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THE LARGE-TIME BEHAVIOR OF SOLUTIONS OF HAMILTON-JACOBI EQUATIONS ON THE REAL LINE*

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Dedicated to Professor Neil S. Trudinger on the occasion of his 65th birthday

Abstract. We investigate the large-time behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations on the real line $\mathbf{R}$. We establish a result on convergence of the solutions to asymptotic solutions as time t goes to infinity.

Key words. Large-time behavior, Hamilton-Jacobi equations, asymptotic solutions.

AMS subject classifications. 35B40, 70H20, 49L25

1. Introduction and main results.

We investigate the large-time behavior of solutions of the Hamilton-Jacobi equation

$$u_t(x,t) + H(x, Du(x,t)) = 0 \quad \text{in } \mathbf{R} \times (0, \infty), \tag{1}$$

with initial condition

$$u|_{t=0} = u_0 \quad \text{on } \mathbf{R}, \tag{2}$$

where $H \in C(\mathbf{R}\times\mathbf{R})$ and $u_0 \in C(\mathbf{R})$ are given functions, $u \in C(\mathbf{R}\times[0,\infty))$ represents the unknown function, and u_t and Du denote the partial derivatives $\partial u/\partial t$ and $\partial u/\partial x$, respectively.

In this note, as far as Hamilton-Jacobi equations are concerned, we mean by solution (resp., subsolution or supersolution) viscosity solution (resp., viscosity subsolution or viscosity supersolution). We refer to [3, 1, 7] for general overviews of viscosity solutions theory.

The large-time behavior of solutions of (1) or more generally

$$u_t(x,t) + H(x, Du(x,t)) = 0 \quad \text{in } \Omega \times (0, \infty), \tag{3}$$

where Ω is an n-dimensional manifold, has been studied by many authors since the works by Kruzkov [18], Lions [19], and Barles [2]. In the last decade it has received much attention under the influence of developments of weak KAM theory introduced by Fathi [9, 11]. We refer for related developments to Namah-Roquejoffre [23], Fathi [10], Roquejoffre [24], Barles-Souganidis [5], Davini-Siconolfi [8], Fujita-Ishii-Loreti [14], Barles-Roquejoffre [4], Ishii [17], Ichihara-Ishii [15, 16], and Mitake [21, 22].

In [10, 23, 24, 5, 8] they studied the asymptotic problem for (3) in the case where Ω is a compact manifold or simply an n-dimensional flat torus. The results obtained there are fairly general and one of them states that if $H(x,p)$ is coercive and strictly

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convex in p, then the solution u of (3) behaves as an asymptotic solution for large t, that is, there is a solution $(c, v) \in \mathbf{R} \times C(\Omega)$ of the additive eigenvalue problem for H

$$H(x, Dv(x)) = c \quad \text{in } \Omega, \tag{4}$$

such that

$$\lim_{t\to\infty} (u(x,t) - (v(x) - ct)) = 0 \quad \text{uniformly for } x \in \Omega. \tag{5}$$

Here and henceforth, for a solution (c, v) of (4), we call the function $v(x) - ct$ an *asymptotic solution* of (3). The strict convexity requirement for H in the above result can be replaced by a condition which is much weaker than the usual strict convexity, for which we refer to [5] (see also [15]). Moreover, as Barles-Souganidis [5] pointed out, the convexity of $H(x, p)$ in p is not enough to guarantee the convergence (5).

If (c, v) is a solution of (4), then we call c and v an (additive) eigenvalue and (additive) eigenfunction for H, respectively.

In the case where $\Omega = \mathbf{R}^n$, there are a few results (e.g., [6, 14, 4, 17, 15, 16]) on the large-time asymptotic behavior of solutions of (3), but the situation is not so clear compared to the case where Ω is compact.

We use the notation: $H[u]$ or $H[u](x)$ for $H(x, Du(x))$ in what follows. For instance, "$H[u] \le 0$ in Ω" means that u is a subsolution of $H(x, Du(x)) = 0$ in Ω. We denote by $\mathcal{S}^-_H(\Omega)$ (resp., $\mathcal{S}^+_H(\Omega)$ or $\mathcal{S}_H(\Omega)$) the set of all subsolutions (resp., supersolutions and solutions) u of $H[u] = 0$ in Ω. We write $\mathcal{S}^-_H$ (resp., $\mathcal{S}^+_H$ or $\mathcal{S}_H$) for $\mathcal{S}^-_H(\Omega)$ (resp., $\mathcal{S}^+_H(\Omega)$ or $\mathcal{S}_H(\Omega)$) when there is no confusion.

In this note we restrict ourselves to the case where $\Omega = \mathbf{R}$ and give an overview on the large-time asymptotic behavior of solutions of (3).

We will always assume the following assumptions (A1)–(A6).

(A1) $H \in C(\mathbf{R}^2)$.

(A2) H is locally coercive in the sense that

$$\lim_{r\to\infty} \inf\{H(x,p) \mid (x,p) \in [-R, R] \times \mathbf{R},\ |p| \ge r\} = \infty \quad \text{for all } R > 0.$$

(A3) $H(x, \cdot)$ is convex on $\mathbf{R}$ for every $x \in \mathbf{R}$.

(A4) $\mathcal{S}^-_H(\mathbf{R}) \ne \emptyset$.

(A5) For any $\phi \in \mathcal{S}_H(\mathbf{R})$ there exist a function $\psi \in C(\mathbf{R})$ and a constant $C > 0$ such that $\psi \in \mathcal{S}^-_{H-C}(\mathbf{R})$ and $\lim_{|x|\to\infty} (\phi - \psi)(x) = \infty$.

(A6) $u_0 \in C(\mathbf{R})$.

Our main theorem (Theorem 3 below) states that, under (A1)–(A6) together with certain additional assumptions, the convergence (5) holds with $c = 0$ on compact sets. Note that if u is a solution of (1) and c is a given constant, then the function $w(x,t) = u(x,t) + ct$ satisfies $w_t + H[w] - c = 0$ in $\mathbf{R} \times (0, \infty)$. Thus, through this simple change of unknown functions, our main theorem applies to the general situation where c in (5) may not be zero.

We denote by $C^{0+1}(X)$ the space of real-valued locally Lipschitz continuous functions on metric space X. If a given function $H \in C(\mathbf{R}^2)$ satisfies (A1)–(A3) and furthermore the condition that there exist a function $\phi_0 \in C^{0+1}(\mathbf{R})$ and three (real) constants $c < B$ and $\rho > 0$ such that

$$\begin{cases} H(x, D\phi_0(x)) \le c \quad \text{a.e. } x \in \mathbf{R}, \\ H(x,p) \le c \implies H(x,p+q) \le B \ \text{ for all } q \in [-\rho,\, \rho], \end{cases}$$

then (A1)–(A5) are satisfied with $H - c$ in replace of H. Indeed, it is clear that (A1)–(A3) hold with $H - c$ in place of H and that $\phi_0 \in \mathcal{S}^-_{H-c}(\mathbf{R})$ and hence (A4) holds with $H - c$ in place of H. (Note here by the convexity of $H(x,p)$ in p that the above condition on ϕ_0 is equivalent to saying that $\phi_0 \in \mathcal{S}^-_H(\mathbf{R})$.) We define the function $g \in C(\mathbf{R})$ by $g(x) = \rho|x|$ and, for any $\phi \in \mathcal{S}^-_{H-c}(\mathbf{R})$, we set $\psi := \phi - g$. Then we have $\psi \in \mathcal{S}^-_{H-B}(\mathbf{R})$ and $\lim_{|x|\to\infty}(\phi - \psi)(x) = \infty$. That is, (A5) holds with $H - c$ in place of H.

Another remark here is that we have $\min_{p\in\mathbf{R}} H(x,p) \le 0$ by (A4), which reads

$$L(x, 0) \ge 0 \quad \text{for all } x \in \mathbf{R},$$

where L denotes the Lagrangian of the Hamiltonian H, i.e., L is the function defined by $L(x,\xi) = \sup_{p\in\mathbf{R}}(\xi p - H(x,p))$.

We define the function $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$d(x,y) = \sup\{w(x) - w(y) \mid w \in \mathcal{S}^-_H(\mathbf{R})\} \quad \text{for } (x,y) \in \mathbf{R} \times \mathbf{R}.$$

It is well-known (see, for instance, [12, 13, 17]) that $d(x,x) = 0$ for all $x \in \mathbf{R}$, $d \in C^{0+1}(\mathbf{R}^2)$, $d(\cdot,y) \in \mathcal{S}^-_H(\mathbf{R}) \cap \mathcal{S}_H(\mathbf{R} \setminus \{y\})$ for all $y \in \mathbf{R}$, and

$$d(x,y) = \inf\Big\{\int_0^t L(\gamma(s), \dot\gamma(s))ds \;\Big|\; t > 0,\ \gamma \in \mathrm{AC}([0,\, t]),\ \gamma(t) = x,\ \gamma(0) = y\Big\}.$$

We define the (projected) Aubry set $\mathcal{A}_H$ for H as the set of those points $y \in \mathbf{R}$ for which $d(\cdot,y) \in \mathcal{S}_H(\mathbf{R})$. See [12, 13, 17] for some properties of $\mathcal{A}_H$. The function $d(\cdot,y)$ can be regarded, in terms of optimal control, as the value function of the optimal hitting problem having y and L as its target point and running cost, respectively.

As a reflection of our one-dimensional domain $\mathbf{R}$, we have:

PROPOSITION 1. (a) *If $x \le y \le z$, then $d(x,z) = d(x,y) + d(y,z)$.* (b) *If $x \ge y \ge z$, then $d(x,z) = d(x,y) + d(y,z)$.*

We postpone the proof of the above proposition till the next section.

We observe that if $x \le 0 < y$, then $d(x,y) - d(0,y) = d(x,0) + d(0,y) - d(0,y) = d(x,0)$ and if $0 < x < y$, then $d(x,y) - d(0,y) = d(x,y) - d(0,x) - d(x,y) = -d(0,x)$, and define $d_+ \in C^{0+1}(\mathbf{R})$ by

$$d_+(x) = \lim_{y\to\infty}(d(x,y) - d(0,y)) \equiv \begin{cases} d(x,0) & \text{for } x \le 0, \\ -d(0,x) & \text{for } x > 0. \end{cases}$$

Also, we observe that if $y < x \le 0$, then $d(x,y) - d(0,y) = d(x,y) - d(0,x) - d(x,y) = -d(0,x)$ and if $y < 0 < x$, then $d(x,y) - d(0,y) = d(x,0) + d(0,y) - d(0,y) = d(x,0)$, and define $d_- \in C^{0+1}(\mathbf{R})$ by

$$d_-(x) = \lim_{y\to-\infty}(d(x,y) - d(0,y)) \equiv \begin{cases} -d(0,x) & \text{for } x \le 0, \\ d(x,0) & \text{for } x > 0. \end{cases}$$

It is easily seen (see also Proposition 7 (a) below) that $d_+, d_- \in \mathcal{S}_H(\mathbf{R})$.

