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# RELATIVE POINCARÉ – HOPF BIFURCATION AND GALLOPING INSTABILITY OF TRAVELING WAVES*

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*Dedicated to Joel Smoller on the occasion of his 70th birthday*

**Abstract.** For a class of reaction–convection–diffusion systems studied by Sattinger, notably including Majda's model of reacting flow, we rigorously characterize the transition from stability to time-periodic "galloping" instability of traveling-wave solutions as a relative Poincaré–Hopf bifurcation arising in ODE with a group invariance– in this case, translational symmetry. More generally, we show how to construct a finite-dimensional center manifold for a second-order parabolic evolution equation inheriting an underlying group invariance of the PDE, by working with a canonical integro-differential equation induced on the quotient space. This reduces the questions of existence and stability of bounded solutions of the PDE to existence and stability of solutions of the reduced, finite-dimensional ODE on the center manifold, which may then be studied by more standard, finite-dimensional bifurcation techniques.

**Key words.** stability of traveling waves, detonation waves, weighted norms, Poincaré–Hopf bifurcation, center manifold reduction, galloping instability

**AMS subject classifications.** 76L05, 35B32, 35B10, 34C20

## 1. Introduction.

In the seminal paper [S], Sattinger studied stability of standing-wave solutions

$$u(x,t) = \bar u(x), \qquad \lim_{z\to\pm\infty} \bar u(z) = u_\pm, \tag{1.1}$$

of the simple class of reaction–convection–diffusion equations

$$u_t = f(u, u_x) + u_{xx}, \qquad u, f \in \mathbb{R}^n, \tag{1.2}$$

having the property (properly speaking, a condition on both wave and system) that the constant solutions $u \equiv u_+$, $u_-$ are time-exponentially stable with respect to weighted norms $\|f\|_{\hat W^{1,\infty}_\eta}$ and $\|f\|_{\hat W^{1,\infty}_{-\eta}}$, respectively, $\eta \geq 0$, where

$$\|f\|_{\hat W^{1,\infty}_\eta} := \|e^{\eta x} f(x)\|_{W^{1,\infty}}. \tag{1.3}$$

We refer to this as the weighted norm property. This analysis includes also general traveling-wave solution $u(x,t) = \bar u(x - st)$, which may be reduced by the change of coordinates $x \to x - st$ to the standing-wave case (1.1).

As discussed in [ZH, Z.1], the weighted norm property captures the phenomenon of "convection-enhanced stability", in which neutrally stable or even slightly unstable modes with respect to the linearized equations about the endstates $u_\pm$ may be stabilized by inward convection combined with the dynamics of the traveling wave. A consequence is that the linearized operator $L$ about the traveling wave solution has a spectral gap with respect to norm

$$\|f\|_{W^{1,\infty}_\eta} := \|e^{\eta(1+|x|^2)^{1/2}} f(x)\|_{W^{1,\infty}}, \tag{1.4}$$

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whereas it typically does not with respect to standard, unweighted norms.

More precisely, introduction of Sattinger's weighted norms (1.4) has the effect of "shifting" the essential spectrum of $L$ away from the imaginary axis into the stable complex half-plane $\Re\lambda < 0$, whereupon a stability analysis may be carried out by standard techniques to show that *spectral stability implies exponential phase-asymptotic orbital stability* (both with respect to $\|\cdot\|_{\hat{W}^{1,\infty}_\eta}$), i.e., time-exponential convergence of a perturbed traveling wave to a specific shift $\bar{u}(x-\delta)$ of the original wave. We refer to [S] for details; see also the related analysis of Section 4, below. For this class of solutions and equations, the introduction of weighted norms thus removes the substantial technical difficulty presented in the original norm by accumulation of essential spectrum at the imaginary axis. For treatments of this difficulty in the general case, see, e.g., [SX, L.1, Z.2]; in particular, see the pointwise Green's function techniques introduced in [ZH, MaZ.1]. For treatments of the additional difficulty of real, or non-definite, viscosity, see [MaZ.2, Z.4].

In this paper, we extend Sattinger's analysis in a natural direction, studying for the same class of equations and solutions the bifurcation of traveling waves at the onset of instability. More precisely, indexing by $\epsilon$ a smoothly-varying one-parameter family of standing-wave solutions

$$\bar{u}^\epsilon(x): \quad \mathcal{F}(\epsilon, \bar{u}^\epsilon) \equiv 0, \tag{1.5}$$

of a smoothly-varying family of equations

$$u_t = \mathcal{F}(\epsilon, u) := f(\epsilon, u, u_x) + u_{xx} \tag{1.6}$$

(possibly shifts of a single equation, written in coordinates $x \to x - s(\epsilon)t$ moving with traveling-wave solutions of varying speeds $s(\epsilon)$), denote by $L(\epsilon, \bar{u}^\epsilon) := \partial\mathcal{F}/\partial u|_{u=\bar{u}^\epsilon}$ the linearized operator about the wave. As pointed out by Sattinger, $L(\epsilon, \bar{u}^\epsilon)$ has always the neutral, zero-eigenfunction $(\partial/\partial x)\bar{u}^\epsilon$, corresponding to translational invariance of the underlying equations; assuming the weighted norm property, the wave is (exponentially) orbitally stable if and only if there are no other neutrally stable modes associated to eigenvalues $\Re\lambda \geq 0$. We wish to investigate the situation that $L(\epsilon, \bar{u}^\epsilon)$ is spectrally stable for $\epsilon < 0$ and spectrally unstable for $\epsilon > 0$, with one or more eigenvalues crossing the imaginary axis at $\epsilon = 0$: specifically, to study associated bifurcation from traveling-wave to more complicated types of solutions.

The case of our particular interest is that of a *relative Poincaré–Hopf bifurcation*, for which a pair of complex conjugate eigenvalues

$$\lambda = \gamma(\epsilon) \pm i\tau(\epsilon) \tag{1.7}$$

crosses the imaginary axis with positive speed at $\epsilon = 0$, i.e.,

$$\gamma(0) = 0, \quad \tau(0) \neq 0, \quad (d\gamma/d\epsilon)(0) > 0; \tag{1.8}$$

the presence of the additional, neutral eigenvalue $\lambda = 0$ arising through translation-invariance distinguishes this from the standard Poincaré–Hopf bifurcation.

Define the weighted Sobolev norm

$$\|f\|^2_{H^2_\eta} := \sum_{j=0}^{2} \|(d/dx)^j f(x)\|^2_{L^2_\eta}, \tag{1.9}$$

inner product

$$\langle f, g\rangle_{H^2_\eta} := \sum_{j=0}^{2} \langle (d/dx)^j f(x), (d/dx)^j g(x)\rangle_{L^2_\eta}, \tag{1.10}$$

and space

$$H^2_\eta := \{f : \|f\|_{H^2_\eta} < +\infty\}, \tag{1.11}$$

where

$$\|f\|_{L^2_\eta} := \|e^{\eta(1+|x|^2)^{1/2}} f(x)\|_{L^2}, \tag{1.12}$$

$$\langle f, g\rangle_{L^2_\eta} := \langle e^{\eta(1+|x|^2)^{1/2}} f(x), g(x)\rangle_{L^2}. \tag{1.13}$$

We make the following assumptions, concerning the spectrum of the linearized operator $L(\epsilon, \bar{u}^\epsilon)$ only. Regarding the essential spectrum, we assume the *weighted norm condition*

$$\Re\sigma(e^{\eta x} L(0, u^0_+) e^{-\eta x}) < 0, \qquad \Re\sigma(e^{-\eta x} L(0, u^0_-) e^{\eta x}) < 0 \tag{1.14}$$

for some $\eta \geq 0$, where $\sigma$ refers to $L^2$ spectrum; hence, by continuity,

$$\Re\sigma(e^{\eta x} L(\epsilon, u^\epsilon_+) e^{-\eta x}) < 0, \qquad \Re\sigma(e^{-\eta x} L(\epsilon, u^\epsilon_-) e^{\eta x}) < 0 \tag{1.15}$$

for $\epsilon$ sufficiently small. As pointed out by Sattinger [S], conditions (1.15) are necessary and sufficient in order that the $W^{k,p}_\eta$ essential spectrum of $L(\epsilon, \bar{u}^\epsilon)$ lie strictly in the stable half-plane $\Re\lambda < 0$, for any $k$, $p$. Indeed, this may be seen as a special case of a standard result of Henry ([He], Theorem A.2, chapter 5) on spectrum of operators with asymptotically constant coefficients, asserting that (i) the rightmost (i.e. largest real part) envelope of the essential spectrum is the envelope of the union of the rightmost envelopes of the limiting, constant-coefficient operators at $\pm\infty$, and (ii) to the right of this envelope, all spectrum consists of eigenvalues of finite multiplicity. Thus, (1.14) precisely enforces the weighted norm property discussed earlier.

Remark 1.1. One may also as in [S] allow different weights $\eta_+$ and $\eta_- \geq 0$ for $x \geq 0$ and $x \leq 0$, without affecting any of the results of this paper. However, in practice this is rarely necessary.

The conditions (1.14) are algebraic requirements on the spectrum of the Fourier symbols of $L(0, u^0_\pm)$, which may in principle be checked by direct calculation. As noted in [S], they hold automatically for traveling waves, or "shock profiles" of scalar viscous conservation laws

$$u_t + h(u)_x = u_{xx}, \tag{1.16}$$

$u \in \mathbb{R}^1$, which, however, are always stable. For systems of conservation laws (1.16), $u \in \mathbb{R}^n$, $n > 1$, they hold for nonclassical "totally compressive" shock profiles [K, ZH] but *not* for standard, Lax-type profiles [ZH]. They hold frequently for traveling-wave solutions of reaction–diffusion equations, with the trivial weight $\eta = 0$; see section 4.3. Also, they are satisfied for "strong detonation" type traveling-wave solutions of Majda's model for reacting flow; see section 4.3.3. These, and reaction–diffusion waves, are the main examples we have in mind.

Regarding the point spectrum, we assume the *bifurcation conditions*

(rPH) For $\epsilon$ sufficiently small, the only eigenvalues of $L^\epsilon$ lying in a neighborhood of the imaginary axis are a single translational eigenvalue zero and a crossing complex conjugate pair as described in (1.7), (1.8).

Then, our main result is as follows.

