have $R/2 < v < 2R$ in $B(x_0, R)$ and $\psi(s)g(s)/\sqrt{2G(s)} \leq 2\Gamma$ for $s \geq \psi^{-1}(2R)$. Hence, we also have

$$|\Delta v| \leq \frac{2\Gamma}{v}[|Dv|^2 + 1]$$

in $B(x_0, R)$. Now set $\bar{v}(x) = v(x_0 + Rx)/R$, so $|\Delta\bar{v}| \leq 4\Gamma[|D\bar{v}|^2 + 1]$ and $1/2 < \bar{v} < 2$ in $B(0, 1)$. It follows from, for example, [15, Theorem 15.8] that $|D\bar{v}(0)| \leq C(\Gamma)$, and hence $|Dv(x_0)| \leq C(\Gamma)$. On the other hand, if $R \geq R_0$, then $\tilde{G}(v)$ is bounded from above by a positive constant in $B(x_0, R)$ and we can apply [15, Theorem 15.8] directly to v.

To proceed, we set $h = v - \rho$ and $b = -\tilde{G}(v)(Dh + 2D\rho)$. Then

$$\Delta h + b^i D_i h = -\Delta\rho$$

in Ω. Our gradient estimate implies that $|b| \leq C/\rho$ and Lemma 2.1 implies that $|\Delta \rho| \leq C\rho^{\delta-1}$. Moreover, Theorems 3.3 and 3.5 give $|h| \leq C\rho^{1+\delta}$. If $\delta < 1$, then it follows from [26, Lemma 7.4] (which is just the divergence form version of the weighted Schauder estimate [15, Lemma 6.20]) that $h \in C^{1,\delta}(\overline{\Omega})$ and that $|Dh| \leq C\rho^{\delta}$. When $\delta = 1$, it's easy to modify the proof of that lemma to see that $|Dh| \leq C\rho$. $\square$

In most references (in particular [1, 2]), a weaker version of this theorem is proved; the authors show that, if Ω satisfies a uniform interior and exterior sphere condition, then $|Dv(x)| \to 1$ as $x \to \partial\Omega$. Here, we show that the direction of the gradient converges to the unit inner normal. For example, we can improve [1, Theorem 3.2] as follows:

PROPOSITION 4.2. *Suppose g satisfies* (0.3) *and* (0.2) *and that*

$$\lim_{t \to \infty} \frac{g(t)}{t^q} = 1$$

for some $q > 1$. If u is a solution of (0.1) *and if $\partial\Omega \in C^{1,\alpha}$, then for any $\varepsilon > 0$, there is a positive constant $\eta(\varepsilon)$ such that*

$$\left| \frac{q-1}{2a_q} \rho^{(q+1)/(q-1)} Du - D\rho \right| \leq \varepsilon$$

if $\rho(x) < \eta(\varepsilon)$, where

(4.1)
$$a_q = \left\{ \frac{q-1}{\sqrt{2(q+1)}} \right\}^{-2/(q-1)}.$$

Proof. From Theorem 4.1, we have

$$\left| \frac{1}{\sqrt{2G}} Du - D\rho \right| \leq K_2 \rho^{\delta},$$

and the triangle inequality gives

$$\left| \frac{q-1}{2a_q} \rho^{(q+1)/(q-1)} Du - D\rho \right| \leq \frac{q-1}{2a_q} \rho^{(q+1)/(q-1)} \left| Du - \sqrt{2G} D\rho \right|$$
$$+ \left| 1 - \left(\frac{q-1}{2a_q} \rho^{(q+1)/(q-1)} \right) \sqrt{2G} \right|.$$

Then Lemma 1.1 and Corollaries 3.4 and 3.6 imply that there is a constant K_3 such that

$$\rho^{(q+1)/(q-1)} \leq K_3/\sqrt{2G}.$$

Hence

$$\frac{q-1}{2a_q} \rho^{(q+1)/(q-1)} \left| Du - \sqrt{2G} D\rho \right| \leq C\rho^{\delta}.$$