We assume only (A6) on initial data u_0 and do not know any existence and uniqueness result concerning solutions u of (1)–(2) which applies in this generality. Our *choice of solution* of (1)–(2) here is the function u given by

$$u(x,t) = \inf\Big\{ \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \;\Big|\; \gamma \in \mathrm{AC}([0,\,t]),\ \gamma(t) = x \Big\}. \tag{6}$$

We understand that formula (6) for $t = 0$ means that $u(x,0) = u_0(x)$. Note that $L(x,\xi)$ may take the value $+\infty$ at some points (x,ξ) and that $L(x,\xi) \geq -H(x,0) \geq -\sup_{|z|\leq R} H(z,0) > -\infty$ for all $R > 0$ and $(x,\xi) \in [-R,\,R] \times \mathbf{R}$. These observations clearly give the meaning of the integral $\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s$ as a real number or $+\infty$. Note that it may happen that $u(x,t) = -\infty$ for some points $(x,t) \in \mathbf{R}\times(0,\infty)$. Noting that $L(x,0) = -\min_{p\in\mathbf{R}} H(x,p) < \infty$ for all $x \in \mathbf{R}$, we see that $u(x,t) \leq L(x,0)t + u_0(x) < \infty$ for all $(x,t) \in \mathbf{R}\times[0,\infty)$. Hence we have $-\infty \leq u(x,t) < \infty$ for all $(x,t) \in \mathbf{R}\times[0,\infty)$. Also we remark (see, e.g., [17, Theorems A.1, A.2]) that if $u \in C(U)$ for some open set $U \subset \mathbf{R}\times(0,\infty)$, then u is a viscosity solution of (1) in U.

We introduce functions u_∞, u_0^- on $\mathbf{R}$ as

$$u_0^-(x) = \sup\{v(x) \mid v \in \mathcal{S}_H^-,\ v \leq u_0 \ \text{ in } \mathbf{R}\},$$
$$u_\infty(x) = \inf\{v(x) \mid v \in \mathcal{S}_H,\ v \geq u_0^- \ \text{ in } \mathbf{R}\}.$$

Note that the set $\{v \in \mathcal{S}_H^- \mid v \leq u_0 \ \text{ in } \mathbf{R}\}$ may be empty, in which case $u_0^-(x) \equiv -\infty$. Otherwise, $u_0^- \in \mathcal{S}_H^-(\mathbf{R})$, and $u_0^- \in C^{0+1}(\mathbf{R})$ because of (A2). Similarly, it may happen that $u_\infty(x) \equiv +\infty$. Otherwise, we have $u_\infty \in \mathcal{S}_H(\mathbf{R})$ and $u_\infty \in C^{0+1}(\mathbf{R})$.

Proposition 2. *Let u be the function given by* (6). (a) *If $u_0^-(x) \equiv -\infty$, then $\liminf_{t\to\infty} u(x,t) = -\infty$ for all $x \in \mathbf{R}$.* (b) *If $u_0^-(x) > -\infty$ and $u_\infty(x) = +\infty$ for all $x \in \mathbf{R}$, then $\lim_{t\to\infty} u(x,t) = +\infty$ for all $x \in \mathbf{R}$.*

We are now ready to state our main result of this note.

Theorem 3. *Assume that $u_0^-(x) > -\infty$ and $u_\infty(x) < \infty$ for all $x \in \mathbf{R}$. Let u be the solution of* (1)–(2) *given by* (6). *Then we have*

$$u(x,t) \to u_\infty(x) \quad \textit{uniformly on bounded intervals of } \mathbf{R} \quad \textit{as } t \to \infty, \tag{7}$$

except the following two cases (a) *and* (b).

$$\text{(a)} \qquad \begin{cases} \sup \mathcal{A}_H < \infty, \\ u_\infty(x) = d_+(x) + c_+ \quad \textit{for all } x > R \quad \textit{and} \quad \textit{some } c_+ \in \mathbf{R},\ R > 0, \\ \liminf\limits_{x\to\infty} (u_0 - u_0^-)(x) = 0 < \limsup\limits_{x\to\infty} (u_0 - u_0^-)(x). \end{cases}$$

$$\text{(b)} \qquad \begin{cases} \inf \mathcal{A}_H > -\infty, \\ u_\infty(x) = d_-(x) + c_- \quad \textit{for all } x < -R \quad \textit{and} \quad \textit{some } c_- \in \mathbf{R},\ R > 0, \\ \liminf\limits_{x\to-\infty} (u_0 - u_0^-)(x) = 0 < \limsup\limits_{x\to-\infty} (u_0 - u_0^-)(x) > 0. \end{cases}$$

The rest of this note is organized as follows. In Section 2 we give some preliminary observations which are needed in our proof of Theorem 3. Section 3 is devoted to the proof of Theorem 3. In Section 4 we discuss two examples and classical convergence results as well as a new twist of "strict convexity" hypothesis on H in connection with Proposition 2 and Theorem 3.

2. Preliminaries. In this section we give some observations on $d_\pm$, $\mathcal{S}_H$, $\mathcal{A}_H$, u_0^-, u_∞, and extremal curves as well as the proof of Propositions 1 and 2. We use the notation: $L[\gamma] \equiv L[\gamma](t)$ for $L(\gamma(t), \dot\gamma(t))$.

Proof of Proposition 1. We prove only assertion (a). Assertion (b) can be proved in a similar way. Let $x \le y \le z$. We know that $d(x,z) \le d(x,y) + d(y,z)$. Fix an $\varepsilon > 0$ and choose a curve $\gamma \in \mathrm{AC}([0,\, t])$, with $t > 0$, so that $\gamma(t) = x$, $\gamma(0) = z$, and

$$d(x,z) + \varepsilon > \int_0^t L[\gamma](s)\, \mathrm{d}s.$$

Choose a $\tau \in [0,\, t]$ so that $\gamma(\tau) = y$, and observe that

$$d(x,z) + \varepsilon > \int_\tau^t L[\gamma]\, \mathrm{d}s + \int_0^\tau L[\gamma]\, \mathrm{d}s \ge d(x,y) + d(y,z).$$

Hence we get $d(x,z) \ge d(x,y) + d(y,z)$, which proves that $d(x,z) = d(z,y) + d(y,z)$. □

We need the following lemmas for the proof of Proposition 2.

LEMMA 4. *There exists a constant $C_R > 0$ for each $R > 0$ and a curve $\eta \in$ $\mathrm{AC}([0,\, T])$ for each $x, y \in [-R,\, R]$ and $T > C_R|x-y|$ such that $\eta(0) = x$, $\eta(T) = y$, and*

$$\int_0^T L(\eta(t), \dot\eta(t))\, \mathrm{d}t \le C_R T.$$

Proof. Fix $R > 0$ and choose constants $\delta > 0$ and $M > 0$ (see for instance [17, Proposition 2.1]), depending on R, such that $L(x,\xi) \le M$ for all $(x,\xi) \in [-R,\, R] \times [-\delta,\, \delta]$. Fix any $x, y \in [-R,\, R]$ and $T > 0$. We define $\eta \in \mathrm{AC}([0,\, T])$ by setting $\eta(t) = x + \frac{t}{T}(y-x)$ for $t \in [0,\, T]$. We observe that $\eta(0) = x$, $\eta(T) = y$, $\eta(t) \in [-R,\, R]$ and $\dot\eta(t) = (y-x)/T$ for all $t \in [0,\, T]$. Hence, if $T > |y-x|/\delta$, then we get $|\dot\gamma(t)| < \delta$ for all $t \in [0,\, T]$ and therefore

$$\int_0^T L(\eta(t), \dot\eta(t))\, \mathrm{d}t = \int_0^T L\Big(\eta(t), \frac{y-x}{T}\Big)\, \mathrm{d}t \le MT.$$

Thus the curve η has the required properties with $C_R = \max\{M,\, 1/\delta\}$. □

LEMMA 5. *Let $U \subset \mathbf{R}$ be an open interval and $v \in \mathrm{USC}(U \times (0,\infty))$ a subsolution of (1) in $U \times (0,\infty)$. Assume that there exists a constant $C_0 > 0$ such that $-C_0 \le v(x,t) \le C_0(1+t)$ for all $(x,t) \in U \times (0,\infty)$. Define $w \in \mathrm{USC}(U)$ by $w(x) = \inf_{t>0} v(x,t)$. Then $w \in \mathcal{S}_H^-(U)$.*

An observation similar to the above lemma can be found in [15, Lemma 4.1].

Proof. We may assume that $v \in \mathrm{USC}(U \times [0,\infty))$ by setting $v(x,0) = \lim_{r\to+0} \sup\{v(y,s) \mid (y,s) \in U \times (0,\infty),\, |y-x| + s < r\}$. Let $\varepsilon > 0$, and consider the sup-convolution v^ε of v defined by

$$v^\varepsilon(x,t) = \sup_{s \ge 0} \left(v(x,s) - \frac{(t-s)^2}{2\varepsilon} \right).$$

Observe that $v^\varepsilon(x,t) \ge v(x,t) \ge -C_0$ for all $(x,t) \in U \times (0,\infty)$.

Fix $(x,t) \in U \times (0,\infty)$. It is clear that there exists an $s \ge 0$ such that $v^\varepsilon(x,t) = v(x,s) - (t-s)^2/(2\varepsilon)$. Fix such an $s \ge 0$, and observe that

$$-C_0 \le v(x,t) \le v^\varepsilon(x,t) = v(x,s) - \frac{(t-s)^2}{2\varepsilon} \le C_0(1+s) - \frac{(t-s)^2}{2\varepsilon}$$
$$\le C_0(1+t+|t-s|) - \frac{(t-s)^2}{2\varepsilon} \le -\frac{(t-s)^2}{4\varepsilon} + C_0(1+t) + \varepsilon C_0^2,$$

and hence

$$|s-t| \le 2\{\varepsilon(2C_0(1+t)+\varepsilon C_0^2)\}^{1/2}.$$

From this last estimate, we see that for each $\tau > 0$ there exists a $\delta > 0$ such that if $t > \tau$ and $0 < \varepsilon < \delta$, then $s > 0$. Fix any $\tau > 0$ and choose such a constant $\delta > 0$. It is now a standard observation that if $\varepsilon \in (0,\, \delta)$, then v^ε is a subsolution of (1) in $U \times (\tau,\infty)$ and $v^\varepsilon \in C^{0+1}(U \times (\tau,\, T))$ for all $T > \tau$. Fix any $\sigma > 0$ and define $w^{\varepsilon,\sigma} \in C(U \times (0,\, \infty))$ by $w^{\varepsilon,\sigma}(x,t) = \inf_{0<s<\sigma} v^\varepsilon(x,t+s)$.