THEOREM 1.2. *Let* (1.6), (1.5) *be a family of traveling-waves and systems as above, satisfying the weighted norm condition* (1.14) *and the bifurcation conditions (rPH), with $f \in C^4$. Then, for $a \geq 0$ sufficiently small and $C > 0$ sufficiently large, there is a $C^1$ function $\epsilon(a)$, $\epsilon(0) = 0$, and a family of solutions*

$$u^a(x,t) = \mathbf{u}^a(x - \alpha^a(t), t), \qquad \alpha^a(t) = \sigma^a t + \theta^a(t), \tag{1.17}$$

*of* (1.6) *with $\epsilon = \epsilon(a)$, where $\mathbf{u}^a(\cdot, t)$ is time-periodic, with $\|\mathbf{u}^a(\cdot,0) - \bar{u}^0\|_{H^2_\eta} = a$ and $\|\mathbf{u}^a(\cdot,t) - \bar{u}^0\|_{H^2_\eta} \leq Ca$ for all $t$, $\sigma^a$ is a constant drift, and $\theta^a \in \mathbb{R}$ is time-periodic. Up to fixed translations in $x$, $t$, these are the only nearby solutions of this form, as measured in $H^2_\eta$. Moreover, if $\Re\sigma(L(0,\bar{u}^0)) \leq 0$, then solutions $\mathbf{u}^a$ are time-exponentially phase-asymptotically orbitally stable with respect to $H^2_\eta$ if $d\epsilon/da > 0$, in the sense that perturbed solutions converge time-exponentially to a specific shift in $x$ and $t$ of the original solution, and unstable if $d\epsilon/da < 0$. If $\Re\sigma(L(0,\bar{u}^0)) \not\leq 0$, then solutions $\mathbf{u}^a$ are unstable for all $a$ sufficiently small.*

Up to translation, Theorem 1.2 describes the standard Poincaré–Hopf picture of a one-parameter family of periodic orbits $u^a$ of amplitude $a$ bifurcating from the family of stationary solutions $\bar{u}^\epsilon$, yielding for generic fixed values of $\epsilon$ a discrete set of periodic orbits $\{u^{a_j}\}$ of increasing amplitude, starting with the trivial, stationary solution $u^0 = \bar{u}^\epsilon$, alternating between stability and instability. Indeed, we shall show that it corresponds exactly to a standard Poincaré–Hopf bifurcation on the quotient space obtained by factoring out translational invariance, with $\{u^{a_j}\}$ corresponding to nested cycles in a two-dimensional submanifold of a three-dimensional (counting $\epsilon$ direction) center manifold.

More generally, let $\mathcal{H}$ denote any subspace of $H^2_\eta$ complementary to the span of the translational zero-eigenfunction $\phi := (\partial/\partial x)\bar{u}^0$ of $L(0,0)$.Then, we have the following more general result, from which Theorem 1.2 follows as a corollary using the standard Poincaré–Hopf Theorem for $2 \times 2$ dynamical systems, as contained, for example, in Theorems 11.12 and 11.16 of [HK]; see also Section 2.4, below.

THEOREM 1.3. *Let* (1.6), (1.5) *be a family of traveling-waves and systems as above, satisfying the weighted norm condition, with $f \in C^4$. Then,*

*(i) the map*

$$(v(x,t), \alpha) \to v(x + \alpha, t) + (\bar{u}^0(x + \alpha) - \bar{u}^0(x)) \tag{1.18}$$

*from $\mathcal{H} \times \mathbb{R}$ to $H^2_\eta$ is invertible for $\|v\|_{H^2_\eta}$ sufficiently small; in $(v, \alpha)$ coordinates,* (1.6) *takes the form*

$$(v, \alpha)' = (G(v, \epsilon), h(v, \epsilon)) \tag{1.19}$$

*independent of $\alpha$, where $G$ and $h$ are appropriate integro-differential operators.*

*(ii) The augmented equations*

$$(v, \epsilon)' = (G(v, \epsilon), 0) \tag{1.20}$$

*on $\mathcal{H} \times \mathbb{R}$ possess a $C^3$ local center manifold at $(v, \epsilon) = (0, 0)$ of finite dimension equal to that of the center subspace of operator $L(0)$, of which one dimension corresponds to variation in the parameter $\epsilon$ and the remaining dimensions correspond to the part of the center subspace of $L(0)$ associated with nonzero eigenfunctions of $L(0)$: that is, the translational zero-eigenfunction has been factored out.*

*(iii) This center manifold contains all globally bounded solutions of (1.20) for $|\epsilon|$ sufficiently small, corresponding to solutions of (1.6) remaining sufficiently close to the set of translates of $\bar{u}^0$. Moreover, if $\Re\sigma(L(0)) \le 0$, then $H^2_\eta$ orbital stability of these as solutions of the original system (1.6) is equivalent to stability as solutions of the finite-dimensional ODE induced on the center manifold, with $\epsilon$ held fixed. If $\Re\sigma(L(0)) \not\le 0$, then all such solutions are unstable.*

Theorem 1.3 reduces the degenerate, infinite-dimensional bifurcation problem for PDE (1.5)–(1.6) to a standard, finite-dimensional bifurcation problem with translational invariance factored out. Besides Theorem 1.2, this also includes an $H^2_\eta$ version of Sattinger's orbital stability theorem ([S], Theorem 4.1), which corresponds to the case that the center subspace of $L(0)$ is just the zero-eigenspace spanned by $(\partial/\partial x)\bar{u}^0$. In this situation, the center manifold in $(Y, \epsilon)$ coordinates consists of the trivial solutions $Y \equiv 0$, $\epsilon \equiv$ constant, which are vacously stable under perturbations with $\epsilon$ held fixed.

Remarks 1.4. 1. Our assumption on essential spectrum are checked for various examples, notably on Majda's model, in section 4.3. However, detection of the relative Poincaré–Hopf bifurcation appears to be a numerical task, in general, since the bifurcation conditions (rPH) are typically not analytically verifiable.

2. The sign of $d\epsilon/da$ near $a = 0$ (determined by the sign of the first nonvanishing derivative $d^k\epsilon/da^k(0)$: generically, $d^2\epsilon/da^2(0)$) is at least numerically computable, by the method described in Section 4 of [MM], so can actually be used to determine stability. This would be a very interesting direction for future investigation, particularly in the context of Majda's equations discussed below.

3. The factorized description of the center manifold given in Theorem 1.3 together with the property of exponential approach yields additional information about the relative Poincaré–Hopf bifurcation of Theorem 1.2. Namely, we may conclude, for nondegenerate $\epsilon \neq 0$ in the range of $\epsilon(a)$ (i.e., for which there exists *some* periodic solution), solutions originating sufficiently near a translate of $\bar{u}^\epsilon$ with respect to $H^2_\eta$ must time-exponentially approach an $(x, t)$-translate of either $\bar{u}^\epsilon$ or else the limit-cycle corresponding to the periodic solution of smallest nonzero amplitude for $\epsilon$, by properties of planar ODE. (Indeed, the proof of the exponential approach property shows, for $\bar{u}^\epsilon$ unstable, that convergence to $\bar{u}^\epsilon$ occurs for a Hausdorff-measure zero set of initial perturbations; see Sections 2.1 and 4.)

4. It is straightforward to carry out a similar analysis in the original $W^{1,\infty}_\eta$ framework of Sattinger; see Remark 4.6. However, the analysis is somewhat simpler in the Sobolev setting; compare especially the linearized estimates of Lemma 4.4 to the detailed estimates of [S]. Of course, our $H^2_\eta$ bounds include also pointwise, $W^{1,\infty}$ bounds, by Sobolev embedding.

5. Similar analysis yields a reduced center manifold for general semilinear second-order elliptic PDE invariant under a bounded symmetry group, provided that the lin-

earized operator about zero have a spectral gap: in particular, for initial–boundary-value problems on bounded domains. An application is to bifurcation from normal (identically zero) to superconducting states in the Ginzberg–Landau model for superconductivity. These equations are invariant under gauge transformations $\Phi_\alpha$ generated by rotations $\Psi \to e^{i\alpha}\Psi$ of a complex order-parameter $\Psi$, which invariance has the effect of doubling the multiplicity of all eigenvalues of the linearized operator about the zero solution; see, e.g., [BR, MW]. Working on a reduced center manifold changes a degenerate multiplicity two bifurcation to a standard transcritical bifurcation that is easily analyzed by standard techniques. Compare the difficulties in the general, even-multiplicity case, as discussed in [MW].

Existence of center manifolds for parabolic PDE has been widely studied; see, e.g., [C, He, MM, G]. In particular, relative Poincaré–Hopf bifurcation has been investigated in [SSW, GLM.1, GLM.2] in the context of "meandering" spiral wave solutions of reaction–diffusion equations in two spatial dimensions, $x \in \mathbb{R}^2$, using considerably more sophisticated techniques. The latter analyses face the additional difficulty that the associated invariance group contains the unbounded operations of spatial rotation on an unbounded domain. Thus, the main novelty of the present work is its simplicity and the application to galloping phenomena: in particular, the accomodation of accumulating essential spectrum in our treatment of detonation waves, as we now describe.

**Galloping waves.** Solutions of form (1.17) correspond to small time-periodic shifts in speed and profile, or "galloping instability", of the base solution $\bar{u}^0$. Galloping instability is a well-known phenomenon in combustion theory; indeed, it is commonly described as the principal longitudinal (i.e., one-dimensional) instability arising in detonation waves. See, e.g., [AT, AlT, B, BL, BN, Er.1, Er.2, F.1, F.2, FD, FW, LS, MT]. One of our main motivations in this analysis was to give a rigorous description of this phenomenon in a simple context. In particular, as pointed out in [LLT] (see also section 4.3.3), Sattinger's weighted norm condition is satisfied for strong detonation-type solutions of Majda's model for reacting flow [M], a model widely used in the qualitative study of detonation, and so our conclusions apply to this model. On the other hand, it is not satisfied for the trivial weight $\eta = 0$, and so the weighted norm construction is truly needed.

Majda's model [M] reads

$$\begin{cases} \partial_t \tilde{u} + \partial_x f(\tilde{u}) = & B\partial_x^2 \tilde{u} + kq\varphi(\tilde{u})\tilde{z}, \\ \partial_t \tilde{z} = & D\partial_x^2 \tilde{z} - k\varphi(\tilde{u})\tilde{z}, \end{cases} \tag{1.21}$$

in which the scalar variable $u$ is a lumped gas-dynamical variable combining aspects of specific volume, particle velocity and temperature, and $z$ corresponds to mass fraction of reactant. The physical constants in (1.21) are $q$, comprising quantities produced in reaction, in particular heat released, and $k$ the rate of reaction. The diffusion coefficients $B$ and $D$ are assumed to be constant and positive. We take $f'(u) > 0, \quad f''(u) > 0$, and $q > 0$, corresponding to an exothermic reaction. The ignition function $\phi$ is a step function, identically zero for $u \leq u_i$, nondecreasing, and for instance identically 1 for $u \geq u^i$.

Previous work on Majda's model notably includes the proof of nonlinear stability of strong detonation waves by Liu and Ying [LYi] in the small heat release limit, the proof of spectral stability of strong detonation waves by Roquejoffre and Vila

[RV] in the small diffusion, or "ZND" limit (referring to the zero-diffusion Zeldovich–von Neumann–Doering model), and a weaker but more general result by Lyng and Zumbrun [LyZ.2] that transition to one-dimensional instability for arbitrary weak or strong detonations may occur *only* by one or more pairs of complex-conjugate eigenvalues crossing the imaginary axis. See [Ly, LyZ.2] for an analogous result for the full reacting Navier–Stokes equations in the ZND limit. Thus, Theorem 1.2 indeed supports the picture of galloping as generic longitudinal instability.