In addition, it follows from Lemma 1.1 and Corollaries 3.4 and 3.6 that there is a positive η_1 such that $\rho < \eta_1$ implies that

$$\left| 1 - \left(\frac{q-1}{2a_q} \rho^{(q+1)(q-1)} \right) \sqrt{2G} \right| \leq \frac{\varepsilon}{2}.$$

The proof is completed by choosing $\eta \leq \eta_1$ so that $C\eta^\delta \leq \varepsilon/2$. $\square$

In fact, [1, Theorem 3.2] applies if u only becomes infinite on part of $\partial\Omega$, a situation we defer to future work.

Similarly, we can improve [1, Theorem 3.3]. In this case, we give a rate of convergence.

PROPOSITION 4.3. *If u is a solution of* (0.1) *with $g(t) = e^t$ and if $\partial\Omega \in C^{1,\alpha}$, then there is a constant K such that*

$$|\rho Du + 2D\rho| \leq K\rho^\alpha.$$

Proof. Here, Example 1.2 gives $Dv = -\frac{1}{2}vDu$, so $|Du| \leq C/\rho$ and $\rho Du + 2D\rho = -2Dv + 2D\rho + (\rho - v)Du$. From Corollaries 3.4 and 3.6, we have $|\rho - v| \leq C\rho^{\alpha+1}$, and combining these inequalities gives the estimate. $\square$

If we only assume that $\lim_{s\to\infty} g(s)/e^s = 1$, then we see that $|\rho Du + 2D\rho|$ converges uniformly to zero as we approach $\partial\Omega$.

5. Higher order estimates. Under stronger hypotheses on g, we can obtain more terms in the asymptotic expansion of ψ. For reasons of technical simplicity, we focus here on second order terms.

THEOREM 5.1. *Suppose $\partial\Omega \in C^{2,\alpha}$. Suppose also that there are constants G^* and H such that*

$$(5.1) \qquad \left| \tilde{G}(v) - \frac{G^*}{v} \right| \leq Hv^{\alpha-1}$$

for v close to zero. Then there is a function $F \in C^\alpha(\overline{\Omega})$ such that

$$(5.2) \qquad |v - d - Fd^2| \leq Cd^{2+\alpha}.$$

Moreover, $F = \Delta d/(4G^* - 2)$ on $\partial\Omega$.

Proof. Again, we write $h = v - \rho$, where now we assume that $\rho = d$ near $\partial\Omega$. We then set $b = -(G^*/v)(Dh + 2Dd)$ and define the operator L by $Lw = \Delta w + b^i D_i w$. From Theorem 4.1, we see that $|Dh| \leq Cd$, so $Lh = f$ for a function f satisfying $|f + \Delta d| \leq Cd^\alpha$. We now use ideas from the proof of [26, Theorem 8.3]. For any $y \in \partial\Omega$, we set

$$F_0(y) = \lim_{\substack{x \to y \\ x \in \Omega}} \frac{-\Delta d(x)}{2 + 2d(x)b^i(x)D_i d(x)}.$$

We note that $db^i D_i d = -(G^*/v)(D_i h + 2D_i d)D_i d$. Since Δd is continuous in $\overline{\Omega}$ and we have $Dh \to 0$ and $d/v \to 1$ as $x \to y$, it follows that $F_0 = \Delta d/(4G^* - 2)$, which is well-defined because $G^* = \Gamma \geq 1$. Now let F be an $H_{2+\alpha}^{(-\alpha)}$ extension of F_0 into Ω and set $w = F\rho^2$. It follows that $|L(h - w)| \leq C\rho^\alpha$. Moreover,

$$(2 + \alpha)(1 + \alpha) + (2 + \alpha)db^i D_i d = (2 + \alpha)(1 + \alpha - (dG^*/v)(D_i h + 2D_i d)D_i d)$$

and this quantity is negative near $\partial\Omega$ since it converges to $(2+\alpha)(1+\alpha-2G^*)$. Hence [26, Lemma 8.2] implies that $|h - w| \leq Cd^{2+\alpha}$, which is the same as (5.2). $\square$