Let $\varepsilon \in (0,\delta)$, and observe that $w^{\varepsilon,\sigma} \in C^{0+1}(U \times (\tau,\, T))$ for all $T > \tau$ and by the convexity of $H(x,p)$ in p that $w^{\varepsilon,\sigma}$ is a subsolution of (1) in $U \times (\tau,\, \infty)$. Note that $w^{\varepsilon,\sigma}(x,t)$ is non-increasing as a function of σ and therefore that if we set $w^\varepsilon(x,t) := \inf_{s>0} v^\varepsilon(x,t+s)$ for $(x,t) \in U \times (0,\, \infty)$, then for any $(x,t) \in U \times (0,\, \infty)$,

$$w^\varepsilon(x,t) = \lim_{r\to+0} \sup\{w^{\varepsilon,\sigma}(y,s) \mid (y,s) \in U \times (0,\, \infty),\, |y-x|+|s-t| < r,\, \sigma > 1/r\}.$$

We now see by the stability of the viscosity property under half relaxed limits that $w^\varepsilon \in \mathrm{USC}(U \times (0,\, \infty))$ is a subsolution of (1) in $U \times (\tau,\, \infty)$. By the definition of w^ε, it is clear that for any $x \in U$, the function $w^\varepsilon(x,t)$ is non-decreasing in $t \in (0,\infty)$, from which we deduce that $w^\varepsilon(\cdot,t) \in \mathcal{S}_H^-(U)$ for all $t > \tau$. In particular, we see that the family $\{w^\varepsilon(\cdot,t) \mid t > \tau\} \subset C^{0+1}(U)$ is locally equi-Lipschitz continuous on U.

Note that $w^\varepsilon(x,t)$ is non-decreasing as a function of ε, that $w^\varepsilon(x,t) \ge \inf_{s>0} v(x,t+s)$ for all $(x,t) \in U \times (0,\, \infty)$ and $\varepsilon > 0$, and that $\inf_{\varepsilon>0} w^\varepsilon(x,t) = \inf\{v^\varepsilon(x,t+s) \mid s > 0,\, \varepsilon > 0\}$ for all $(x,t) \in U \times (0,\infty)$. It is now easy to see by using the convexity of H that if we set $z(x,t) := \inf_{\varepsilon>0} w^\varepsilon(x,t)$, then $z(x,t) = \inf_{0<\varepsilon<\delta} w^\varepsilon(x,t)$ for all $(x,t) \in U \times (0,\infty)$ and $z(\cdot,t) \in \mathcal{S}_H^-(U)$ for all $t > \tau$. Since $\tau > 0$ is arbitrary, we see that $z(\cdot,t) \in \mathcal{S}_H^-(U)$ for all $t > 0$. Setting $w(x) := \inf_{t>0} z(x,t)$ for $x \in U$, we see that $w(x) = \inf_{t>0} v(x,t)$ for all $x \in U$ and moreover that $w \in \mathcal{S}_H^-(U)$. ∎

LEMMA 6. *Let* $\phi \in \mathcal{S}_H^-$ *and* $\gamma \in \mathrm{AC}([0,\, t])$. *Then*

$$\phi(\gamma(t)) - \phi(\gamma(0)) \le \int_0^t L[\gamma]\, \mathrm{d}s.$$

For a proof of the above lemma we refer, for instance, to [17, Proposition 2.5].

Proof of Proposition 2. We begin with (a). Assume that $u_0^-(x) \equiv -\infty$. We suppose that there exists an $x_0 \in \mathbf{R}$ such that $\liminf_{t\to\infty} u(x_0,t) > -\infty$, and will get a contradiction. By translation, we may assume that $x_0 = 0$.

We show first that for each $R > 0$ there exists a constant $M_R > 0$ such that $u(x,t) \ge -M_R$ for all $(x,t) \in [-R,R] \times [0,\infty)$. For this we fix $R > 0$ and choose constants $\tau > 0$ and $C_0 > 0$ so that $u(0,t) \ge -C_0$ for all $t \ge \tau$. Let $C_R > 0$ be the constant from Lemma 4 and fix any $(x,\, t) \in [-R,\, R] \times [0,\, \infty)$. By Lemma 4, we may choose a curve $\eta \in \mathrm{AC}([0,\, T_R])$, with $T_R := RC_R + \tau$, so that $\eta(0) = x$, $\eta(T_R) = 0$, and

$$\int_0^{T_R} L[\eta]\,\mathrm{d}s \le C_R T_R.$$

Fix any $\gamma \in \mathrm{AC}([0,\, t])$ so that $\gamma(t) = x$, and define $\zeta \in \mathrm{AC}([0,\, t + T_R])$ by

$$\zeta(s) = \begin{cases} \gamma(s) & \text{for } 0 \le s \le t, \\ \eta(s-t) & \text{for } t \le s \le t + T_R. \end{cases}$$

We observe that

$$\begin{aligned} -C_0 \le u(0, t+t_R) &\le \int_0^t L[\gamma]\,\mathrm{d}s + \int_0^{t_R} L[\eta]\,\mathrm{d}s + u_0(\zeta(0)) \\ &\le C_R T_R + \int_0^t L[\gamma]\,\mathrm{d}s + u_0(\gamma(0)), \end{aligned}$$

from which we deduce that $u(x,t) \ge -C_0 - C_R T_R$. Thus we conclude that $u(x,t) \ge -M_R$ for all $(x,t) \in [-R,\, R] \times [0,\, \infty)$, where $M_R := C_0 + C_R T_R$.

Next we observe from (6) that $u(x,t) \le L(x,0)t + u_0(x)$ for all $(x,t) \in \mathbf{R}\times[0,\infty)$. Since $L(x,0) = -\min_{p\in\mathbf{R}} H(x,p)$ is a continuous function of x because of (A1) and (A2), we see that u is locally bounded on $\mathbf{R}\times[0,\infty)$ and hence by [17, Theorem A.1] for instance that u^* is a viscosity subsolution of (1), where u^* is the upper semicontinuous envelope of u, i.e., $u^*(x,t) := \lim_{r\to+0} \sup\{u(y,s) \mid (y,s) \in \mathbf{R}\times[0,\infty),\ |y-x|+|s-t| < r\}$. Set $w(x) = \inf_{t>0} u^*(x,t)$ for $x \in \mathbf{R}$. According to Lemma 5, we have $w \in \mathcal{S}_H^-(\mathbf{R})$. Also, since $u^*(x,t) \le L(x,0)t + u_0(x)$ for all $(x,t) \in \mathbf{R}\times(0,\,\infty)$, we have $w(x) \le u_0(x)$ for all $x \in \mathbf{R}$. Now we see that $u_0^-(x) \ge w(x) > -\infty$ for all $x \in \mathbf{R}$. This is a contradiction, which proves (a).

We now turn to (b). Assume that $u_0^-(x) > -\infty$ and $u_\infty(x) = +\infty$ for all $x \in \mathbf{R}$. We suppose that $\liminf_{t\to\infty} u(x_0,t) < \infty$ for some $x_0 \in \mathbf{R}$, and will obtain a contradiction.

Define the function u^- on $\mathbf{R}\times[0,\infty)$ by

$$u^-(x,t) = \inf\Big\{\int_0^t L[\gamma](s)\,\mathrm{d}s + u_0^-(\gamma(0)) \;\Big|\; \gamma \in \mathrm{AC}([0,\, t]),\ \gamma(t) = x\Big\}. \tag{8}$$

Since $u_0^- \le u_0$ in $\mathbf{R}$, we have $u^-(x,t) \le u(x,t)$ for all $(x,t) \in \mathbf{R}\times[0,\infty)$. Note that the function u^- satisfies the dynamic programming principle

$$u^-(x,t+s) = \inf\Big\{\int_0^t L[\gamma](r)\,\mathrm{d}r + u^-(\gamma(0),s) \;\Big|\; \gamma \in \mathrm{AC}([0,\, t]),\ \gamma(t) = x\Big\}.$$

The term inside the above infimum sign can be $\infty - \infty$, which we agree to mean $+\infty$. Since $u_0^- \in \mathcal{S}_H^-$, by Lemma 6, we have for all $\gamma \in \mathrm{AC}([0,t])$,

$$u_0^-(\gamma(t)) - u_0^-(\gamma(0)) \le \int_0^t L[\gamma](s)\,\mathrm{d}s.$$

Consequently, we get

$$u_0^-(x) \le u^-(x,t) \qquad \text{for all } (x,t) \in \mathbf{R}\times[0,\infty).$$

This together the dynamic programming principle yields

$$u^-(x,t+s) \geq \inf\Big\{\int_0^t L[\gamma](r)\,\mathrm{d}r + u_0^-(\gamma(0)) \;\Big|\; \gamma \in \mathrm{AC}([0,\,t]),\, \gamma(t)=x\Big\} = u^-(x,t)$$

for all $x \in \mathbf{R}$ and $t, s \in [0, \infty)$. Thus we see that the function $u^-(x,t)$ is non-decreasing in t for any $x \in \mathbf{R}$.

We may assume without any loss of generality that $x_0 = 0$. We choose a constant $C_1 > 0$ so that $\liminf_{t\to\infty} u(0,t) \leq C_1$. By the monotonicity of $u^-(0,t)$, we have

$$u^-(0,t) \leq C_1 \quad \text{for all } t \geq 0.$$

Fix any $R > 0$. By the dynamic programming principle and Lemma 4 with $T = C_R R + 1$, we get for all $(x,t) \in [-R,\, R] \times [0,\, \infty)$,

$$u^-(x,t+T) \leq C_R T + u^-(0,t) \leq C_R T + C_1,$$

where $C_R > 0$ is the constant from Lemma 4. Hence we get

$$u^-(x,t) \leq K_R \quad \text{for all } (x,t) \in [-R,\, R] \times [0,\infty),$$

where $K_R := C_R T + C_1$.