Fickett and Davis ([FD], p. 231) state that high dependence of reaction rate on temperature is associated with galloping instability. In [M], the reaction rate intervenes as $k\phi(u)$, where $u$ is analogous to temperature. Hence in Theorem 1.2, the bifurcation parameter could be the reaction rate; in the light of Liu and Ying's result [LYi], it could also be the heat release $q$; in the light of Roquejoffre and Vila's result [RV], it could be the diffusion $D$, or perhaps a parameter controlling the shape of $\phi$ (note: the shape of $\phi$ indeed enters the analysis; see [RV]).

Remark 1.5. Though one-dimensional instabilities are well known to occur for the full reacting Navier–Stokes equations, it is not known whether this can in fact occur for the simpler Majda's model. A very interesting numerical problem would be to resolve this question by exhaustive search: note that point spectrum may be rapidly determined numerically using shooting algorithms as described, e.g., in [Br.1, Br.2, BrZ, BDG].

Provided that such instability can be found, with accompanying relative Poincaré–Hopf bifurcation, a second interesting problem would be to calculate the sign of $d^2\epsilon/da^2$ at the transition point $\epsilon = 0$. Positivity ("supercritical" case [HK]) would mean that small-amplitude galloping solutions are stable, implying bifurcation from stable traveling wave to stable galloping wave as seen in experiment. Negativity ("subcritical" case [HK]) would indicate that small-amplitude galloping solutions are unstable, suggesting a more complicated, global bifurcation.

**Breathers.** A related phenomenon encompassed by our analysis is that of "breathers" arising through Hopf bifurcation of traveling pulse solutions of singularly perturbed reaction–diffusion systems, as studied numerically in, e.g., [IIM, NM, IN]. These pulse solutions, satisfing the weighted norm condition (1.14) with $\eta = 0$, feature a singularly perturbed "front" and "back" layer structure. Under the bifurcation conditions (rPH) (verified in different cases either numerically, or via the singular perturbation structure), they are observed numerically to bifurcate to time-periodic "breathing" solutions in which the front and back layers oscillate back and forth. Our results give precise and rigorous justification also to these numerical observations.

**Discussion/open problems.** As mentioned above, one of our main motivations in this analysis was to understand the onset of galloping instability in detonation waves. More generally, this paper is intended as the initiation of a larger study of bifurcations associated with instability in detonation waves, including also transverse, or multi-dimensional instabilities of spinning or cellular type as described, e.g., in [FD, B], arising for detonations in cylindrical domains with finite (circular or rectangular, respectively) cross-section. As discussed formally in [Er.2, KS] in the context of the ZND model, these appear also to be relative Poincaré–Hopf bifurcations (with respect to axial translation), though of a more complicated sort.

Similarly as in the study of stability of detonations or shock waves, the difficulty (and interest) of these problems comes from the implicit presence of multiple scales, as discussed in [Z.1, Z.3, Z.4]. In particular, there is no spectral gap between essential spectrum and the one-dimensional translational zero-eigenvalue of the linearized operator about the traveling wave, so that the associated bifurcation problems are of a nonstandard, degenerate type. Indeed, even the extension of the present results from the class of equations considered by Sattinger to the full, reacting Navier–Stokes equations appears to involve considerable technical difficulty. In particular, since stable modes in this case are only time-algebraically decaying in any useful norm, [ZH], we do not necessarily expect the existence of a center manifold; rather, we hope to construct directly the periodic solutions involved in Poincaré–Hopf bifurcation at the same time that we establish their stability by combining high-order approximate solutions with detailed stability estimates of the type carried out in [Z.2, MaZ.2, Z.4] in the context of stability of traveling waves.

In short, we propose to rigorously verify a formal viscous weakly nonlinear approximation analogous to that carried out in [Er.2, MR, BMR] in the (inviscid) ZND setting. This appears to be a very interesting direction both for the viscous theory in itself and for possible implications regarding the standard (inviscid) ZND approximation.

**Plan of the paper.** We state the Center Manifold theorem (Proposition 2.1) and the Poincaré-Hopf bifurcation theorems in $\mathbb{R}^2$ and $\mathbb{R}^N$ (Theorems 2.4 and 2.6) in section 2, paying special attention to stability issues. In section 3, we describe how symmetry can be exploited to factorize a differential equation, and in effect reduce the space dimension by one, in a finite-dimensional setting (Lemma 3.3). This leads to Theorem 3.6 that describes a reduced center manifold, or a center manifold with symmetry, and to Theorem 3.8 that describes a relative Poincaré-Hopf bifurcation, or Poincaré-Hopf bifurcation with extra zero eigenvalue arising through group invariance. Next in section 4, we prove Theorems 1.3 and 1.2 that are infinite-dimensional analogues of Theorem 3.6 and 3.8. Finally, in section 4.3 we check that the assumption of Theorem 1.2 on essential spectrum holds for various model systems.

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## 2. The center manifold theorem and Poincaré-Hopf bifurcation in finite dimension.

**2.1. Center Manifold theory.** We begin by stating a basic version of the Center Manifold Theorem for finite-dimensional ODE, following the notes of Bressan [Bre]. Consider a finite-dimensional, autonomous ODE

$$X'(t) = F(X(t)), \qquad X,\, F \in \mathbb{R}^n. \tag{2.1}$$

Proposition 2.1 (Center Manifold Theorem [Bre]). *Let $F \in C^{k+1}$, $k \geq 1$, with $F(0) = 0$, and let $\Sigma_c$ denote the center subspace of $dF(0)$, defined as the range of the total eigenprojection $\Pi_c$ associated with all eigenvalues of real part zero. Then, for some $\delta > 0$, there exists a local $C^k$ invariant center manifold $\mathcal{C}$ with the following properties.*

*(i)There exists a $C^k$ function $\phi : \Sigma_c \to \mathbb{R}^n$ with $\Pi_c\phi(x) = x$, such that $\mathcal{C}$ is the image under $\phi$ of $\Sigma_c \cap B(0, \delta)$.*

*(ii) $\mathcal{C}$ contains all globally bounded solutions contained in a sufficiently small ball about the origin.*

*(iii) The manifold $\mathcal{C}$ is locally invariant under the flow of* (2.1)*, i.e.,* $X(0) \in \mathcal{C}$ *implies* $X(t) \in \mathcal{C}$ *for* $|t|$ *sufficiently small.*

*(iv)* $\mathcal{C}$ *is tangent to* $\Sigma_c$ *at the origin.*

*(v) (exponential approximation property) Given any solution* $X(t)$ *of* (2.1) *such that* $|X(t)| < \delta$ *as* $t \to \infty$*, there exists a trajectory* $t \to Y(t)$ *on the center manifold and* $\eta > 0$ *such that*

$$e^{\eta t}|X(t) - Y(t)| \to 0 \quad \text{as } t \to +\infty.$$

*Proof.* See [Bre], or Appendix A. □

One can rewrite (2.1) as

$$X' = AX + G(X), \tag{2.2}$$

where the right-hand side is a Taylor expansion of $F$, i.e., $G(0)$, $dG(0) = 0$. The first step of the proof consists in truncating the nonlinearity: let

$$\rho : x \mapsto \begin{cases} 1, \ |x| \le 1, \\ 0, \ |x| \ge 2, \end{cases}, \qquad G^\delta(x) := \rho\Big(\frac{|x|}{\delta}\Big) G(x),$$

and consider the equation

$$X' = AX + G^\delta(X). \tag{2.3}$$

There exists projectors $\pi_c, \pi_s, \pi_u$, such that

$$\mathbb{R}^n = \text{Range } \pi_s \oplus \text{Range } \pi_c \oplus \text{Range } \pi_u, \tag{2.4}$$

and positive numbers $\omega, \beta$, such that

$$\|e^{tA}\pi_s\| \le C_\beta e^{-\beta t}, \quad t \ge 0, \qquad \|e^{tA}\pi_u\| \le C_\beta e^{\beta t}, \quad t \le 0, \tag{2.5}$$

and

$$\|e^{tA}\pi_c\| \le C_\omega e^{\omega |t|}, \qquad t \in \mathbb{R}, \tag{2.6}$$

for some $C_\beta, C_\eta > 0$. A possible choice is to let $\pi_c := \Pi_c$ the projector over the center subspace of $A$, to let $\pi_s$ and $\pi_u$ be the spectral projectors of $A$ over the stable and unstable spaces, to let $\beta < \beta_0$, where $\beta_0$ is the spectral gap of $A$, defined as

$$\beta_0 := \min\{\Re\lambda, \lambda \in \sigma(A) \text{ and } \Re\lambda \ne 0\},$$

and to let $0 < \omega < \beta$.

The proofs of points (i) and (v) of Proposition 2.1 given by Bressan in [Bre] yield the following more precise statement.

Proposition 2.2. *For* $\delta > 0$ *sufficiently small, the truncated equations* (2.3) *possess a unique, global center manifold* $\hat{\mathcal{C}}$*. Moreover, given any solution* $X(t)$ *of* (2.3) *that grows sufficiently slowly in forward time:* $X \in \mathcal{S}^+_\eta$ *for some* $\omega < \eta < \beta$*, where* $\omega$ *and* $\beta$ *are as in* (2.5) *and* (2.6)*,*

$$\mathcal{S}^+_\eta := \{t \mapsto z(t) \in \mathbb{R}^n, \sup_{t \ge 0} e^{-\eta t}|z(t)| < \infty\},$$

*there exists a trajectory $t \to Y(t)$ on $\hat{\mathcal{C}}$, such that*

$$\sup_{t\geq 0} |X(t) - Y(t)| \leq C|X(0)|, \tag{2.7}$$

*where $C$ depends only on $G^\delta$, $\omega$ and $\eta$, and*

$$e^{\eta t}|X(t) - Y(t)| \to 0 \quad \text{as } t \to +\infty, \tag{2.8}$$

*Proof.* See Appendix A. ◻

Proposition 2.1 has the important implication that in studying existence of sufficiently small bounded solutions (e.g., traveling or standing waves, or periodic solutions) of (2.1), it suffices to consider the reduced flow along the center manifold $\mathcal{C}$. The following proposition shows that stability of these solutions also may be determined entirely through consideration of the reduced flow.

PROPOSITION 2.3. *Under the assumptions of Proposition 2.1, if $\Re\sigma(dF(0)) \leq 0$, then globally bounded solutions of (2.1) confined to a sufficiently small neighborhood of the origin are stable within the center manifold if and only if they are stable under perturbations in $\mathbb{R}^n$. If $\Re\sigma(dF(0)) \not\leq 0$, then all solutions on the center manifold originating from a sufficiently small neighborhood of the origin are unstable.*

*Proof.* The first follows by Proposition 2.2; see Appendix A. The second assertion follows by routine, essentially linear estimates. ◻

**2.2. Poincaré-Hopf bifurcation.** We start by stating a theorem that describes apparition of periodic orbits and discusses their stability, in a simple $2 \times 2$ setting. This paragraph is inspired, sometimes literally, by chapter 11 of [HK].