Note that, in the special case that $g(u) = u^q$ (in which case $\tilde{G}(v) = G^*/v$), this theorem is the same as [19, Theorem 1.1], which generalizes [11, Theorem 1.1]. Similarly, when $g(u) = e^u$, this theorem is the same as [18, Theorem 1.1]. On the other hand, the relationship between (5.1) and the conditions in [4] or [5] is not so clear because those works introduce some additional structure functions, in particular, $\int_0^t \sqrt{G(s)}\, ds$.

6. Uniqueness. It follows from the argument in [27] that solutions of (0.1) are unique if $g(s)/s$ is an increasing function of s and the boundary is sufficiently smooth. The proof is a simple maximum principle argument which uses weak asymptotic information (namely that the ratio of two solutions approaches 1 at the boundary). When $g(s)/s$ is not necessarily increasing, other methods have been used which rely on other conditions for g. (We refer to [28] and [29] for a discussion of these other methods.) In [16], the authors suggest that these structure conditions can be dispensed with, at least for smooth domains. Since there are known examples (see, e.g. [10, 20]) of nonsmooth domains in which uniqueness fails even for $g(s) = s^q$ with $q > 1$, we consider an intermediate smoothness situation, and we show how to implement the program of [16] in sufficiently smooth domains. In the applications, we shall see a relation between the smoothness of the domain and the structure of g in order to infer uniqueness, although we do not make any claims that our conditions are sharp.

We start with a simple result, which was proved for a more general class of operators in [16, Theorem 1.2]. Here, we reproduce (for the reader's convenience) the proof of [12, Lemma 2.4], which is close in spirit (but not in detail) to the proof of [21, Lemma 3.1].

THEOREM 6.1. *Suppose u_1 and u_2 are solutions of* (0.1) *such that, for any $\varepsilon > 0$, there is an $\eta > 0$ such that*

$$|u_1(x) - u_2(x)| < \varepsilon$$

if $d(x) < \eta$. Then $u_1 \equiv u_2$ in Ω.

Proof. Let $\varepsilon > 0$ be given, set $v_\varepsilon = u_1 - u_2 - \varepsilon$, and write Ω_ε for the subset of Ω on which $v_\varepsilon > 0$. Suppose that Ω_ε is non-empty. Then it is an open subset of Ω and $\overline{\Omega_\varepsilon}$ is a compact subset of Ω. On Ω_ε, we have $u_1 > u_2$ and hence

$$\Delta v_\varepsilon = \Delta u_1 - \Delta u_2 = g(u_1) - g(u_2) > 0.$$

Therefore v_ε can't have a maximum in Ω_ε. But $v_\varepsilon \leq 0$ on $\partial\Omega_\varepsilon$, so $v_\varepsilon \leq 0$ in Ω_ε, which means that Ω_ε is empty. It follows that $u_1 \leq u_2 + \varepsilon$ for any $\varepsilon \geq 0$ and hence $u_1 \leq u_2$ in Ω.

A similar argument shows that $u_2 \leq u_1$ in Ω, and therefore $u_1 \equiv u_2$. ∎

Our next theorem extends one which was first proved by Loewner and Nirenberg as part of [27, Theorem 4] (see also [3, Theorem 2.4]). Although we may only a simple change in the proof, our version is applicable to a wide range of examples.

THEOREM 6.2. *Suppose u_1 and u_2 are solutions of* (0.1) *such that, for any $\varepsilon > 0$, there is an $\eta > 0$ such that*

$$\left|\frac{u_1(x)}{u_2(x)} - 1\right| < \varepsilon$$

if $d(x) < \eta$. Suppose also that there is a constant K such that $u_1 + K$ and $u_2 + K$ are both positive in Ω and such that $g(z)/(z + K)$ is an increasing function of z for $z \geq \min\{\min u_1, \min u_2\}$. Then $u_1 \equiv u_2$ in Ω.