Since $u_0^- \in C^{0+1}(\mathbf{R})$, we have $u^- \in C^{0+1}(\mathbf{R} \times [0,\, \infty))$. Indeed, we fix $R > 0$, $x, y \in [-R,\, R]$ with $x \neq y$, and $t \geq 0$, and observe by using the dynamic programming principle and Lemma 4, with $T > C_R|x-y|$, that for all $x, y \in [-R,\, R]$ and $t \geq 0$,

$$u^-(y,t) \leq u^-(y,t+T) \leq u^-(x,t) + C_R T. \tag{9}$$

Thus we have

$$|u^-(y,t) - u^-(x,t)| \leq C_R^2 |x-y| \quad \text{for all } x, y \in [-R,\, R] \text{ and } t \geq 0.$$

On the other hand, using the dynamic programming principle and Lemma 4, we have for $x \in [-R,\, R]$ and $t, s \in [0,\, \infty)$,

$$u^-(x,t) \leq u^-(x,t+s) \leq u^-(x,t) + C_R s,$$

and hence $|u^-(x,t) - u^-(x,s)| \leq C_R|t-s|$ for all $x \in [-R,\, R]$ and $t, s \in [0,\, \infty)$. Thus we conclude that $u^- \in C^{0+1}(\mathbf{R} \times [0,\, \infty))$. It is now standard to see that if we set $w(x) = \lim_{t\to\infty} u^-(x,t)$, then $w \in C^{0+1}(\mathbf{R})$ and $w \in \mathcal{S}_H(\mathbf{R})$. The monotonicity of the function $u^-(x,t)$ in t guarantees that $u_0^- \leq w$ in $\mathbf{R}$. Therefore we see that $u_\infty(x) \leq w(x) < \infty$ for all $x \in \mathbf{R}$, which is a contradiction. ∎

PROPOSITION 7. (a) $d_\pm \in \mathcal{S}_H(\mathbf{R})$. (b) *If* $x \leq y$, *then* $d(x,y) = d_+(x) - d_+(y)$. (c) *If* $x \geq y$, *then* $d(x,y) = d_-(x) - d_-(y)$. (d) *The function* $d_+ - d_-$ *is non-increasing on* $\mathbf{R}$.

Proof. (a) Since $d(\cdot, y) \in \mathcal{S}_H(\mathbf{R} \setminus \{y\})$ for any $y \in \mathbf{R}$, by the stability of the viscosity property, we see that $d_\pm \in \mathcal{S}_H(\mathbf{R})$. (b) Let $x \leq y < z$, and observe that $d(x,z) - d(0,z) = d(x,y) + d(y,z) - d(0,z)$. Hence, sending $z \to \infty$, we get $d_+(x) = d(x,y) + d_+(y)$, that is, if $x \leq y$, then $d(x,y) = d_+(x) - d_+(y)$. (c) An argument parallel to (b) readily yields $d(x,y) = d_-(x) - d_-(y)$ for $x \geq y$. (d) Let $x < y$ and observe that $d_-(x) - d_-(y) \leq d(x,y) = d_+(x) - d_+(y)$, from which we get $(d_+ - d_-)(x) \geq (d_+ - d_-)(y)$. □

PROPOSITION 8. *We have*

$$u_0^-(x) = \inf\{u_0(y) + d(x,y) \mid y \in \mathbf{R}\} \quad \textit{for all } x \in \mathbf{R}.$$

Proof. We denote by w the function defined by the right hand side of the above equality. Let $v \in \mathcal{S}_H^-(\mathbf{R})$ satisfy $v \leq u_0$ in $\mathbf{R}$. Then we have $v(x) \leq v(y) + d(x,y) \leq u_0(y)+d(x,y)$ for all $x \in \mathbf{R}$. Hence we get $v(x) \leq w(x)$ and consequently $u_0^-(x) \leq w(x)$ for all $x \in \mathbf{R}$. On the other hand, if $w(x_0) > -\infty$ for some $x_0 \in \mathbf{R}$, then we see that $w \in C^{0+1}(\mathbf{R})$ and $w \in \mathcal{S}_H^-(\mathbf{R})$. It is clear that $w(x) \leq u_0(x)$ for all $x \in \mathbf{R}$. Therefore we have $w(x) \leq u_0^-(x)$ for all $x \in \mathbf{R}$. Thus we have $w(x) = u_0^-(x)$ for all $x \in \mathbf{R}$. □

Let $I \subset \mathbf{R}$ be an interval and $\phi \in \mathcal{S}_H^-$. We call a function (curve) $\gamma \in C(I)$ an extremal curve on I for ϕ if for any $a, b \in I$, with $a < b$, we have

$$\gamma \in \mathrm{AC}([a,\, b]) \quad \text{and} \quad \phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b L[\gamma](s)\,\mathrm{d}s. \tag{10}$$

We denote by $\mathcal{E}(I,\phi)$ the set of all extremal curves on I for ϕ. When $0 \in I$, for $y \in \mathbf{R}$, we denote by $\mathcal{E}(I,\phi,y)$ the set of those $\gamma \in \mathcal{E}(I,\phi)$ which satisfy $\gamma(0) = y$.

PROPOSITION 9. *Let $\phi \in \mathcal{S}_H$ and $y \in \mathbf{R}$. Then $\mathcal{E}((-\infty,\, 0], \phi, y) \neq \emptyset$.*

We can adapt the proof of [17, Corollary 6.2] to the above lemma. We will not give the details of the proof here, and instead give a key observation:

LEMMA 10. *Let $\phi \in \mathcal{S}_H$ and $t > 0$. Then, for any $x \in \mathbf{R}$,*

$$\phi(x) = \inf\Big\{\int_0^t L[\gamma]\,\mathrm{d}s + \phi(\gamma(0)) \;\Big|\; \gamma \in \mathrm{AC}([0,\, t]),\, \gamma(t) = x\Big\}. \tag{11}$$

Proof. Thanks to (A5), we may choose a function $\psi \in C^{0+1}(\mathbf{R})$ and a constant $C > 0$ so that $\psi \in \mathcal{S}_{H-C}^-$ and $\lim_{|x|\to\infty}(\psi - \phi)(x) = -\infty$. Then, we apply [17, Theorem 1.1], with ϕ_0 and ϕ_1 replaced by ϕ and ψ, respectively, to conclude that the solution $u(x,t) := \phi(x)$ of (1)–(2) can be represented as

$$u(x,t) = \inf\Big\{\int_0^t L[\gamma]\,\mathrm{d}s + \phi(\gamma(0)) \;\Big|\; \gamma \in \mathrm{AC}([0,\, t]),\, \gamma(t) = x\Big\},$$

which shows that (11) holds true. (In [17, Theorem 1.1], the Hamiltonian $H(x,p)$ is assumed to be strictly convex in p, but this assumption is actually superfluous and can be replaced by our convexity assumption (A3).) □

PROPOSITION 11. *$\mathcal{A}_H = \mathcal{E}_H$, where $\mathcal{E}_H$ denotes the set of equilibria, that is, $\mathcal{E}_H = \{x \in \mathbf{R} \mid L(x,0) = 0\}$.*

LEMMA 12. *Let $y \in \mathbf{R}$ and $\delta > 0$. Then we have $y \in \mathcal{A}_H$ if and only if*

$$\inf\Big\{\int_0^t L[\gamma]\,\mathrm{d}s \;\Big|\; t \geq \delta,\, \gamma \in \mathrm{AC}([0,\, t]),\, \gamma(t) = \gamma(0) = y\Big\} = 0.$$

We refer to [17, Proposition A.3] (see also [12, 13]) for a proof of the above lemma.

Proof of Proposition 11. Let $z \in \mathcal{A}_H$, and we need to show that $L(z,0) \le 0$. Fix any $\varepsilon \in (0,1)$. Let $\delta > 0$ be a constant to be fixed later on. According to Lemma 12, for any $n \in \mathbf{N}$ there exists a $\gamma_n \in \mathrm{AC}([0,T_n])$, with $T_n \ge \delta$, such that $\gamma_n(0) = \gamma_n(T_n) = z$ and

$$\int_0^{T_n} L(\gamma_n, \dot\gamma_n)\,\mathrm{d}s < \frac{1}{n}.$$

We claim that we may assume by choosing $\delta > 0$ small enough that

$$\max_{0 \le s \le T_n} |\gamma_n(s) - z| \le \varepsilon.$$

To see this, we first consider the case where $\max_{0 \le s \le T_n}(\gamma_n(s) - z) > \varepsilon$. It is easily seen that there are $0 \le s_n < t_n \le \sigma_n < \tau_n \le T_n$ such that $\gamma_n(s_n) = \gamma_n(\tau_n) = z$, $\gamma_n(t_n) = \gamma_n(\sigma_n) = z + \varepsilon$, and $\gamma_n(s) \in (z,\, z+\varepsilon)$ for all $s \in (s_n,\, t_n) \cup (\sigma_n,\, \tau_n)$. Observe that

$$0 = d(z,\, z) \le \int_0^{s_n} L[\gamma_n]\,\mathrm{d}s.$$

Similarly we have

$$\int_{t_n}^{\sigma_n} L[\gamma_n]\,\mathrm{d}s \ge 0 \quad \text{and} \quad \int_{\tau_n}^{T_n} L[\gamma_n]\,\mathrm{d}s \ge 0.$$

Therefore we get

$$\frac{1}{n} > \int_0^{T_n} L[\gamma_n]\,\mathrm{d}s \ge \int_{s_n}^{t_n} L[\gamma_n]\,\mathrm{d}s + \int_{\sigma_n}^{\tau_n} L[\gamma_n]\,\mathrm{d}s.$$

We define $\tilde\gamma_n \in \mathrm{AC}([0,\, \widetilde{T}_n])$, with $\widetilde{T}_n := t_n - s_n + \tau_n - \sigma_n$, by setting $\tilde\gamma_n(s) = \gamma_n(s + s_n)$ for $s \in [0,\, t_n - s_n]$ and $\tilde\gamma_n(s) = \gamma_n(s + \sigma_n - t_n + s_n)$ for $s \in [t_n - s_n,\, \widetilde{T}_n]$, and note that

$$\max_{0 \le s \le \widetilde{T}_n} |\tilde\gamma_n(s) - z| = \varepsilon, \quad \tilde\gamma_n(t_n - s_n) = z + \varepsilon, \quad \text{and} \quad \int_0^{\widetilde{T}_n} L[\tilde\gamma_n]\,\mathrm{d}s < \frac{1}{n}.$$

By (A1), there exists a constant $C_\varepsilon > 0$ such that $\varepsilon L(x,\xi) \ge (|\xi| - C_\varepsilon)$ for all $(x,\xi) \in [z-1,\, z+1] \times \mathbf{R}$. We compute that

$$\begin{aligned}
2\varepsilon &= |\tilde\gamma_n(t_n - s_n) - \tilde\gamma_n(0)| + |\tilde\gamma_n(\widetilde{T}_n) - \tilde\gamma_n(t_n - s_n)| \\
&\le \int_0^{t_n - s_n} \left| \frac{\mathrm{d}\tilde\gamma_n(s)}{\mathrm{d}s} \right| \mathrm{d}s + \int_{t_n - s_n}^{\widetilde{T}_n} \left| \frac{\mathrm{d}\tilde\gamma_n(s)}{\mathrm{d}s} \right| \mathrm{d}s \\
&\le \int_0^{\widetilde{T}_n} (\varepsilon L[\tilde\gamma_n] + C_\varepsilon)\,\mathrm{d}s < \varepsilon + C_\varepsilon \widetilde{T}_n.
\end{aligned}$$

Hence we have $\widetilde{T}_n \ge \varepsilon / C_\varepsilon$. We now fix $\delta = \varepsilon / C_\varepsilon$ and observe that $\tilde\gamma_n(0) = \tilde\gamma(\widetilde{T}_n) = z$,

$$\int_0^{\widetilde{T}_n} L[\tilde\gamma_n]\,\mathrm{d}s < \frac{1}{n}, \quad \text{and} \quad \max_{0 \le s \le \widetilde{T}_n} |\tilde\gamma_n(s) - z| \le \varepsilon.$$

Similarly, if $\min_{0\le s\le T_n}(\gamma_n(s)-z) < -\varepsilon$, then we can build a $\tilde\gamma_n \in \mathrm{AC}([0,\, \widetilde{T}_n])$, with $\widetilde{T}_n \ge \delta$, so that $\tilde\gamma_n(0) = \tilde\gamma_n(\widetilde{T}_n) = z$,

$$\max_{0\le s\le \widetilde{T}_n} |\tilde\gamma_n(s) - z| \le \varepsilon, \quad \text{and} \quad \int_0^{\widetilde{T}_n} L[\tilde\gamma_n]\,\mathrm{d}s < \frac{1}{n}.$$

Thus we may assume by replacing γ_n if necessary that $\max_{0\le s\le T_n}|\gamma_n(s) - z| \le \varepsilon$.