Let $F$ be a $C^3$ vector field in $\mathbb{R}^2$, depending on the scalar parameter $\epsilon$, such that for all $\epsilon$, $F(\epsilon, 0) = 0$. Consider the differential equation in $\mathbb{R}^2$ :

$$X'(t) = F(\epsilon, X(t)). \tag{2.9}$$

THEOREM 2.4 (Poincaré-Hopf bifurcation in $\mathbb{R}^2$). *Assume that the spectrum of $\partial_x F(\epsilon, 0)$ consists of two conjugate eigenvalues $\lambda(\epsilon), \bar{\lambda}(\epsilon) = \gamma(\epsilon) \pm i\tau(\epsilon)$, such that*

$$\gamma(0) = 0, \qquad \tau(0) \neq 0, \qquad (d\gamma/d\epsilon)(0) > 0. \tag{2.10}$$

*Then, for any sufficiently small $a > 0$, there exists a unique nontrivial periodic orbit $X_a$ of (2.9) with initial radius $a : |X_a(0)| = a$, and parameter value $\epsilon(a)$. The function $a \to \epsilon(a)$ is $\mathcal{C}^1$, with $d\epsilon/da(0) = 0$; if $d\epsilon/da > 0$, $X_a$ is orbitally asymptotically stable; if $d\epsilon/da < 0$, it is unstable.*

*Proof.* The equation is

$$X' = \begin{pmatrix} \gamma(\epsilon) & \tau(\epsilon) \\ -\tau(\epsilon) & \gamma(\epsilon) \end{pmatrix} X + G(\epsilon, X),$$

where $G(\epsilon, 0) \equiv 0, \partial_x G(\epsilon, 0) \equiv 0$. It follows from the assumptions $F \in C^3$, $\gamma(0) = 0$, $\tau(0) \neq 0$, that $\gamma$ and $\tau$ are $C^2$ in $\epsilon$, in a neighbourhood of 0. In polar coordinates:

$$\begin{aligned} r'(t) &= \gamma(\epsilon) r + R(\epsilon, r, \cos\theta, \sin\theta), \\ \theta'(t) &= \tau(\epsilon) + \Theta(\epsilon, r, \cos\theta, \sin\theta), \end{aligned} \tag{2.11}$$

where $R$ and $\Theta$ have simple expressions in terms of $G$. As $\tau(\epsilon) \neq 0$, for small $\epsilon$, the second equation of (2.11) is locally invertible. The system thus reduces to a scalar equation with periodic coefficients:

$$r'(\theta) = \frac{\gamma(\epsilon)}{\tau(\epsilon)} r(\theta) + \mathcal{R}(\epsilon, r(\theta), \theta). \tag{2.12}$$

Consider an orbit $r$ of (2.12), with initial datum $r(0) = a$. The orbit $r$ is $2\pi$–periodic if and only if $a$ is a fixed point of the Poincaré map

$$\Pi(\epsilon, a) := e^{2\pi(\gamma/\tau)(\epsilon)} a + \int_0^{2\pi} e^{(2\pi - \theta')(\gamma/\tau)(\epsilon)} \mathcal{R}(\epsilon, r(\theta'), \theta')\, d\theta'. \tag{2.13}$$

As $\mathcal{R}(\epsilon, 0, \theta) \equiv 0$, $\partial_r \mathcal{R}(\epsilon, 0, \theta) \equiv 0$, properties inherited from the corresponding properties of $G$, the condition that guarantees the existence of a unique $\epsilon(a)$ such that $\Pi(\epsilon(a), a) = a$, for small $a > 0$, is $d(\gamma/\tau)/d\epsilon(0) \neq 0$, by the Implicit Function Theorem applied to the reduced equation $0 = g(\epsilon(a), a)$,

$$g(\epsilon(a), a) := \Pi(\epsilon, a)/a - 1.$$

(From $\mathcal{R}(\epsilon, 0, \theta) \equiv 0$, $\partial_r \mathcal{R}(\epsilon, 0, \theta) \equiv 0$, we obtain $g(0,0) = 0$, $g_\epsilon(0,0) = 2\pi d(\gamma/\tau)/d\epsilon(0)$.) This condition holds by (2.10). Then $r(\epsilon(a), \theta)$ defines a $2\pi$–periodic orbit of (2.12) with initial datum $a$, and

$$\{(r(\epsilon(a), \theta)\cos\theta, r(\epsilon(a), \theta)\sin\theta), \theta \geq 0\}$$

is a periodic orbit of (2.9).

Solution $r_a$, hence $X_a$, is stable if $\left|\partial_a \Pi(\epsilon, a)|_{\epsilon = \epsilon(a)}\right| < 1$ and unstable if $\left|\partial_a \Pi(\epsilon, a)|_{\epsilon = \epsilon(a)}\right| < 1$. Computing $\partial_a \Pi(0,0) = \partial_a(ag + a)(0,0) = 1$, we see that $\partial_a \Pi(0,0) > -1$ for $a$ sufficiently small, and so stability is determined by the sign of $1 - \partial_a \Pi(\epsilon, a)|_{\epsilon = \epsilon(a)}$, or

$$1 - \partial_a(ag + a)|_{\epsilon = \epsilon(a)} = a g_a|_{\epsilon = \epsilon(a)}.$$

Recalling that $d\epsilon/da = -g_a/g_\epsilon$ by the Implicit Function Theorem, we find that the righthand side may be expressed as $-a(d\epsilon/da) g_\epsilon|_{\epsilon = \epsilon(a)}$. Since also, $g_\epsilon(0,0) = 2\pi \frac{d(\gamma/\tau)}{d\epsilon}(0) > 0$, so that $g_\epsilon > 0$ for $a$ sufficiently small, and since $a > 0$, we find as claimed that, for $a > 0$ sufficiently small, the sign of $1 - \partial_a \Pi(\epsilon, a)|_{\epsilon = \epsilon(a)}$ and thus stability is determined by the sign of $d\epsilon/da$. For an explicit expression of the stability condition in terms of $F$, see [HK].

Finally, note that (2.12) is valid also for $r < 0$, and smooth on all $r$. Since the map $(a, \epsilon) \to \hat{a} = -a + O(a^2)$ advancing the flow by $\pi$ evidently preserves periodic orbits, we have

$$\epsilon(a) = \epsilon(\hat{a}(a, \epsilon(a)))$$

and so $d\epsilon/da(0) = (d\epsilon/da)(0)(\partial\hat{a}/\partial a)(0,0) = -d\epsilon/da(0)$, from which we may conclude that $d\epsilon/da(0) = 0$. □

Remarks 2.5. 1. Stability of the Poincaré map corresponds to orbital stability of the periodic orbit. Thus there is no contradiction with the stability assertion in the above theorem and the second assertion of Proposition 2.3.

2. If $\frac{d\gamma}{d\epsilon}(0) < 0$, then the periodic orbit is unstable if $d\epsilon/da > 0$ and orbitally asymptotically stable if $d\epsilon/da < 0$.

3. The sign of $d^2\epsilon/da^2(0)$ is numerically computable; it can be expressed in terms of derivatives of $F$ up to order three ([HK], pp. 359-360).

The relation of stability to sign of $d\epsilon/da$ yields for each fixed value of $\epsilon$ the expected phase portrait of alternating stable and unstable solutions; see Figure 1.

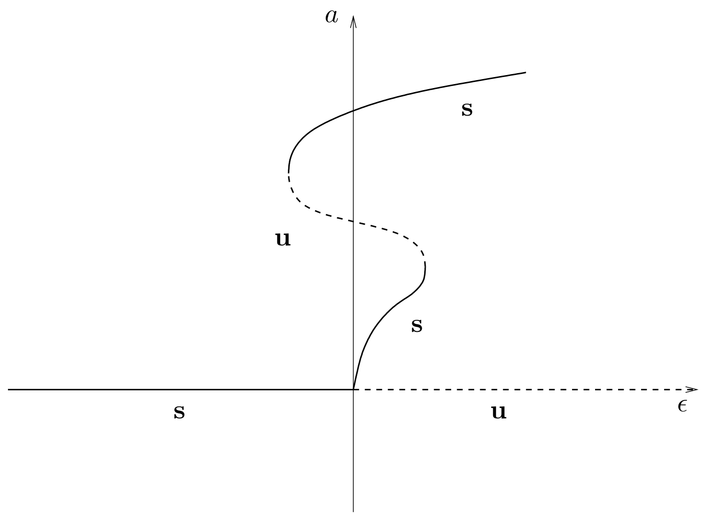


FIG. 1. *Alternation of stable/unstable solutions.*

Next we combine Proposition 2.1 and Theorem 2.2 to obtain the well-known Poincaré-Hopf bifurcation theorem in $\mathbb{R}^n$.

Let $F$ be a $C^4$ vector field in $\mathbb{R}^n$, depending on the scalar parameter $\epsilon$, such that for all $\epsilon$, $F(\epsilon, 0) = 0$. Consider the differential equation in $\mathbb{R}^n$ :

$$X'(t) = F(\epsilon, X(t)). \tag{2.14}$$

Let $\Psi$ be the flow of $F$. For any given $(t_0, X_0) \in \mathbb{R} \times \mathbb{R}^n$, the mapping

$$t \in I_{t_0,X_0} \mapsto \Psi(t, X_0) \in \mathbb{R}^n,$$

solves (2.14) with the initial datum $X(t_0) = X_0$, over some open time interval $I_{t_0,X_0}$.