Proof. First, we note that, because u_1 and u_2 tend to infinity near $\partial\Omega$, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{u_1(x) + K}{u_2(x) + K} - 1\right| < \varepsilon$$

if $d(x) < \delta$. We now argue as in [27]. Define v by

$$v(x) = \begin{cases} \frac{u_1(x)+K}{u_2(x)+K} & \text{if } x \in \Omega, \\ 1 & \text{if } x \in \partial\Omega. \end{cases}$$

Then $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and it satisfies the differential equation $Lv = f$ in Ω with L defined by

$$Lw = \Delta w - \frac{2}{u_2+K} Du \cdot Dw$$

and

$$f(x) = \left(\frac{g(u_1(x))}{u_1(x)+K} - \frac{g(u_2(x))}{u_2(x)+K} \right) v(x).$$

Since $f \geq 0$ wherever $v > 1$ and $f \leq 0$ wherever $v < 1$, it follows from the maximum principle that v cannot have a maximum wherever $v > 1$ or a minimum wherever $v < 1$. It follows that $v \equiv 1$, which implies that $u_1 \equiv u_2$. $\square$

Let us note that, because g is locally Lipschitz, the condition $g(z)/(z + K)$ is increasing is equivalent to $g'(z)(z + K) \geq g(z)$ for all $z \geq -K$. We first observe that this condition is satisfied if there are constants $t_1 > t_0$ and K_1 such that $g'(z)(z + K_1) \geq g(z)$ for $z \geq t_1$ and if

$$\inf\{g'(z) : t_2 \leq z \leq t_1\} > 0$$

for any $t_2 \in (t_0, t_1)$. In this case, by choosing $t_2 = \min\{\min u_1, \min u_2\}$, we have $g'(z)(z + K) \geq (z + K) \inf\{g'(z) : t_2 \leq z \leq t_1\}$ for any $K > -t_2$. By choosing K sufficiently large, we see that $g'(z)(z+K) \geq g(t_2) \geq g(z)$ if $z \leq t_1$ and $g'(z)(z+K) \geq g'(z)(z + K_1) \geq g(z)$ if $z \geq t_1$ provided we take $K \geq K_1$. In particular, if g is convex, we can apply Theorem 6.2.

7. Applications to examples. We are now ready to examine some examples. Unless otherwise specified, we assume that $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$.

First, we suppose that $g(s) = s^q$ for some $q > 1$. It then follows from Corollaries 3.4 and 3.6 that there is a positive constant K such that

$$d - Kd^{1+\alpha} \leq \psi(u) \leq d + Kd^{1+\alpha}.$$

The explicit formula for ψ then says that

$$a_q d^{2/(1-q)}[1 + Kd^{\alpha}]^{2/(1-q)} \leq u \leq a_q d^{2/(1-q)}[1 - Kd^{\alpha}]^{2/(1-q)}$$

as long as $Kd^{\alpha} < 1$. Therefore

$$u = a_q d^{2/(1-q)} + O(d^{2/(1-q)+\alpha}).$$

Since s^q/s is an increasing function of s, we easily obtain uniqueness from Theorem 6.2 in this situation because

$$\left| \frac{u}{a_q d^{2/(1-q)}} - 1 \right| = O(d^{\alpha}).$$

If $\partial\Omega \in C^{2,\alpha}$, then we can apply Theorem 5.1 to conclude that

$$u = a_q(d + Fd^2)^{2/(1-q)} + O(d^{2/(1-q)+1+\alpha}),$$

where F is a Hölder continuous function such that

$$F = \frac{2q+6}{q-1}\Delta d$$

on $\partial\Omega$. Note that the estimate for v from Theorem 5.1 is, after taking into account the different notation, really the same as [19, Theorem 1.1]; we have replaced the $o(1)$ term in that theorem by the term $O(d^\alpha)$ but the proof of [19, Theorem 1.1] also gives this result.