Next, let $R > 0$ and set

$$L_R(x,\xi) = \max_{|p|\le R}(\xi p - H(x,p)).$$

Observe that L_R is continuous on $\mathbf{R}\times\mathbf{R}$, $L_R(x,\xi) \le L(x,\xi)$ for all (x,ξ), and $L_R(x,\xi) \to L(x,\xi)$ as $R\to\infty$ for all (x,ξ). Let ω_R be a modulus of the function H on $[z-1,\, z+1]\times[-R,\, R]$ and observe that for all $x, y \in [z-1,\, z+1]$ and $\xi\in\mathbf{R}$,

$$|L_R(x,\xi) - L_R(y,\xi)| \le \max_{|p|\le R}|H(x,p) - H(y,p)| \le \omega_R(|x-y|).$$

We compute that

$$\begin{aligned}
L_R(z,0) &= L_R\Big(z, \frac{1}{T_n}\int_0^{T_n}\dot\gamma_n(t)\,\mathrm{d}t\Big) \le \frac{1}{T_n}\int_0^{T_n} L_R(z,\dot\gamma_n(t))\,\mathrm{d}t \\
&\le \frac{1}{T_n}\int_0^{T_n} L_R(\gamma_n(t),\dot\gamma_n(t))\,\mathrm{d}t + \omega_R(\max_{0\le t\le T_n}|\gamma_n(t) - z|) \\
&\le \frac{1}{T_n}\int_0^{T_n} L(\gamma_n(t),\dot\gamma_n(t))\,\mathrm{d}t + \omega_R(\max_{0\le t\le T_n}|\gamma_n(t) - z|) \\
&< \frac{1}{nT_n} + \omega_R(\max_{0\le t\le T_n}|\gamma_n(t) - z|) \le \frac{1}{n\delta} + \omega_R(\varepsilon).
\end{aligned}$$

Sending $n\to\infty$ and then $\varepsilon\to+0$, we get $L_R(z,0)\le 0$, from which we conclude by sending $R\to\infty$ that $L(z,0)\le 0$. The proof is complete. ∎

3. Proof of Theorem 3. This section is devoted to the proof of Theorem 3. We assume all the hypotheses of Theorem 3 in what follows. Let u be the function on $\mathbf{R}\times[0,\,\infty)$ given by (6) and u^+ denote the function on $\mathbf{R}$ defined by

$$u^+(x) = \limsup_{t\to\infty} u(x,t).$$

LEMMA 13. *For all $x\in\mathbf{R}$ we have*

$$u^+(x) = \lim_{r\to+0}\sup\{u(y,s) \mid s > r^{-1},\ |y-x| < r\}, \tag{12}$$

$$u_\infty(x) \le \lim_{r\to+0}\inf\{u(y,s) \mid s > r^{-1},\ |y-x| < r\}. \tag{13}$$

Inequality (13) is a modification of (18) in [15, Lemma 4.1].

Proof. By Lemma 4 and the dynamic programming principle, we get

$$u(y,t+T) \le u(x,t) + C_R T \quad \text{for all } x,y\in[-R,\,R],\ t\ge 0 \text{ and } T > C_R|x-y|,$$

where $C_R > 0$ is a constant depending only on R, from which we easily obtain (12) for all $x \in \mathbf{R}$.

Let u^- be the function on $\mathbf{R} \times [0,\infty)$ defined by (8). As in the proof of Proposition 2, we have $u^- \in C^{0+1}(\mathbf{R} \times [0,\infty))$, $u^- \leq u$ in $\mathbf{R} \times [0,\infty)$, and $u_\infty(x) = \lim_{t\to\infty} u^-(x,t)$. Therefore we have

$$\begin{aligned} u_\infty(x) &= \lim_{r\to+0} \inf\{u^-(y,s) \mid s > r^{-1},\, |y-x| < r\} \\ &\leq \lim_{r\to+0} \inf\{u(y,s) \mid s > r^{-1},\, |y-x| < r\}, \end{aligned}$$

which completes the proof. ∎

In order to show that $u(x,t) \to u_\infty(x)$ uniformly on bounded intervals of $\mathbf{R}$, due to the above lemma, we only need to prove that $u^+(x) \leq u_\infty(x)$ for all $x \in \mathbf{R}$. We fix $y \in \mathbf{R}$ and will prove that $u_0^-(y) \leq u_\infty(y)$. By Proposition 9, we may choose a $\gamma \in \mathcal{E}((-\infty,0],\, u_\infty,\, y)$. We first divide our considerations into two cases.

Case 1: $\operatorname{dist}(\gamma((-\infty,0]), \mathcal{A}_H) = 0$ and Case 2: $\operatorname{dist}(\gamma(-\infty,0]), \mathcal{A}_H) > 0$, where we set $\operatorname{dist}(\gamma((-\infty,0]), \mathcal{A}_H) = \infty$ when $\mathcal{A}_H = \emptyset$. We first treat Case 1.

LEMMA 14. *In Case* 1, *we have* $u^+(y) \leq u_\infty(y)$.

Proof. Since $\gamma((-\infty,0])$ is an interval and $\mathcal{A}_H$ is a closed set (see. e.g., [12, 13, 17]), it is not hard to see that there exists a $z \in \mathcal{A}_H$ such that $\operatorname{dist}(\gamma((-\infty,0]), z) = 0$. Fix such a $z \in \mathcal{A}_H$ and set $R = |z|+1$. Let $C_R > 0$ be the constant from Lemma 4. Fix any $\varepsilon \in (0,1)$, and choose an $r > 0$ so that $|\gamma(-r)-z| < \varepsilon$ and $u_\infty(z) \leq u_\infty(\gamma(-r))+\varepsilon$. By Lemma 4, we may choose a curve $\eta \in \mathrm{AC}([0,\tau])$, with $\tau = C_R|z-\gamma(-r)|+\varepsilon$, so that $\eta(0) = z$, $\eta(\tau) = \gamma(-r)$, and

$$\int_0^\tau L[\eta]\,\mathrm{d}t \leq C_R\,\tau = C_R^2(|z-\gamma(-r)|+\varepsilon) \leq 2C_R^2\,\varepsilon.$$

In view of Proposition 8 and the variational representation for d, we have

$$u_0^-(z) = \inf\Big\{\int_0^t L[\zeta]\,\mathrm{d}s + u_0(\zeta(0)) \;\Big|\; t > 0,\ \zeta \in \mathrm{AC}([0,t]),\ \zeta(t) = z\Big\}.$$

Hence we may choose a curve $\zeta \in \mathrm{AC}([0,\sigma])$, with $\sigma > 0$, so that $\zeta(\sigma) = z$ and

$$u_0^-(z) + \varepsilon > \int_0^\sigma L[\zeta]\,\mathrm{d}s + u_0(\zeta(0)).$$

Let $t > r+\tau+\sigma$ and define the curve $\mu \in \mathrm{AC}([-t,0])$ as follows: we set $T = t-(r+\tau+\sigma)$ and

$$\mu(s) = \begin{cases} \gamma(s) & \text{for } s \in [-r,0], \\ \eta(s+r+\tau) & \text{for } s \in (-(r+\tau),-r], \\ z & \text{for } s \in (-(r+\tau+T),-(r+\tau)], \\ \zeta(s+t) & \text{for } s \in [-t,-t+\sigma] \equiv [-t,-(r+\tau+T)]. \end{cases}$$

We compute that

$$\begin{aligned} u(y,t) &\leq \int_{-t}^0 L[\mu]\,\mathrm{d}s + u_0(\mu(-t)) \\ &\leq \int_{-r}^0 L[\gamma]\,\mathrm{d}s + \int_0^\tau L[\eta]\,\mathrm{d}s + \int_0^T L(z,0)\,\mathrm{d}s + \int_0^\sigma L[\zeta]\,\mathrm{d}s + u_0(\zeta(0)) \\ &< u_\infty(y) - u_\infty(\gamma(-r)) + 2C_R^2\varepsilon + u_0^-(z) + \varepsilon \leq u_\infty(y) + 2(C_R^2+1)\varepsilon, \end{aligned}$$

where we have used the fact that $u_0^-(z) \le u_\infty(z) \le u_\infty(\gamma(-r)) + \varepsilon$, and conclude that $u^+(y) \le u_\infty(y)$. ∎

Now, we turn to Case 2 and begin with a few lemmas.

LEMMA 15. *Let* $c \in \mathbf{R}$. *Assume that* $d_+ + c \ge u_0^-$ *on* $\mathbf{R}$ *and* $\inf_{\mathbf{R}}(d_+ + c - u_0^-) = 0$. *Then* $\lim_{x\to\infty}(d_+(x) + c - u_0^-(x)) = 0$.

Proof. Suppose on the contrary that $\limsup_{x\to\infty}(d_+(x) + c - u_0^-(x)) > 0$ and choose a $\delta > 0$ and a sequence $x_n \to \infty$ such that $d_+(x_n) + c - u_0^-(x_n) \ge \delta$ for all $n \in \mathbf{N}$. We show that $d_+(x) + c - u_0^-(x) \ge \delta/2$ for all $x \in \mathbf{R}$, which is an obvious contradiction to the assumption that $\inf_{\mathbf{R}}(d_+ + c - u_0^-) = 0$.