THEOREM 2.6 (Poincaré-Hopf bifurcation in $\mathbb{R}^n$). *Assume that the spectrum of $\partial_x F(\epsilon, 0)$ is strictly contained in $\{z, \Re z \neq 0\}$ for small $\epsilon$, except for two conjugate eigenvalues $\lambda(\epsilon), \bar\lambda(\epsilon) = \gamma(\epsilon) \pm i\tau(\epsilon)$ that cross the imaginary axis as in (2.10). Then, there exists a neighborhood $\mathcal{U}$ of the origin in a submanifold of $\mathbb{R}^n$, such that for all $X_0 \in \mathcal{U}$, there exists a unique periodic orbit of (2.14) with initial datum $X_0$, which stability can generically be determined through a computable condition involving $F$ and its derivatives up to order three. Moreover, the only periodic orbits of $F$ in a neighbourhood of the origin are the above.*

*Proof.* First, extend $F$ to a vector field in $\mathbb{R}^{n+1}$ by letting $\tilde F := (0, F)$, and consider the differential equation in $\mathbb{R}^{n+1}$, $\tilde X' := (\epsilon, X)' = \tilde F(\epsilon, X)$. By Proposition 2.1, there exists a three-dimensional center manifold $\mathcal{C}$ tangent to the flow $\tilde\Psi$ of $\tilde F$ in a neighbourhood of the origin in $\mathbb{R}^{n+1}$. The manifold $\mathcal{C}$ is the graph of a $C^k$ map $\phi$ over a neighborhood of the origin in $\tilde{\Sigma}_C$, the three-dimensional vector space spanned by the eigenvectors associated with $\lambda(0)$ and $\bar{\lambda}(0)$, and the $\epsilon$–axis. Let $\tilde{\Pi}_C$ be the projection onto $\tilde{\Sigma}_C$. Define the reduced vector field $F^\sharp$ on $\tilde{\Sigma}_C$ by $F^\sharp := \tilde{\Pi}_C \tilde{F}(\phi)$. Because $\mathcal{C}$ is locally invariant under $\tilde{\Psi}$, the flow of $F^\sharp$ is

$$\Psi^\sharp(t, \tilde{X}_0) := \tilde{\Pi}_C \tilde{\Psi}(t, \phi(\tilde{X}_0)).$$

Indeed, one has $\partial_t \Psi^\sharp(t, \tilde{X}_0) = \tilde{\Pi}_C \partial_t \tilde{\Psi}(t, \phi(\tilde{X}_0)) = \tilde{\Pi}_C \tilde{F} \Psi(t, \Phi(\tilde{X}_0))$, and, as $\Psi(t, \phi(\tilde{X}_0)) \in \mathcal{C}$ for small $|t|$ :

$$\Psi(t, \phi(\tilde{X}_0)) = \phi \tilde{\Pi}_C \Psi(t, \phi(\tilde{X}_0)),$$

hence

$$\partial_t \Psi^\sharp(t, \tilde{X}_0) = \tilde{\Pi}_C \tilde{F} \phi \tilde{\Pi}_C \Psi(t, \phi(\tilde{X}_0)) = \tilde{\Pi}_C \tilde{F} \phi \Psi^\sharp(t, \tilde{X}_0) = F^\sharp \Psi^\sharp(t, \tilde{X}_0).$$

We now prove that the reduced field $F^\sharp$ in $\tilde{\Sigma}_C$ inherits the spectral properties of $\tilde{F}$. First, note that $(\epsilon, 0)$ is an equilibrium of $\tilde{F}$, hence it lies on the center manifold: $\phi(\epsilon, 0) = (\epsilon, 0)$. Second, $\phi \tilde{\Pi}_C$, in restriction to $\mathcal{C}$, is the identity map. Hence $d\phi(\tilde{X})\tilde{\Pi}_C = \mathrm{Id}$, for $\tilde{X}$ close to the origin, where Id is the identity in $T_{\tilde{X}}\mathcal{C} = \tilde{\Sigma}_C$. This implies

$$\begin{aligned} d\phi(\epsilon, 0) dF^\sharp(\epsilon, 0) &= d\phi(\epsilon, 0) \tilde{\Pi}_C d\tilde{F}(\phi(\epsilon, 0)) d\phi(\epsilon, 0) \\ &= d\tilde{F}(\epsilon, 0) d\phi(\epsilon, 0), \end{aligned}$$

hence

$$\partial_X \phi(\epsilon, 0) \partial_X F^\sharp|_{\Sigma_C}(\epsilon, 0) = \partial_X F(\epsilon, 0) \partial_X \phi(\epsilon, 0),$$

where $\Sigma_C$ is the subspace of $\tilde{\Sigma}_C$ spanned by the eigenvectors associated with $\lambda(0)$ and $\bar{\lambda}(0)$. It follows that the spectrum of $\partial_x F^\sharp|_{\Sigma_C}(\epsilon, 0)$ is contained in the spectrum of $\partial_x \tilde{F}(\epsilon, 0)$ and that $\sigma(\partial_X F^\sharp|_{\Sigma_C}(0, 0)) = \{\lambda(0), \bar{\lambda}(0)\}$, and eventually that $\sigma(\partial_X F^\sharp|_{\Sigma_C}(\epsilon, 0)) = \{\lambda(\epsilon), \bar{\lambda}(\epsilon)\}$, for small $\epsilon$.

By Theorem 2.2, if $X_C \in \Sigma_C$ is close enough to the origin, then for small $\epsilon$, $\tilde{X}_C = (\epsilon, X_C)$ is included in a closed orbit of $F^\sharp$. Then $t \to \phi(\Psi^\sharp(t, \tilde{X}_C))$ defines a closed orbit of $\tilde{F}$. Conversely, if $X_0 \in \mathbb{R}^n$ is close enough to the origin and $\epsilon$ sufficiently small, if $\tilde{X}_0 = (\epsilon, X_0)$ belongs to a closed orbit of $\tilde{F}$, then necessarily $\tilde{X}_0 \in \mathcal{C}$, as $\mathcal{C}$ contains all globally bounded orbits, and $t \to \tilde{\Pi}_C \tilde{\Psi}(t, \tilde{X}_0)$ is the closed orbit of $F^\sharp$ in $\tilde{\Sigma}_C$ with initial datum $\tilde{\Pi}_C \tilde{X}_0$. This gives existence and uniqueness of periodic orbits of $F$, for small $\epsilon$, in a neighbourhood of the origin in $\mathbb{R}^n$. □

Proposition 2.7. *Under the assumptions of Proposition 2.6, if the spectrum of $\partial_x F(0, \epsilon)$ is strictly contained in $\{z, \Re z < 0\}$ for small $\epsilon$, except for the two eigenvalues $\lambda(\epsilon), \bar{\lambda}(\epsilon)$, then for small $\epsilon$, periodic orbits confined to a sufficiently small neigborhood of the origin on the center manifold are orbitally stable (resp. asymptotically orbitally stable) under perturbations within the center manifold if and only if they are stable (resp. asymptotically orbitally stable) under perturbations in $\mathbb{R}^n$.*

*Proof.* This follows from Proposition 2.3, applied to the vector field $\tilde{F}(\epsilon, X) = (F(\epsilon, X), 0)$. Indeed, the assumption in Proposition 2.3 deals only with spectrum at the bifurcation point $(0, 0)$. □

Theorem 2.2 asserts that stability of closed orbits in $\tilde{\Sigma}_C$ is determined by a computable condition involving derivatives of $F^\sharp$ up to order three, which can be expressed in terms of derivatives of $F$ up to order three. Indeed, the first terms of the Taylor expansion of $\phi$, whose existence is given by the Implicit Function Theorem, can be computed in terms of derivatives of $F$. An explicit expression of this stability condition in terms of $F$ is given in [MM], section 4.

**3. Ordinary differential equations with symmetries and relative Poincaré–Hopf bifurcation.** We next discuss relative bifurcations and ODE with symmetries in a way suited for our later analysis. For a general reference, see, e.g. [A].

**3.1. General reduction process.** Consider (2.1), where $F \in C^k$, and let $\Psi$ be the flow of $F$.

ASSUMPTION 3.1. *Suppose that there exists an additive group* $\{\Phi_\alpha\}_{\alpha\in\mathbb{R}}$*,*

$$\Phi_\alpha \cdot \Phi_\beta = \Phi_{\alpha+\beta},$$

*of* $\mathcal{C}^1$ *transformations* $\mathbb{R}^n \to \mathbb{R}^n$*, with* $\alpha \mapsto \Phi_\alpha \in \mathcal{C}^1$*, that leaves* $\Psi$ *invariant, that is*

$$\Phi_\alpha(\Psi(t, t_0, X_0)) = \Psi(t, t_0, \Phi_\alpha(X_0)),$$

*for any* $\alpha, t \in I_{t_0,X_0}$*, and such that*

$$\frac{\partial \Phi_\alpha}{\partial \alpha}\Big|_{\alpha=0}(0) \neq 0. \tag{3.1}$$

REMARK 3.2. As a consequence of invariance, $\Phi_\alpha X(t)$ is a solution of (2.1) if and only if $X(t)$ is a solution.

Let us define coordinates $X = (Y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $\frac{\partial \Phi_\alpha}{\partial \alpha}\big|_{\alpha=0}(0)$ is transverse to the hyperplane $z = 0$.

LEMMA 3.3. *Under Assumption* 3.1*, in a neighbourhood of the origin, the dynamics of* (2.1) *can be locally described in terms of a vector field in* $\mathbb{R}^{n-1}$*, specifically,* (3.3)–(3.5) *below.*

That is, group invariance effectively reduces the dimension of the phase space by one.

*Proof.* Consider the map $T : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^n$, defined as

$$T\begin{pmatrix} Y \\ \alpha \end{pmatrix} := \Phi_\alpha \begin{pmatrix} Y \\ 0 \end{pmatrix}. \tag{3.2}$$

As $\Phi_0 = \mathrm{Id}$,

$$dT(0) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & * \\ 0 & a \end{pmatrix},$$

where $a$ is the component of $\frac{\partial \Phi_\alpha}{\partial \alpha}\big|_{\alpha=0}(0)$ in the $z$ direction, which, by Assumption 3.1, does not vanish. Thus $T$ is a local diffeomorphism on $\mathcal{V}$, a neighbourhood of the origin in $\mathbb{R}^n$.

Consider a trajectory $t \to X(t)$ of (2.1), such that for some $t_0$, $X(t_0) \in \mathcal{V}$. There exist $\mathcal{C}^\infty$ maps $t \mapsto Y(t) \in \mathbb{R}^{n-1}$ and $t \mapsto \alpha(t) \in \mathbb{R}$, such that

$$X(t) = T\begin{pmatrix} Y(t) \\ \alpha(t) \end{pmatrix} = \Phi_{\alpha(t)}\begin{pmatrix} Y(t) \\ 0 \end{pmatrix}.$$

in a neighbourhood of $t_0$. Fix now a time $\tilde{t}$ close to $t_0$ and let $\alpha(\tilde{t}) = \alpha_0$. Consider

$$Z(t) := \Phi_{-\alpha_0}\Phi_{\alpha(t)}\begin{pmatrix} Y(t) \\ 0 \end{pmatrix} = \Phi_{\alpha_0}X(t).$$

The group property implies

$$Z(t) = T\begin{pmatrix} Y(t) \\ \alpha(t) - \alpha_0 \end{pmatrix},$$

so that

$$Z'(\tilde{t}) = dT\begin{pmatrix} Y(\tilde{t}) \\ 0 \end{pmatrix}\begin{pmatrix} Y'(\tilde{t}) \\ \alpha'(\tilde{t}) \end{pmatrix}.$$

Besides, by Remark 3.2, $Z$ solves (2.1), hence

$$Z'(\tilde{t}) = F\begin{pmatrix} Y(\tilde{t}) \\ 0 \end{pmatrix}.$$

For $Y$ close enough to the origin, by continuity, $dT(Y,0)$ is invertible. We obtain

$$\begin{pmatrix} Y'(\tilde{t}) \\ \alpha'(\tilde{t}) \end{pmatrix} = dT\begin{pmatrix} Y(\tilde{t}) \\ 0 \end{pmatrix}^{-1} F\begin{pmatrix} Y(\tilde{t}) \\ 0 \end{pmatrix},$$

thus proving the lemma. ▯

Introducing notations

$$\frac{\partial \Phi_\alpha}{\partial \alpha}\Big|_{\alpha=0}\begin{pmatrix} Y \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{v}(Y) \\ v_n(Y) \end{pmatrix},$$

where, from Assumption 3.1, $v_n(Y) \neq 0$ for $Y$ small enough, we have

$$dT\begin{pmatrix} Y \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \tilde{v}(Y) \\ 0 & v_n(Y) \end{pmatrix},$$

and

$$dT\begin{pmatrix} Y \\ 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & -\frac{\tilde{v}(Y)}{v_n(Y)} \\ 0 & \frac{1}{v_n(Y)} \end{pmatrix}.$$

Now if we let $F = (\tilde{F}, F_n)$, then the equation for $Y$ is

$$Y'(t) = G(Y(t)), \tag{3.3}$$

with

$$G(Y) := \tilde{F}(Y,0) - \frac{\tilde{v}(Y)}{v_n(Y)}F_n(Y,0). \tag{3.4}$$

Besides, the shift satisfies

$$\alpha'(t) = h(Y(t)) := \frac{F_n(Y(t), 0)}{v_n(Y(t))}. \tag{3.5}$$

For similar, but slightly more general formulae, see [O]. The explicit, triangular form derived here (special to the additive case) will be quite helpful in the more complicated PDE analysis of Section 4.