In similar fashion, if $g(s) = e^s$, we obtain

$$|u + 2\ln d - \ln 2| \leq \max\{\ln(1 + Kd^\alpha), -\ln(1 - Kd^\alpha)\},$$

so $|u + 2\ln d - \ln 2| = O(d^\alpha)$. Uniqueness now follows from Theorem 6.1.

Now, suppose that there is an increasing function g_0 such that

$$\lim_{t \to t_1^+} g_0(t) = 0$$

and $g(t) > 0$ if $t > t_1$ for some constant $t_1 \in \mathbb{R} \cup \{-\infty\}$. Suppose also that there are constants $t_2 > t_1$, δ, and θ satisfying $0 < \delta \leq \theta$ such that

$$2 + \eta \leq \frac{sg_0(s)}{G_0(s)} \leq 2 + \theta \text{ if } s \geq t_2.$$

(In other words, g_0 satisfies (0.2) and (1.1).) Suppose also that there are positive constants $A \geq B$ such that

$$Bg_0(t) \leq g(t) \leq Ag_0(t)$$

for $t \geq t_2$. This time, we also assume that $2B(\eta + 2) > A\alpha\theta$ (to guarantee that $\alpha < 2\gamma$). Writing ϕ for the inverse function to ψ, we see that $t^{2/\eta}\phi(t)$ is an increasing function of t. It follows from Corollary 3.4 that $v \leq d + Kd^{1+\alpha}$, so

$$u \geq \phi(d + Kd^{1+\alpha}) \geq (1 + Kd^\alpha)^{-2/\eta}\phi(d) \geq \phi(d)(1 - O(d^\alpha)).$$

A similar argument using Corollary 3.6 gives a corresponding upper bound for u and hence

$$|u - \phi(d)| = \phi(d)O(d^\alpha).$$

If $g(s)/s$ is an increasing function of s, then we obtain a uniqueness result immediately. Otherwise, we need to show that $\phi(d)d^\alpha \to 0$ as $d \to 0$. This will be the case if $2 < \alpha\eta$. Since $\alpha \leq 1$ in any case, it follows that this method proves uniqueness only if $\eta > 2$.

Now suppose that there are constants t_1 and H along with a function h satisfying $0 \leq h(s) \leq He^s$ for $s \geq t_1$ such that $g(s) = e^s + h(s)$. (Note that this information is not enough to determine t_0 other than seeing that $t_1 \leq t_0$, but we do assume that h is chosen so that g satisfies (0.2).) For example, we could take

$$h(s) = \begin{cases} (1 + \sin s)e^s & \text{if } s \geq 0, \\ (1 + s)e^s & \text{if } s < 0, \end{cases}$$

in which case $t_0 = -1$. Then, from Lemma 1.1, we see that there is a $t_2 \geq t_1$ such that $(2 + H)^{-1/2} \leq e^{t/2}\psi(t)/\sqrt{2} \leq 2$ for $t \geq t_2$. Hence $|\phi(s) + 2\ln s| \leq C(H)$ for

$s \in (0, s_0)$ with $s_0 = \psi(t_2)$ and $C(H)$ a suitable positive constant. In addition, $\phi'(s) = -\sqrt{2G \circ \phi(s)}$ satisfies $-C_1(H)/s \leq \phi'(s) \leq -C_2(H)/s$ for suitable positive constants $C_1 \geq C_2$. From Corollaries 3.4 and 3.6, we see that there are numbers $d_1 \in (d - Kd^{1+\alpha}, d)$ and $d_2 \in (d, d + Kd^{1+\alpha})$ such that

$$u \geq \phi(d + Kd^{1+\alpha}) = \phi(d) + \phi'(d_1)Kd^{1+\alpha} \geq \phi(d) - C_1 Kd^{1+\alpha}/d_1$$

and

$$u \leq \phi(d - Kd^{1+\alpha}) = \phi(d) - \phi'(d_2)Kd^{1+\alpha} \leq \phi(d) + C_2 Kd^{1+\alpha}/d_2.$$

If d is small enough that $Kd^\alpha \leq 1/2$, we infer that $|u - \phi(d)| = O(d^\alpha)$. Hence Theorem 6.1 again provides a uniqueness result in this case. (Note that, in [16], the corresponding structure is $g(s) = e^s + As^q$ for some positive exponent q.)