Fix any $x \in \mathbf{R}$, and choose an n so that $x \le x_n$ and then a $y_n \in \mathbf{R}$ in view of Proposition 8 so that $u_0^-(x_n) + \delta/2 > u_0(y_n) + d(x_n, y_n)$. Noting that $d(x, x_n) = d_+(x) - d_+(x_n)$, we compute that

$$\begin{aligned} u_0^-(x) &\le u_0(y_n) + d(x, y_n) \le u_0(y_n) + d(x, x_n) + d(x_n, y_n) \\ &< u_0^-(x_n) + \frac{\delta}{2} + d(x, x_n) \le d_+(x_n) + c - \frac{\delta}{2} + d_+(x) - d_+(x_n) \\ &= d_+(x) + c - \frac{\delta}{2}, \end{aligned}$$

and conclude that $d_+(x) + c - u_0^-(x) \ge \delta/2$. ∎

LEMMA 16. *In Case* 2, *the set* $\gamma((-\infty, 0])$ *is unbounded.*

Proof. On the contrary we suppose that $\gamma((-\infty, 0])$ is bounded. We may choose a sequence $\{t_n\} \subset (-\infty, 0]$ so that $t_{n+1} \le t_n - 1$ for all $n \in \mathbf{N}$ and $\{\gamma(t_n)\}$ is convergent. Set $z := \lim_{n\to\infty} \gamma(t_n)$. Observe that as $n \to \infty$,

$$\int_{t_{n+1}}^{t_n} L(\gamma, \dot\gamma)\, \mathrm{d}t = u_\infty(\gamma(t_n)) - u_\infty(\gamma(t_{n+1})) \to 0.$$

Fix any $n \in \mathbf{N}$. By Lemma 4, there are curves $\eta_n \in \mathrm{AC}([0, \tau_n])$ and $\zeta_n \in \mathrm{AC}([0, \sigma_n])$, with $\tau_n > 0$ and $\sigma_n > 0$, such that $\eta_n(0) = \zeta_n(\sigma_n) = z$, $\eta_n(\tau_n) = \gamma(t_{n+1})$, $\zeta_n(0) = \gamma(t_n)$, and

$$\int_0^{\tau_n} L[\eta_n]\, \mathrm{d}t \le C_0|\gamma(t_{n+1}) - z| + \frac{1}{n},$$
$$\int_0^{\sigma_n} L[\zeta_n]\, \mathrm{d}t \le C_0|\gamma(t_n) - z| + \frac{1}{n},$$

where $C_0 > 0$ is a constant independent of n. We set $T_n = t_n - t_{n+1} + \tau_n + \sigma_n$ and define the curve $\gamma_n \in \mathrm{AC}([0, T_n])$ by

$$\gamma_n(t) = \begin{cases} \eta_n(t) & \text{for } t \in [0, \tau_n], \\ \gamma(t + t_{n+1} - \tau_n) & \text{for } t \in (\tau_n, \tau_n + t_n - t_{n+1}], \\ \zeta_n(t - (\tau_n + t_n - t_{n+1})) & \text{for } t \in (\tau_n + t_n - t_{n+1}, T_n]. \end{cases}$$

Observe that $\gamma_n(0) = \gamma_n(T_n) = z$ and

$$\begin{aligned} \int_0^{T_n} L[\gamma_n]\, \mathrm{d}t &\le u_\infty(\gamma(t_n)) - u_\infty(\gamma(t_{n+1})) \\ &\qquad + C_0(|\gamma(t_n) - z| + |\gamma(t_{n+1}) - z|) + \frac{2}{n} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and conclude by Lemma 12 that $z \in \mathcal{A}_H$. This is a contradiction. ∎

In what follows we divide our considerations concerning Case 2 into two subcases:
Case 2a: $\sup \gamma((-\infty, 0]) = \infty$ and Case 2b: $\inf \gamma((-\infty, 0]) = -\infty$.
We now deal with Case 2a.

LEMMA 17. *In Case* 2a*, we have* $[y, \infty) \cap \mathcal{A}_H = \emptyset$*. Moreover, the function* γ *is decreasing on* $(-\infty, 0]$ *and there exists a constant* $c \in \mathbf{R}$ *such that* $u_\infty(x) = d_+(x) + c$ *for all* $x \geq y$*.*

Proof. Since $\sup \gamma((-\infty, 0]) = \infty$ and y is in the interval $\gamma((-\infty, 0])$, we see that $[y, \infty) \subset \gamma((-\infty, 0])$ and hence $\operatorname{dist}([y, \infty), \mathcal{A}_H) \geq \operatorname{dist}(\gamma((-\infty, 0]), \mathcal{A}_H) > 0$. That is, we have $[y, \infty) \cap \mathcal{A}_H = \emptyset$.

To see that γ is decreasing, we suppose on the contrary that there exist $a < b \leq 0$ such that $\gamma(a) \leq \gamma(b)$. Since $\gamma([a, b])$ is a compact interval and $[y, \infty) \subset \gamma((-\infty, 0])$, we see that there exists an $a' \in (-\infty, a]$ such that $\gamma(a') = \gamma(b)$. Then we have

$$\int_{a'}^{b} L[\gamma]\, \mathrm{d}t = u_\infty(\gamma(b)) - u_\infty(\gamma(a')) = 0,$$

which implies that $\gamma(a') \in \mathcal{A}_H \cap [y, \infty)$. This is a contradiction, which ensures that γ is decreasing on $(-\infty, 0]$.

It is now clear that $\gamma((-\infty, 0]) = [y, \infty)$. Fix $x \in [y, \infty)$ and choose a (unique) $t_x \in (-\infty, 0]$ so that $\gamma(t_x) = x$. We have

$$\begin{aligned} d_+(y) - d_+(x) &\leq \int_{t_x}^{0} L[\gamma]\, \mathrm{d}t \\ &= u_\infty(y) - u_\infty(x) \leq d(y, x) = d_+(y) - d_+(x), \end{aligned}$$

where the last equality is a consequence of Proposition 7 (b). Therefore we get

$$u_\infty(x) = d_+(x) + c, \qquad \text{with } c := u_\infty(y) - d_+(y).$$ ∎

LEMMA 18. *In Case* 2a*, let* $\beta, z \in \mathbf{R}$ *be such that* $y \leq \beta < z$*. Then there exists a curve* $\eta \in \mathcal{E}((-\infty, \tau], d_-, \beta)$*, with* $\tau > 0$*, such that* $\eta(\tau) = z$*. Moreover,* η *is increasing on* $[0, \tau]$*.*

Proof. By Proposition 9, we may choose a $\zeta \in \mathcal{E}((-\infty, 0], d_-, z)$. By continuity, there is a $T > 0$ such that $(-\infty, \beta) \cap \zeta([-T, 0]) = \emptyset$. We fix such a $T > 0$, and will show that that ζ is increasing on $[-T, 0]$. Suppose on the contrary that $\zeta(a) \geq \zeta(b)$ for some $a, b \in [-T, 0]$ satisfying $a < b$. By Proposition 7, we have $d(\zeta(b), \zeta(a)) = d_+(\zeta(b)) - d_+(\zeta(a))$ and $d(\zeta(a), \zeta(b)) = d_-(\zeta(a)) - d_-(\zeta(b))$. Also, we have

$$d_+(\zeta(b)) - d_+(\zeta(a)) = \int_a^b L[\zeta]\, \mathrm{d}s = d_-(\zeta(b)) - d_-(\zeta(a)) \leq d(\zeta(b), \zeta(a)).$$

From these we conclude that

$$\int_a^b L[\zeta]\, \mathrm{d}s = d(\zeta(b), \zeta(a)) = -d(\zeta(a), \zeta(b)),$$

which yields

$$\begin{aligned} 0 &= d(\zeta(b),\, \zeta(a)) + d(\zeta(a),\, \zeta(b)) \\ &= \inf\Big\{ \int_0^t L[\eta]\, \mathrm{d}s \;\Big|\; t \geq b-a,\, \eta \in \mathrm{AC}([0,\, t]),\, \eta(t) = \eta(0) = \zeta(b) \Big\}. \end{aligned}$$

This implies that $\zeta(b) \in \mathcal{A}_H \subset (-\infty,\, y)$, which is a contradiction.

Next, we show that $\beta \in \zeta((-\infty,\, 0])$. Suppose on the contrary that $\beta \notin \zeta((-\infty,\, 0])$. Then, since $\zeta((-\infty,\, 0])$ is an interval and $z \in \zeta((-\infty,\, 0])$, we infer that $(-\infty, \beta] \cap \zeta((-\infty,\, 0]) = \emptyset$. Therefore, ζ is increasing on $(-\infty,\, 0]$ and $\inf \zeta((-\infty,\, 0]) \geq \beta$. Set $\alpha := \lim_{t\to-\infty} \zeta(t)$ and note that $\alpha \in [\beta,\, z)$. Now the proof of Lemma 16 guarantees that $\alpha \in \mathcal{A}_H$, which yields a contradiction, $\alpha \in \mathcal{A}_H \subset (-\infty,\, y)$.

We choose a $\tau > 0$ so that $\zeta(-\tau) = \beta$ and $(-\infty,\, \beta) \cap \zeta([-\tau,\, 0]) = \emptyset$. We see immediately that $\zeta([-\tau,\, 0]) = [\beta,\, z]$ and ζ is increasing on $[-\tau,\, 0]$. We define the curve $\eta \in \mathcal{E}((-\infty,\, \tau], d_-)$ by $\eta(s) = \zeta(s-\tau)$. The curve η has all the required properties. ∎

Since $u_0^- \leq u_0$ on $\mathbf{R}$, we have $\liminf_{x\to\infty}(u_0(x) - u_0^-(x)) \geq 0$. Because of one of assumptions of Theorem 3, we have only two cases to consider.

Case (i): $\liminf_{x\to\infty}(u_0(x) - u_0^-(x)) > 0$ and Case (ii): $\lim_{x\to\infty}(u_0(x) - u_0^-(x)) = 0$.

Proposition 19. *In Case* (i)*, we have* $u^+(y) \leq u_\infty(y)$.

Proof. We choose a $\delta > 0$ so that $\liminf_{x\to\infty}(u_0(x) - u_0^-(x)) > \delta$ and then a $\beta > y$ so that $u_0(x) - u_0^-(x) > \delta$ for all $x \geq \beta$. We have

$$u_0^-(x) \leq u_0^-(z) + d(x,z) < u_0(z) + d(x,z) - \delta \quad \text{ for all } x \in \mathbf{R} \text{ and } z \geq \beta,$$

and therefore, by Proposition 8, we get

$$u_0^-(x) = \inf_{z\leq\beta}(u_0(z) + d(x,z)) \quad \text{ for all } x \in \mathbf{R}.$$

In particular, we have for all $x \geq \beta$,

$$u_0^-(x) = \inf_{z\leq\beta}(u_0(z) + d_-(x) - d_-(z)) = d_-(x) + b,$$

where $b := \inf_{z\leq\beta}(u_0(z) - d_-(z))$. Since $u_\infty(x) \geq u_0^-(x)$ for all $x \in \mathbf{R}$, we have

$$d_+(x) - d_-(x) + c - b \geq 0 \quad \text{ for all } x \geq \beta,$$

where c is the constant from Lemma 17.