EXAMPLES 3.4. 1. Linear ODE are invariant under the additive group of scaling transformations $\Phi_\alpha X := e^\alpha X$. In this case, the phase space reduction we describe corresponds to projectivation, and the reduced variable $Y$ defined in (3.2) to standard Plücker coordinates. Consider indeed the linear equation

$$X'(t) = AX(t), \tag{3.6}$$

and trajectories in a neighbourhood of $X_0 := (0, \dots, 0, 1)$. The change of variable (projectivation)

$$X = (x_1, \dots, x_n) \longrightarrow \Big(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, 1\Big),$$

is a diffeomorphism in a neighborhood of $X_0$. Consider the additive group $\Phi_\alpha(X) := e^\alpha X$. Then equation (3.6) is invariant under the action of $\Phi_\alpha$ and $\frac{\partial \Phi_\alpha}{\partial \alpha}|_{\alpha=0}(X_0) = -X_0 \neq 0$. The above lemma asserts that in a neighbourhood of $X_0$, (3.6) reduces to a nonlinear equation in $Y := (\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n})$, and that the time evolution of $\alpha := -\ln x_n$ depends on $Y$ only. Introducing notations for the block decomposition of $A$ :

$$A = \begin{pmatrix} A_{n-1} & a_{n-1} \\ l_{n-1} & a_n \end{pmatrix},$$

where $A_{n-1} \in \mathcal{L}(\mathbb{R}^{n-1})$, one can directly check that

$$Y' = A_{n-1}Y + a_{n-1} - Y(l_{n-1}Y + a_n), \tag{3.7}$$

and

$$\alpha' = l_{n-1}Y + a_n. \tag{3.8}$$

(3.7) and (3.8) correspond to (3.4) and (3.5). See, e.g., [AGJ, GZ] for related calculations in projectivized coordinates.

2. In cylindrical coordinates $X = (r, \theta, z)$, we may express rotations in the $r$-$\theta$ plane via the additive group $\Phi_\alpha X := (r, \theta + \alpha, z)$, $\alpha \in \mathbb{R}$. In these coordinates, invariance of $X' = F(X)$ with respect to $\{\Phi_\alpha\}$ means that $F = F(r, z)$ is independent of $\theta$, and, taking $Y = (r, z)$, we obtain the simple reduced equations $Y' = G(r, z)$, $\alpha' = h(r, z)$.

REMARK 3.5. 1. Though we assumed a one-parameter group invariance, the above calculations extend readily to multi-parameter additive groups.

2. In the case of invariance under a general, not necessarily additive Lie group, similar but more complicated formulae apply. In particular, we still obtain reduced dynamics (3.3) driving the evolution of $\alpha$; however, the $\alpha$-equation in general depends

on $\alpha$. (Think, for example, of $SL(3)$, for which there does not exist an additive covering group as in Example 3.4.2.)

As a simple application of the reduced flow, consider a rotationally invariant system of ODE in the plane undergoing a Poincaré–Hopf bifurcation from a steady solution $X \equiv 0$. Working in polar coordinates $(r, \theta)$ we obtain

$$r' = f(r, \epsilon), \quad \theta' = g(r, \epsilon),$$

$f(\cdot, \epsilon)$ odd, with $f(0, \epsilon) = 0$, $\partial_r f(0, \epsilon) = 0$, $\partial_\epsilon \partial_r f(0, 0) \neq 0$, and $\partial_r g(0, 0) = \tau \neq 0$. Restricting attention to the reduced flow $r' = f(r, \epsilon)$, we find that the planar, Poincaré–Hopf bifurcation reduces to a *scalar pitchfork bifurcation*; indeed, the Poincaré–Hopf bifurcation diagram of Figure 1 may be recognized as the positive half of the scalar pitchfork bifurcation diagram.

For example, in the simple case

$$r' = \epsilon r - \beta r^3,$$

$r \equiv 0$ bifurcates at $\epsilon = 0$ to a family of periodic solutions $r^\epsilon \equiv a$, $\theta = g(a)t$, with $\epsilon(a) = \beta a^2$, stable if $0 > \partial_r f(a, \epsilon(a)) = -2\beta a^2$, or $\beta > 0$, and unstable if $0 < \partial_r f(a, \epsilon(a)) = -2\beta a^2$, or $\beta < 0$.

**3.2. Center Manifolds with symmetry.** A recurring issue for ODE with symmetry is the construction of center manifolds respecting the underlying symmetry. For, the usual construction by fixed-point iteration described in Section 2.1 and Appendix A, based on linearization about a single rest point, is typically not compatible with a group symmetry. Indeed, the usual introduction of an artificial cutoff on the nonlinear term typically destroys the group symmetry, guaranteeing that the constructed center manifold does *not* respect the symmetry.

A simple solution is to work instead in the reduced coordinates of Section 3.1, for which the group symmetry is in effect built in, constructing a center manifold for the reduced flow (3.3), with $\alpha$-dynamics determined through (3.5). See, for example, the center-manifold analysis of [FreS] for linear eigenvalue ODE, carried out in projectivized coordinates as described in Remark 3.4.1 above.

Consider (2.1), where $F \in C^{k+1}$ and $F(0) = 0$, and let $\Psi$ be the flow of $F$. Assume that Assumption 3.1 holds. Assume in addition that the map $X \mapsto \frac{d\Phi}{d\alpha}|_{\alpha=0}(X)$ is $C^{k+1}$. We use Lemma 3.3 and the reduced equations (3.3) with a vector field $G$ defined in (3.4). We use global coordinates $X = (Y, z) = Y \oplus z e_0$, where $e_0$ is not orthogonal to $\frac{d\Phi}{d\alpha}|_{\alpha=0}(0)$, and local coordinates induced by the symmetry $X = T(Y, \alpha)$ in a neighborhood of the origin ($T$ as in (3.2)). Let $\Sigma_c^r$ denote the center subspace of $dG(0)$.

Then we have the following theorem:

THEOREM 3.6. *For some $\delta > 0$, there exists a local $C^k$ invariant center manifold $\mathcal{C}_r$ that is embedded in a hyperplane of $\mathbb{R}^n$, and such that*

*(i) The manifold $\mathcal{C}_r$ is a graph over $\Sigma_c^r \cap B(0, \delta)$.*

*(ii) The manifold $\mathcal{C}_r$ is locally invariant by the flow of* (2.1)*, that is: if $X_0 = T(Y_0, \alpha_0)$ is such that $Y_0 \in \mathcal{C}_r$, then for $|t|$ sufficiently small, $X(t) = T(Y(t), \alpha(t))$, and $Y(t) \in \mathcal{C}_r$.*

*(iii) If $X$ is a globally bounded solution in a neighborgood of the origin, then its first coordinate $Y$ belongs to $\mathcal{C}_r$.*

*(iv) The center manifold is tangent to $\Sigma_c^r$ at the origin.*

*(v) Given any solution $X(t)$ of (2.1) that is eventually confined to a suitably small neighborhood of the origin in $\mathbb{R}^n$, there exists a trajectory $t \to Y_r(t)$ on $\mathcal{C}_r$ and a fixed asymptotic shift $\Delta\alpha \in \mathbb{R}$, such that*

$$e^{\eta t}|X(t) - T(Y_r(t), \alpha(Y_r(t)) + \Delta\alpha)| \to 0, \tag{3.9}$$

*as $t \to \infty$, where $\alpha(Y_r)$ is defined by (3.5).*

*Proof.* We use Lemma 3.3. Point $(i)$ follows from Proposition 2.1 applied to the reduced equations (3.3). With Proposition 2.1 and Lemma 3.3, the proofs of $(ii)$, $(iii)$ and $(iv)$ are straightforward. If $X = T(Y, \alpha(Y))$ is small for large $t$, then $Y$ is small as well, hence, by point $(v)$ of Proposition 2.1, $Y$ is exponentially approximated by some trajectory $Y_r$ on $\mathcal{C}_r$. Now, with the notations of section 3.1,

$$\alpha(Y(t)) - \alpha(Y_r(t)) = \Delta\alpha - \int_t^\infty (h(Y(t')) - h(Y_r(t')))\, dt',$$

where

$$\Delta\alpha := a(Y(0)) - \alpha(Y_r(0)) + \int_0^\infty (h(Y(t')) - h(Y_r(t')))\, dt'.$$

As $h$ is Lipschitz in a neighborhood of the origin, there exists $\eta > 0$ (strictly smaller than the spectral gap of $dF(0)$) such that

$$e^{\eta t}|\alpha(Y(t)) - \alpha(Y_r(t)) - \Delta\alpha| \to 0, \qquad \text{as } t \to \infty,$$

and (3.9) follows by regularity of $T$. ∎

For an equation (2.1) with symmetry, one can define orbital stability of the orbits as in the case of periodic orbits. That is, orbital stability of $X$ as a solution of (2.1) means stability of its first coordinate $Y$ as a solution of (3.3), and asymptotic orbital stability of $X$ means asymptotic stability of $Y$ and convergence of the shift $\alpha$ as $t \to \infty$.

PROPOSITION 3.7. *Under the assumptions of Theorem 3.6, if $\Re\sigma(dG(0)) \leq 0$, then globally bounded solutions $X = T(Y, \alpha)$ of (3.3) confined to a sufficiently small neighborhood of the origin are orbitally stable (resp. asymptotically orbitally stable) if and only if $Y$ is stable (resp. asymptotically stable) as solution of the reduced ODE on the center manifold $\mathcal{C}_r$.*

*Proof.* Let $Y$ be a trajectory of (3.3) that is stable within $\mathcal{C}_r$ and contained in a small neighborhood of the origin. By Proposition 2.3, $Y$ is also stable with respect to perturbations in $\mathbb{R}^{n-1}$. That is, the corresponding trajectory $X$ of (2.1) is orbitally stable. If $Y$ is small enough and asymptotically stable, then by point (iv) of Theorem 3.6, the perturbation converges exponentially, hence the shift is eventually convergent, and the corresponding trajectory in $\mathbb{R}^n$ is orbitally asymptotically stable. ∎

**3.3. Application to periodic orbits.** We now combine Theorem 2.6 with Lemma 3.3.

Consider (2.14), where $F$ be a $C^{k+1}$ vector field in $\mathbb{R}^n$, depending on the scalar parameter $\epsilon$, such that for all $\epsilon$, $F(\epsilon, 0) = 0$. Let $\Psi$ denote the flow of $F$.