Now, let us suppose that there are positive constants t_1 and $\eta \leq \theta$ such that $h = \ln(g)$ satisfies

$$\eta \leq \frac{sh'(s)}{h(s)} \leq \theta$$

for $s \geq t_1$. If $\alpha < 2\eta/\theta$, we infer that there are numbers $d_1 \in (d - Kd^{1+\alpha}, d)$ and $d_2 \in (d, d + Kd^{1+\alpha})$ such that

$$u \geq \phi(d + Kd^{1+\alpha}) = \phi(d) + \phi'(d_1)Kd^{1+\alpha} \geq \phi(d) + 2Kd_1\phi'(d_1)d^\alpha$$

and

$$u \leq \phi(d - Kd^{1+\alpha}) = \phi(d) - \phi'(d_2)Kd^{1+\alpha} \leq \phi(d) - 2Kd_2\phi'(d_2)d^\alpha$$

for some positive constant K and $d \leq 1/(2K)^{1/\alpha}$. If we set $\delta_j = \phi(d_j)$ for $j = 1, 2$, then have

$$|\phi'(d_j)d_j| = \sqrt{2G(\delta_j)}\psi(\delta_j) \leq \frac{4}{\eta}\frac{\delta_j}{h(\delta_j)} \leq C\delta_j$$

provided δ_j is sufficiently large (which means d is sufficiently small). Now, $\psi(s) \leq (2/\eta)[h(s)/s\sqrt{2G(s)}]$, so $\psi(s) \leq s^{-2/\alpha}$ if s is sufficiently large. Hence, if d is sufficiently small, then

$$\delta_j d^\alpha \leq 2\delta_j(d_j^{\alpha/2})d^{\alpha/2} \leq 2d^{\alpha/2}.$$

It follows that $|u - \phi(d)| = O(d^{\alpha/2})$, which again implies uniqueness. Note that, in fact, we have $|u - \phi(d)| = O(d^\delta)$ for any $\delta \in (0, 1)$. Moreover, if $\eta \geq 1$, then $\delta_j/h(\delta_j)$ is bounded, so we obtain $|u - \phi(d)| = O(d^\alpha)$ in this case.

Finally, let us suppose that g has the form (1.3) for some $\theta > 0$. If we set $\Theta = (\sqrt{2}/\theta)^{2/\theta}$, then we can write

$$\phi(t) = \exp(\Theta t^{-2/\theta}),$$

so Corollary 3.4 implies that there is a positive constant K such that

$$u(x) \geq \phi(d + Kd^{1+\alpha}) = \exp(\Theta d^{-2/\theta}(1 + Kd^\alpha)^{-2/\theta}).$$

Now, the mean value theorem implies that there is a constant $d_1 \in (0, d)$ such that

$$(1 + Kd^\alpha)^{-2/\theta} = 1 - \frac{2}{\theta}(1 + Kd_1^\alpha)^{-1-(2/\theta)}Kd^\alpha,$$

and hence

$$(1 + Kd^\alpha)^{-2/\theta} \geq 1 - \frac{2K}{\theta}d^\alpha$$

if d is sufficiently small. Therefore, for small d, we have

$$u(x) \geq \phi(d)\exp(-\frac{2K}{\theta}d^{\alpha-2/\theta}).$$

Similarly, for small d, we also have

$$u(x) \leq \phi(d)\exp(\frac{2K}{\theta}d^{\alpha-2/\theta}).$$

Hence $u/\phi(d) \to 1$ if $\alpha > 2/\theta$, and we obtain a uniqueness result in this case.

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