Fix any $\varepsilon > 0$. By the definition of b, we may choose an $\alpha \in (-\infty, \beta]$ so that $b + \varepsilon > u_0(\alpha) - d_-(\alpha)$. Since $\gamma(0) = y < \beta$ and $\lim_{t\to-\infty} \gamma(t) = \infty$, we may choose a $\sigma > 0$ so that $\gamma(-\sigma) = \beta$. Since $d(\beta,\alpha) = d_-(\beta) - d_-(\alpha)$, we may choose a $\zeta \in \mathrm{AC}([0,\rho])$, with $\rho > 0$, so that $\zeta(0) = \alpha$, $\zeta(\rho) = \beta$, and

$$d_-(\beta) - d_-(\alpha) + \varepsilon > \int_0^\rho L[\zeta]\, \mathrm{d}s.$$

Fix any $t > 0$ and set $z = \gamma(-t-\sigma)$. In view of Lemma 18, we may choose an $\eta \in \mathcal{E}((-\infty, \tau], d_-, \beta)$, with $\tau > 0$, such that $\eta(\tau) = z$. Remark that η is increasing on $[0, \tau]$. Set $T = \min\{\tau, t\}$. We define the function f on $[0, T]$ by $f(s) = \eta(s) - \gamma(s-t-\sigma)$, and observe that $f(0) = \beta - \gamma(-t-\sigma) < \beta - \gamma(-\sigma) = 0$ and that if $T = \tau$, then $f(T) = z - \gamma(\tau - t - \sigma) > z - \gamma(-t-\sigma) = 0$ and if $T = t$, then $f(T) = \eta(t) - \gamma(-\sigma) > \eta(0) - \beta = 0$. By the continuity of f, we may choose a $\lambda \in (0, T)$ so that $f(\lambda) = 0$, that is, $\eta(\lambda) = \gamma(\lambda - t - \sigma)$.

We define $\mu \in \mathrm{AC}([-(t+\sigma+\rho), 0])$ by

$$\mu(s) = \begin{cases} \gamma(s) & \text{for } s \in [\lambda - (t+\sigma),\, 0], \\ \eta(s+t+\sigma) & \text{for } s \in [-(t+\sigma),\, \lambda-(t+\sigma)], \\ \zeta(s+t+\sigma+\rho) & \text{for } s \in [-(t+\sigma+\rho),\, -(t+\sigma)]. \end{cases}$$

Observe that $\mu(0) = y$ and $\mu(-(t+\sigma+\rho)) = \zeta(0) = \alpha$, and compute that

$$\begin{aligned}
&\int_{-(t+\sigma+\rho)}^{0} L[\mu]\,\mathrm{d}s + u_0(\mu(-(t+\sigma+\rho))) \\
&= \int_0^{\rho} L[\zeta]\,\mathrm{d}s + \int_0^{\lambda} L[\eta]\,\mathrm{d}s + \int_{\lambda-(t+\sigma)}^{0} L[\gamma]\,\mathrm{d}s + u_0(\alpha) \\
&< d_-(\beta) - d_-(\alpha) + \varepsilon + d_-(\eta(\lambda)) - d_-(\eta(0)) \\
&\quad + d_+(\gamma(0)) - d_+(\gamma(\lambda - (t+\sigma))) + u_0(\alpha) \\
&= d_+(y) + d_-(\eta(\lambda)) - d_+(\eta(\lambda)) + u_0(\alpha) - d_-(\alpha) + \varepsilon \\
&< d_+(y) + d_-(\eta(\lambda)) - d_+(\eta(\lambda)) + b + 2\varepsilon.
\end{aligned}$$

As noted above, we have

$$d_+(\eta(\lambda)) - d_-(\eta(\lambda)) + c - b \ge 0,$$

and therefore

$$u(y, t+\sigma+\rho) < d_+(y) + c + 2\varepsilon = u_\infty(y) + 2\varepsilon,$$

from which we conclude that $u^+(y) \le u_\infty(y)$. ∎

The switch-back construction of μ in the proof above is adapted from [16].

Proposition 20. *In Case* (ii)*, we have* $u^+(y) \le u_\infty(y)$*.*

Proof. Fix any $\varepsilon > 0$. By assumption, there exists an $R > y$ such that if $x \ge R$, then $u_0(x) \le u_0^-(x) + \varepsilon$. Since $\lim_{t\to-\infty} \gamma(t) = \infty$, there exists a $T > 0$ such that if $t \ge T$, then $\gamma(-t) \ge R$. Fix any $t \ge T$ and compute that

$$\begin{aligned}
u(y,t) &\le \int_{-t}^{0} L[\gamma]\,\mathrm{d}s + u_0(\gamma(-t)) \le u_\infty(y) - u_\infty(\gamma(-t)) + u_0^-(\gamma(-t)) + \varepsilon \\
&\le u_\infty(y) - u_\infty(\gamma(-t)) + u_\infty(\gamma(-t)) + \varepsilon = u_\infty(y) + \varepsilon.
\end{aligned}$$

From this we conclude that $u_\infty(y) \le u_0^-(y)$. ∎

We may treat Case 2b by an argument parallel to the above, to conclude that $u^+(y) \le u_\infty(y)$. The proof of Theorem 3 is now complete. ∎

4. Concluding remarks. We first discuss two examples in connection with Theorem 3 and Proposition 2. Barles-Souganidis [5] gave a simple example of Hamiltonian H and initial data u_0 for which convergence (5) does not hold. In the example H and u_0 are given, respectively, by $H(p) = |p+1| - 1$ and $u_0(x) = \sin x$ for $p, x \in \mathbf{R}$. The solution u of (1)–(2) is then given by $u(x,t) := \sin(x-t)$, for which (5) does not hold with any asymptotic solution $v(x) - ct$, and all assumptions (A1)–(A6) are satisfied. Noting that $H(p) \le 0$ if and only if $p \in [-2, 0]$, we see that $d_+(x) = -2x$ and $d_-(x) = 0$ for all $x \in \mathbf{R}$ and that $\mathcal{A}_H = \emptyset$. Also, it is easily seen that $u_0^-(x) = \inf_{y\in\mathbf{R}}(u_0(y) + d(x,y)) = -1$ and $u_\infty(x) = -1$ for all $x \in \mathbf{R}$. Hence we have $u_\infty(x) = d_-(x) - 1$ for all $x \in \mathbf{R}$, $\liminf_{x\to-\infty}(u_0 - u_0^-)(x) = 0$, and $\limsup_{x\to-\infty}(u_0 - u_0^-)(x) = 2$. These explicitly violate one of assumptions of Theorem 3.

Lions-Souganidis [20] examined the following Hamilton-Jacobi equation $\frac{1}{2}|Dv|^2 - f(x) = 0$ in $\mathbf{R}$, where f is given by $f(x) = 2 + \sin x + \sin\sqrt{2}x$. Note that $f(x) > 0$ for all $x \in \mathbf{R}$ and $\inf_{\mathbf{R}} f = 0$. The Lagrangian L of $H(x,p) := \frac{1}{2}|p|^2 - f(x)$ is given by $L(x,\xi) = \frac{1}{2}|\xi|^2 + f(x)$ and satisfies $L(x,\xi) > 0$ for all (x,ξ), which implies that $\mathcal{A}_H = \emptyset$. The function d, d_+, and d_- are given, respectively, by

$$d(x,y) = \Big|\int_y^x \sqrt{2f(s)}\,\mathrm{d}s\Big|, \quad d_+(x) = -\int_0^x \sqrt{2f(s)}\,\mathrm{d}s, \quad \text{and} \quad d_-(x) = -d_+(x).$$

Consider the evolution equation $u_t + H(x, Du) = 0$ together with initial data $u_0(x) \equiv 0$. We write u for the solution of this problem as usual. It is easy to see that $u_0^-(x) = \inf_{y\in\mathbf{R}} d(x,y) = 0$ and $u_\infty(x) = +\infty$ for all $x \in \mathbf{R}$. Proposition 2 ensures that $\lim_{t\to\infty} u(x,t) = \infty$ for all $x \in \mathbf{R}$ and u does not "converge" to any asymptotic solution in this case.

Next we discuss two existing convergence results in light of Theorem 3. In [17], the Cauchy problem for (3), with $\Omega = \mathbf{R}^n$, are treated and, in addition to (A1)–(A6), it is there assumed that there exist functions ϕ_0, $\sigma_0 \in C(\mathbf{R}^n)$ such that $H[\phi_0] \le -\sigma_0$ in $\mathbf{R}^n$ and $\lim_{|x|\to\infty} \sigma_0(x) = \infty$. Most of results in [17] are concerned with solutions u of (3) with $\Omega = \mathbf{R}^n$ for which $u_\infty(x) \ge \phi_0(x) - C_0$ for all x and for some constant $C_0 \in \mathbf{R}$.

We restrict ourselves to the case when $n = 1$, and assume that (A1)–(A6) hold, that there exist functions ϕ_0, $\sigma_0 \in C(\mathbf{R})$ having the properties described above, and that $u_\infty(x) \ge \phi_0(x) - C_0$ for all x and for some constant $C_0 \in \mathbf{R}$. We show as a consequence of Theorem 3 that convergence (7) holds. The first thing to note is that if $\sup \mathcal{A}_H < \infty$, then $d_+(x) - \phi_0(x) \to -\infty$ as $x \to \infty$. Indeed, assuming that $\mathcal{A}_H \subset (-\infty, \beta)$ for some $\beta \in \mathbf{R}$, for any $\gamma \in \mathcal{E}((-\infty, 0], d_+, \beta)$, we see, as in the proof of Lemma 18, that γ is decreasing on $(-\infty, 0]$ and $\gamma(s) \to \infty$ as $s \to -\infty$. Moreover, for $t > 0$, we get

$$d_+(\gamma(0)) - d_+(\gamma(-t)) = \int_{-t}^0 L[\gamma]\,\mathrm{d}s \ge \phi_0(\gamma(0)) - \phi_0(\gamma(-t)) + \int_{-t}^0 \sigma_0(\gamma(s))\,\mathrm{d}s.$$

Since $\int_{-t}^0 \sigma_0\,\mathrm{d}s \to \infty$ as $t \to \infty$, we conclude that $(\phi_0 - d_+)(x) \to \infty$ as $x \to \infty$. Similarly, if $\inf \mathcal{A}_H > -\infty$, then we have $(d_- - \phi_0)(x) \to \infty$ as $x \to -\infty$. These observations guarantee that, under our current hypotheses, there is no possibility that either $u_\infty(x) = d_+(x) + c_+$ for all $x > r$ and for some constants c_+ and $r \in \mathbf{R}$, or $u_\infty(x) = d_-(x) + c_-$ for all $x < r$ and for some constants c_- and $r \in \mathbf{R}$. Now, Theorem 3 ensures that convergence (7) holds.