Assume that the spectrum of $\partial_x F(\epsilon, 0)$ is contained in $\{z, \Re z \neq 0\}$ for small $\epsilon$, except for the simple eigenvalue 0 and two conjugate eigenvalues $\lambda(\epsilon), \bar{\lambda}(\epsilon) = \gamma(\epsilon) \pm i\tau(\epsilon)$, such that (2.10) holds. Assume in addition that Assumption 3.1 holds, and that $\frac{\partial \Phi_\alpha}{\partial \alpha}|_{\alpha=0}(0) =: e_0$ generates the Kernel of $\partial_x F(\epsilon, 0)$. This assumption is natural in the context of the partial differential equations (1.2), see section 4. We use again global coordinates $X = (Y, z) = Y \oplus ze_0$, and local coordinates induced by the symmetry $X = T(Y, \alpha)$ in a neighborhood of the origin ($T$ as in (3.2)). With Lemma 3.3, (2.14) reduces to

$$Y' = G(Y, \epsilon) \in \mathbb{R}^{n-1}, \qquad \alpha' = h(Y, \epsilon) \in \mathbb{R}. \tag{3.10}$$

Then we have the following theorem:

THEOREM 3.8 (relative Poincaré-Hopf bifurcation). *For any small $a > 0$, there exists an orbit $X$ of* (2.14) *for the value $\epsilon(a)$ of the parameter such that $X(0) = T(Y(0), \alpha(0))$, $|Y(0)| = a$, $X = T(Y, \alpha)$ where $Y$ is periodic and $\alpha(t) = \alpha^0 t + \beta(t)$, where $\alpha^0$ is a constant drift and $\beta$ is periodic.*

*Proof.* As $\frac{\partial \Phi_\alpha}{\partial \alpha}|_{\alpha=0}(0)$ generates $\ker \partial_x F(\epsilon, 0)$, for all $\epsilon$, the spectrum of $\partial_x G(0, \epsilon))$ coincides with the spectrum $\sigma(\partial_x F(0, \epsilon)$, except for the eigenvalue 0. With Theorem 3.6, the equations (2.14) possess a 3-dimensional reduced center manifold, where two directions correspond to the neutral eigenvalues $\pm i\tau(0)$ of $\partial_x F(0, 0)$, and one direction corresponds to variation of the parameter $\epsilon$. Now with Theorem 2.2, for $a$ small enough, there exists a periodic orbit $Y$ of the reduced equations (3.10) issued from $Y(0)$, with $|Y(0)| = a$, for the value $\epsilon(a)$ of the parameter. Let $\tau$ be a period. Remark that $(Y(t), 0) = \Phi_{\alpha(t)} X(t)$, and that $\alpha = \alpha(Y, \epsilon)$ is such that

$$\alpha(Y(t), \epsilon(a)) - \frac{t}{\tau} \int_0^\tau h(Y(t'), \epsilon(a)) dt'$$

is periodic. ∎

For the orbits of (2.14) described above, one can define orbital stability as orbital stability of the periodic $Y$ coordinate, and asymptotic orbital stability as asymptotic orbital stability of $Y$ and convergence of the shift $\alpha$ as $\tau \to \infty$. Then the following proposition is a straighforward consequence of Proposition 2.7:

PROPOSITION 3.9. *Under the assumptions of Theorem 3.6, if the spectrum of $\partial_x F(0, \epsilon)$ is contained strictly contained in $\{z, \Re z < 0\}$ for small $\epsilon$, except for the two eigenvalues $\lambda(\epsilon), \bar{\lambda}(\epsilon)$ and the eigenvalue 0, then for small $\epsilon$, bounded solutions $X = T(Y, \alpha)$ of* (3.3)*, with $Y$ periodic, that are confined to a sufficiently small neighborhood of the origin are orbitally stable (resp. asymptotically orbitally stable) if and only if $Y$ is orbitally stable (resp. asymptotically stable) as a periodic orbit of the reduced ODE on the center manifold $\mathcal{C}_r$.*

*Proof.* Let $Y$ be an orbitally periodic orbit on the reduced center manifold. Then with Proposition 2.7, $Y$ is stable under perturbations in all of $\mathbb{R}^{n-1}$, for $\epsilon$ held fixed, and the corresponding orbit $X$ of (2.14) is orbitally stable. If $Y$ is asymptotically orbitally stable, then the perturbation converges exponentially by point (iv) of Theorem 3.6, hence the shift is eventually convergent, and $X$ is asymptotically orbitally stable. ∎

**4. Applications to stability of traveling waves for partial differential equations.** We now turn to the analysis of the PDE bifurcation problem (1.6), with associated group invariance

$$\tilde\Phi_\alpha(\tilde u) := \tilde u(x+\alpha, t). \tag{4.1}$$

Following the approach of Section 3, introduce the related group invariance

$$\begin{aligned}\Phi_\alpha(u,\epsilon) &:= \big(\tilde\Phi_\alpha(u+\bar u^0) - \bar u^0, \epsilon\big)\\ &= \big(u(x+\alpha) + (\bar u^0(x+\alpha) - \bar u^0(x)), \epsilon\big)\end{aligned} \tag{4.2}$$

on the perturbation equations

$$\partial_t \begin{pmatrix} u \\ \epsilon \end{pmatrix} = \begin{pmatrix} L(0)u + M(0)\epsilon + \mathcal{G}(u,\epsilon) \\ 0 \end{pmatrix} \tag{4.3}$$

about the steady-state $(\bar u^0, 0)$ at bifurcation point $\epsilon = 0$, where

$$L(\epsilon) := \frac{\partial \mathcal{F}}{\partial u}(\epsilon, \bar u^\epsilon), \quad M(\epsilon) := \frac{\partial \mathcal{F}}{\partial \epsilon}(\epsilon, \bar u^\epsilon),$$

and

$$\mathcal{G}(\epsilon, u) = g(\epsilon, u, u_x, x) := f(\epsilon, u, u_x) - L(0)u - M(0)\epsilon \tag{4.4}$$

is a quadratic-order Taylor remainder.

Notice that the Kernel of $L(0)$ is one-dimensional, generated by

$$\phi := \frac{\partial \bar u^0}{\partial x} = \frac{d\Phi_\alpha}{d\alpha}_{|\alpha=0.}$$

This justifies an assumption made in Theorem 3.8.

Associated with the linearized operator $L(0)$, define the spectral projections

$$\Pi_2 := \phi\,\langle\tilde\phi, \cdot\rangle, \qquad \Pi_1 := \mathrm{Id} - \Pi_2, \tag{4.5}$$

onto the range of right zero-eigenfunction $\phi := (\partial/\partial x)\bar u^0$ of $L(0)$ and its complementary $L(0)$-invariant space, where $\tilde\phi$ denotes the dual, left zero-eigenfunction, and $\langle\cdot,\cdot\rangle$ denotes standard $L^2$ inner product.

LEMMA 4.1. *Under the assumed regularity $f \in C^4$ and the weighted norm condition* (1.14), $\Pi_j$, $j = 1, 2$ *are bounded as operators from $H^s_\eta$ to itself for* $0 \le s \le 5$.

*Proof.* We have only to observe that this is equivalent to the corresponding statement for the projections associated with the conjugated operator $L_\eta(0) := e^{\eta(1+|x|^2)^{1/2}} L(0) e^{\eta(1+|x|^2)^{1/2}}$, for which by assumption there is a spectral gap between the essential spectrum and the imaginary axis: in particular, $\lambda = 0$. (In unweighted Sobolev norms, the statement may in general be false.) ∎

Decomposing

$$u =: v \oplus w\phi, \qquad v \in \mathcal{H} := \mathrm{Range}\,\Pi_1,\ w \in \mathbb{R}, \tag{4.6}$$

coordinatize $u$ alternatively as $(v, w)$. In these coordinates, (4.3) take the form

$$\partial_t \begin{pmatrix} v \\ w \\ \epsilon \end{pmatrix} = \begin{pmatrix} \Pi_1(L(0)v + M(0)\epsilon + \mathcal{G}(v + w\phi, \epsilon)) \\ \pi_2(M(0)\epsilon + \mathcal{G}(v + w\phi, \epsilon)) \\ 0 \end{pmatrix}, \tag{4.7}$$

where we have defined $\pi_2 u := \langle \tilde{\phi}, u \rangle$. Here, we are using $\Pi_2 L = 0$, $\pi_2 L = 0$ to simplify the second row of the righthand side.

Now, as in (3.2), introduce the map $T : H^2_\eta \times \mathbb{R} \to H^2_\eta \times \mathbb{R}$, defined as

$$T \begin{pmatrix} v \\ \alpha \\ \epsilon \end{pmatrix} := \Phi_\alpha \begin{pmatrix} v \\ 0 \\ \epsilon \end{pmatrix}.$$

We claim that $T$ is a local diffeomorphism in a neighborhood of the origin in $H^2_\eta \times \mathbb{R}$. Indeed, remark that

$$\frac{\partial \Phi_\alpha}{\partial \alpha}|_{\alpha=0} \begin{pmatrix} v \\ 0 \\ \epsilon \end{pmatrix} = \begin{pmatrix} \Pi_1 \partial_x v \\ \langle \tilde{\phi}, \phi + \partial_x v \rangle \\ 0 \end{pmatrix}.$$

For $\|v\|_{L^2_\eta}$ small enough,

$$\langle \tilde{\phi}, \phi + \partial_x v \rangle = \langle \tilde{\phi}, \phi \rangle - \langle \partial_x \tilde{\phi}, v \rangle \neq 0; \tag{4.8}$$

in particular,

$$\langle \tilde{\phi}, \frac{\partial \Phi_\alpha}{\partial \alpha}|_{\alpha=0}(0) \rangle = \langle \tilde{\phi}, \phi \rangle = 1 \neq 0.$$