Let us consider the Cauchy problem (1)–(2) in the case where the functions $H(x,p)$ in x and u_0 are periodic with period 1. In addition to (A1)–(A6), we assume as in [15] (see also [5]) that there exists a function $\omega_0 \in C([0,\infty))$ satisfying $\omega_0(0)=0$ and $\omega_0(r)>0$ for all $r>0$ such that for all $(x,p)\in \mathbf{R}^2$ satisfying $H(x,p)=0$ and for all $\xi \in D_2^- H(x,p)$ and $q\in\mathbf{R}$, if $\xi q>0$, then

$$H(x,p+q) \geq \xi q + \omega_0(\xi q). \tag{14}$$

Note that if $v\in\mathcal{S}_H^-$ (resp., $v\in\mathcal{S}_H$), then $v(\cdot+1)\in\mathcal{S}_H^-$ (resp., $v(\cdot+1)\in\mathcal{S}_H$). Hence, by the definition of u_0^- and u_∞, we infer that u_0^- and u_∞ are periodic with period 1. Note also by the periodicity of $H(x,p)$ in x that $d(x+1,y+1)=d(x,y)$ for all x, $y\in\mathbf{R}$. In order to apply Theorem 3, we assume that $\sup\mathcal{A}_H<\infty$ and $u_\infty(x)=d_+(x)+c_+$ for all $x\geq R$ and for some constants c_+, $R\in\mathbf{R}$. By the above periodicity of d, we deduce that $\mathcal{A}_H=\emptyset$ and $u_\infty(x)=d_+(x)+c_+$ for all $x\in\mathbf{R}$.

Fix any $y\in\mathbf{R}$ and choose a $\gamma\in\mathcal{E}((-\infty,0],d_+,y)$. As in the proof of Lemma 18, we see that γ is decreasing on $(-\infty,0]$ and $\sup\gamma((-\infty,0])=\infty$. We may choose a $\tau>0$ so that $\gamma(-\tau)=y+1$. We extend $\dot\gamma|_{(-\tau,0]}$ to $\mathbf{R}$ by periodicity and integrating the resulting periodic function, we may assume that $\gamma(t-\tau)=\gamma(t)+1$ for all $t\in\mathbf{R}$.

We assume that

$$0=\liminf_{x\to\infty}(u_0-u_0^-)(x)<\limsup_{x\to\infty}(u_0-u_0^-)(x).$$

(Otherwise, by Theorem 3, we know that $u^+(y)\leq u_\infty(y)$.) By the periodicity of u_0^- and u_∞, we have $\min_{[x,x+1)}(u_0-u_0^-)=0$ for all $x\in\mathbf{R}$. Moreover we have $\min_{s\in[t,t+\tau)}(u_0-u_0^-)(\gamma(-s))=0$ for all $t\in\mathbf{R}$.

It has been proved in [15] that there exist a constant $\delta>0$ and a non-decreasing function $\omega\in C([0,\infty))$ satisfying $\omega(0)=0$ such that for any $0\leq\varepsilon\leq\delta$, we have

$$\int_{-t/(1+\varepsilon)}^0 L[\gamma_\varepsilon]\,\mathrm{d}s \leq u_\infty(\gamma_\varepsilon(0)) - u_\varepsilon(\gamma_\varepsilon(-t/(1+\varepsilon)) + t\varepsilon\omega(\varepsilon), \tag{15}$$

where $\gamma_\varepsilon(s):=\gamma((1+\varepsilon)s)$ for all $s\in\mathbf{R}$.

We fix any $t\geq\tau/\delta$. Choose a $\sigma\in[t,t+\tau)$ so that $(u_0-u_0^-)(\gamma(-\sigma))=0$ and then an $\varepsilon\geq0$ so that $\frac{\sigma}{1+\varepsilon}=t$. Note that $\varepsilon=\frac{\sigma}{t}-1=\frac{\sigma-t}{t}\leq\frac{\tau}{t}\leq\delta$. Therefore, by (15), we get

$$\begin{aligned}\int_{-t}^0 L[\gamma_\varepsilon]\,\mathrm{d}s &\leq u_\infty(\gamma_\varepsilon(0)) - u_\infty(\gamma_\varepsilon(-t)) + \sigma\varepsilon\omega(\varepsilon)\\ &\leq u_\infty(y) - u_\infty(\gamma(-\sigma)) + \frac{\sigma\tau}{t}\omega(\frac{\tau}{t})\\ &\leq u_\infty(y) - u_\infty(\gamma(-\sigma)) + \frac{\tau(t+\tau)}{t}\omega(\frac{\tau}{t})\\ &\leq u_\infty(y) - u_0^-(\gamma(-\sigma)) + \tau(1+\delta)\omega(\frac{\tau}{t}),\end{aligned}$$

and furthermore

$$\begin{aligned}u(y,t) &\leq \int_{-t}^0 L[\gamma_\varepsilon]\,\mathrm{d}s + u_0(\gamma_\varepsilon(-t))\\ &\leq u_\infty(y) - u_0^-(\gamma(-\sigma)) + u_0(\gamma(-\sigma)) + \tau(1+\delta)\omega(\frac{\tau}{t})\\ &= u_\infty(y) + \tau(1+\delta)\omega(\frac{\tau}{t}).\end{aligned}$$

Thus we obtain $u^+(y) \le u_\infty(y)$. Similarly, if we assume that $\inf \mathcal{A}_H > -\infty$ and $u_\infty(x) = d_-(x) + c_-$ for all $x \ge R$ for some constant c_-, $R \in \mathbf{R}$ and also that $0 = \liminf_{x\to-\infty}(u_0 - u_0^-)(x) < \limsup_{x\to-\infty}(u_0 - u_0^-)(x)$, then we get $u^+(y) \le u_\infty(y)$. These observations and Theorem 3 guarantee that convergence (7) holds.

We continue to consider the Cauchy problem (1)–(2), where the functions $H(\cdot, p)$ and u_0 are periodic with period 1. Now we assume in addition to (A1)–(A6) that there exists a function $\omega_0 \in C([0, \infty))$ satisfying $\omega_0(0) = 0$ and $\omega_0(r) > 0$ for all $r > 0$ such that for all $(x, p) \in \mathbf{R}^2$ satisfying $H(x, p) = 0$ and for all $\xi \in D_2^- H(x, p)$ and $q \in \mathbf{R}$, if $\xi q < 0$, then

$$H(x, p + q) \ge \xi q + \omega_0(|\xi q|). \tag{16}$$

We will show that convergence (7) holds under these hypotheses, which seems to be a new observation.

We argue as in the previous result and thus assume that $\sup \mathcal{A}_H < \infty$ and $u_\infty(x) = d_+(x)+c_+$ for all $x > R$ and for some constants c_+, $R \in \mathbf{R}$. We then observe that $\mathcal{A}_H = \emptyset$ and $u_\infty(x) = d_+(x)+c_+$ for all $x \in \mathbf{R}$ and that $\liminf_{x\to\infty}(u_0-u_0^-)(x) < \limsup_{x\to\infty}(u_0 - u_0^-)(x)$. Fix any $y \in \mathbf{R}$ and choose a $\gamma \in \mathcal{E}(\mathbf{R}, d_+, y)$ so that $\gamma(t - \tau) = \gamma(t) + 1$ for all $t \in \mathbf{R}$ and for some constant $\tau > 0$. A careful review of [15, Lemmas 3.1, 3.2, Proposition 3.4] reveals that there exist a constant $\delta \in (0, 1)$ and a non-decreasing function $\omega \in C([0, \infty))$ satisfying $\omega(0) = 0$ such that for any $0 \le \varepsilon \le \delta$ and $t > 0$, we have

$$\int_{-t/(1-\varepsilon)}^0 L[\eta_\varepsilon]\,\mathrm{d}s \le u_\infty(\eta_\varepsilon(0)) - u_\infty(\eta_\varepsilon(-t/(1-\varepsilon)) + t\varepsilon\omega(\varepsilon), \tag{17}$$

where $\eta_\varepsilon(s) := \gamma((1-\varepsilon)s)$ for all $s \in \mathbf{R}$.

As before we fix any $t \ge \tau/\delta$ and choose a $\sigma \in (t-\tau,\, t]$ so that $(u_0-u_0^-)(\gamma(-\sigma)) = 0$ and then an $\varepsilon \ge 0$ so that $\frac{\sigma}{1-\varepsilon} = t$. Note that $\varepsilon = 1 - \frac{\sigma}{t} = \frac{t-\sigma}{t} \le \frac{\tau}{t} \le \delta$. Hence by (17) we get

$$\begin{aligned}\int_{-t}^0 L[\eta_\varepsilon]\,\mathrm{d}s &\le u_\infty(\eta_\varepsilon(0)) - u_\infty(\eta_\varepsilon(-t)) + \sigma\varepsilon\omega(\varepsilon)\\ &\le u_\infty(y) - u_\infty(\gamma(-\sigma)) + \frac{\sigma\tau}{t}\omega(\frac{\tau}{t})\\ &\le u_\infty(y) - u_0^-(\gamma(-\sigma)) + \tau\omega(\frac{\tau}{t}),\end{aligned}$$

and consequently

$$\begin{aligned}u(y,t) &\le \int_{-t}^0 L[\eta_\varepsilon]\,\mathrm{d}s + u_0(\eta_\varepsilon(-t))\\ &\le u_\infty(y) - u_0^-(\gamma(-\sigma)) + u_0(\gamma(-\sigma)) + \tau\omega(\frac{\tau}{t})\\ &= u_\infty(y) + \tau\omega(\frac{\tau}{t}),\end{aligned}$$

from which we get $u^+(y) \le u_\infty(y)$. Similarly, if we assume that $\inf \mathcal{A}_H > -\infty$ and $u_\infty(x) = d_-(x) + c_-$ for all $x \ge R$ for some constants c_-, $R \in \mathbf{R}$ and also that $0 = \liminf_{x\to-\infty}(u_0 - u_0^-)(x) < \limsup_{x\to-\infty}(u_0 - u_0^-)(x)$, then we get $u^+(y) \le u_\infty(y)$. Theorem 3 now guarantees that convergence (7) holds.

For possible relaxations of the periodicity of $H(\cdot, p)$ and u_0 in the above convergence results, we refer to [15] as well as [6, Théorème 1].

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