Using $\Phi_0 = \mathrm{Id}$, we find that

$$dT(0) = \begin{pmatrix} \mathrm{Id}_{\mathcal{L}(\mathcal{H})} & 0 & 0 \\ 0 & \langle \tilde{\phi}, \phi \rangle & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

an isomorphism. We compute:

$$dT \begin{pmatrix} v \\ 0 \\ \epsilon \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{Id}_{\mathcal{L}(\mathcal{H})} & \frac{-\Pi_1 \partial_x v}{\langle \tilde{\phi}, \phi + \partial_x v \rangle} & 0 \\ 0 & \frac{1}{\langle \tilde{\phi}, \phi + \partial_x v \rangle} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider now a local solution $t \mapsto (u, \epsilon)$ to (4.3). The same computations as in section 3.1 show that the equations for the local coordinates $(v, \alpha, \epsilon)$ (such that $(u, \epsilon) = T(v, \alpha, \epsilon)$) are

$$\partial_t \begin{pmatrix} v \\ \alpha \\ \epsilon \end{pmatrix} = dT \begin{pmatrix} v \\ 0 \\ \epsilon \end{pmatrix}^{-1} \begin{pmatrix} \Pi_1(L(0)v + M(0)\epsilon + \mathcal{G}(v, \epsilon)) \\ \pi_2(M(0)\epsilon + \mathcal{G}(v, \epsilon)) \\ 0 \end{pmatrix}, \tag{4.9}$$

that is

$$\partial_t \begin{pmatrix} v \\ \epsilon \end{pmatrix} = L_0 \begin{pmatrix} v \\ \epsilon \end{pmatrix} + \mathcal{G}_0 \begin{pmatrix} v \\ \epsilon \end{pmatrix}, \quad \partial_t \alpha = \frac{\pi_2(M(0)\epsilon + \mathcal{G}(v, \epsilon))}{\langle \tilde{\phi}, \phi + \partial_x v \rangle}, \tag{4.10}$$

where

$$L_0 := \begin{pmatrix} \Pi_1 L(0) & \Pi_1 M(0) \\ 0 & 0 \end{pmatrix},$$
$$\mathcal{G}_0 \begin{pmatrix} v \\ \epsilon \end{pmatrix} := \begin{pmatrix} \Pi_1 \mathcal{G}(v,\epsilon) - \langle \tilde{\phi}, \phi + \partial_x v\rangle^{-1} \Pi_1 \partial_x v \langle \tilde{\phi}, M(0)\epsilon + \mathcal{G}(v,\epsilon)\rangle \\ 0 \end{pmatrix}.$$

LEMMA 4.2. *Under the assumed regularity $f \in C^4$ and the weighted norm condition* (1.14), *both $\mathcal{G}$ and $\mathcal{G}_0$ are Frechet differentiable of order 4 considered as functions from $H^2_\eta$ to $H^1_\eta$: $\mathcal{G}$ on the whole space and $\mathcal{G}_0$ for $|v|_{H^2_\eta}$ sufficiently small.*

*Proof.* Differentiability of $\mathcal{G}$ follows by direct calculation taking into account the favorable effect of the weighted norm; see [S]. Differentiability of $\mathcal{G}_0$ follows similarly, using also (4.8) and the fact (see Lemma 4.1) that $\Pi_j$ as bounded linear operators from each $H^s_\eta$ to itself are infinitely differentiable in the Frechet sense. ∎

**4.1. Preliminaries.** Let $\rho$ be a smooth truncation function, as in section 2.1, and let

$$\mathcal{G}_0^\delta(v,\epsilon) := \rho\Big(\frac{|v|_{H^2_\eta}}{\delta}\Big)\mathcal{G}_0(v,\epsilon),$$

for small $\delta > 0$. Remark that, for $v$ in a neighborhood of 0 in $H^2_\eta$, the intregro-differential term $\mathcal{G}_0$ in the reduced equations above is as regular as $\mathcal{G}$.

LEMMA 4.3. *The map $\mathcal{G}_0^\delta : H^2_\eta \times \mathbb{R} \to H^1_\eta \times \mathbb{R}$ is $C^4$ and its Lipschitz norm with respect to $v$ can be made arbitrarily small as $\delta, \epsilon \to 0$.*

*Proof.* The norm in $H^2_\eta$ is a quadratic form, hence the map

$$v \in H^2_\eta \mapsto \rho\Big(\frac{|v|_{H^2_\eta}}{\delta}\Big) \in \mathbb{R}_+,$$

is smooth, and $\mathcal{G}_0^\delta$ is as regular as $\mathcal{G}_0$. Now

$$\begin{aligned} |\mathcal{G}_0^\delta(v_1,\epsilon) - \mathcal{G}_0^\delta(v_2,\epsilon)|_{H^1_\eta} &\le |\rho\Big(\frac{|v_1|_{H^2_\eta}}{\delta}\Big) - \rho\Big(\frac{|v_2|_{H^2_\eta}}{\delta}\Big)|_{L^\infty} |\mathcal{G}_0(v_1,\epsilon)|_{H^1_\eta} \\ &\quad + |\rho\Big(\frac{|v_2|_{H^2_\eta}}{\delta}\Big)|_{L^\infty} |\mathcal{G}_0(v_1,\epsilon) - \mathcal{G}_0(v_2,\epsilon)|_{H^1_\eta} \\ &\le 3|v_1 - v_2|_{H^2_\eta} \sup_{|v|_{H^2_\eta}<\delta} |\mathcal{G}_0(v,\epsilon)|_{H^1_\eta}, \end{aligned}$$

and $\sup_{|v|_{H^2_\eta}<\delta} |\mathcal{G}_0(v,\epsilon)|_{H^1_\eta} = O(\epsilon^2 + \delta^2)$. ∎

LEMMA 4.4. *Under the assumptions of Theorem 1.2,*

$$\begin{aligned} \|e^{tL_0}\Pi_s\|_{H^1_\eta \to H^2_\eta} &\le C(1 + t^{-1/2})e^{-\beta t}, \\ \|e^{tL_0}\Pi_c\|_{H^1_\eta \to H^3_\eta} &\le C_\omega e^{\omega t}, \\ \|e^{-tL_0}\Pi_u\|_{H^1_\eta \to H^3_\eta} &\le C_\omega e^{\beta t}, \end{aligned} \tag{4.11}$$

*for some $\beta > 0$, and for all $\omega > 0$.*

*Proof.* By definition of $L_0$, $\sigma(L_0) = \sigma(\Pi_1 L(0)) \cup \{0\}$. Let

$$\Pi_c := \int_{\Gamma_c} (\lambda - L_0)^{-1}\, d\lambda, \qquad \Pi_u := \int_{\Gamma_u} (\lambda - L_0)^{-1}\, d\lambda,$$

where $\Gamma_c$ is a contour enclosing only the neutral eigenvalues and $\Gamma_u$ a contour enclosing only the unstable eigenvalues. By the weighted norm property (1.14), and the properties of asymptotically constant-coefficient operators described in the introduction just below, there are finitely many neutral and unstable eigenvalues, so we may choose these contours to be bounded. Further, we assume that $|\Re\Gamma_c| \le \omega$ and $\Re\Gamma_u \ge \beta$. Let $\tilde\Sigma_c$ be the range of $\Pi_c$ and $\tilde\Sigma_u$ be the range of $\Pi_u$; one has $\tilde\Sigma_c = \Sigma_c \oplus \mathbb{R}$, where $\Sigma_c$ is the neutral eigenspace of $\Pi_1 L(0)$. Define also $\Pi_s := \mathrm{Id} - \Pi_c - \Pi_u$, a projector onto the stable (including essential) spectrum. As (1.2) is strictly parabolic, $L_0$ is sectorial and we have the inverse Laplace transform representations

$$\begin{aligned} e^{tL(0)}\Pi_c &:= \int_{\Gamma_c} e^{\lambda t}(\lambda - L_0)^{-1}\, d\lambda,\\ e^{tL(0)}\Pi_u &:= \int_{\Gamma_u} e^{\lambda t}(\lambda - L_0)^{-1}\, d\lambda,\\ e^{tL(0)}\Pi_s &:= \int_{\Gamma_s} e^{\lambda t}(\lambda - L_0)^{-1}\, d\lambda, \end{aligned} \tag{4.12}$$

where $\Gamma_s$ denotes a sectorial contour bounding the stable spectrum to the right [Pa], without loss of generality $\Re\Gamma_s \le -\beta$.

Applying the resolvent formula $L(\lambda - L)^{-1} = \lambda(\lambda - L)^{-1} - Id$, we obtain in the standard way

$$e^{tL(0)}\Pi_j := \int_{\Gamma_j} \lambda e^{\lambda t}(\lambda - L_0)^{-1}\, d\lambda,$$

from which we obtain immediately the second two stated bounds, and, by a scaling argument [Pa], the bound

$$\|e^{tL_0}\Pi_s\|_{H^1_\eta \to H^3_\eta} \le \|Le^{tL_0}\Pi_s\|_{H^1_\eta \to H^1_\eta} \le C(1 + t^{-1})e^{-\beta t}. \tag{4.13}$$

Recalling the standard bound $\|e^{tL_0}\Pi_s\|_{H^1_\eta \to H^1_\eta} \le Ce^{-\beta t}$, and interpolating between $|\cdot|_{H^1_\eta}$ and $|\cdot|_{H^3_\eta}$, we obtain the first stated bound. □

Corollary 4.5. *Under the assumptions of Theorem 1.2,*

$$\begin{aligned} \|e^{tL_0}\Pi_s\mathcal{G}_0^\delta\|_{H^2_\eta \to H^2_\eta} &\le C(1 + t^{-1/2})e^{-\beta t},\\ \|e^{tL_0}\Pi_c\mathcal{G}_0^\delta\|_{H^2_\eta \to H^2_\eta} &\le C_\omega e^{\omega t},\\ \|e^{-tL_0}\Pi_u\mathcal{G}_0^\delta\|_{H^2_\eta \to H^2_\eta} &\le C_\omega e^{\beta t}, \end{aligned} \tag{4.14}$$

*for some* $\beta > 0$, *and for all* $\omega > 0$.

Remark 4.6. Sectorial estimates analogous to (4.11) may be obtained in the $W^{k,\infty}$ framework (indeed, for any $W^{k,p}$) by a more detailed analysis involving pointwise bounds on the resolvent kernel, as described, for example, in [S, ZH, Z.1].

Likewise, we may define a truncation analogous to $\mathcal{G}_0^\delta(v,\epsilon)$ in the $W^{1,\infty}$ framework by first truncating $f(v,v_x)$ in more standard fashion as

$$\rho(v,v_x)f(v,v_x),$$

and afterwards carrying out the reduction to the quotient space. These two modifications are all that is needed to convert our analysis to a $W^{1,\infty}$ framework.

**4.2. Proofs of the main theorems.**

*Proof.* [Proof of Theorem 1.3.] It remains to prove $(ii)$ and $(iii)$. As in the finite-dimensional case, we start by truncating the nonlinear term. As shown in Lemma 4.3, this can be done in a smooth way in $H^2_\eta$. Then the center manifold is the graph of the map $\phi$ defined on $\tilde\Sigma_c$ as $\phi(u_c,\epsilon)=(v_c(0),\epsilon)$, where $v_c$ is the unique solution of

$$\begin{aligned} v(x,t) = {} & e^{tL_0}u_c + \int_0^t e^{(t-t')L_0}\Pi_c\mathcal{G}_0^\delta(v(x,t'))dt' \\ & + \int_{-\infty}^t e^{(t-t')L_0}\Pi_s\mathcal{G}_0^\delta(v(x,t'))dt' \\ & - \int_t^{\infty} e^{(t-t')L_0}\Pi_u\mathcal{G}_0^\delta(v(x,t))dt', \end{aligned} \tag{4.15}$$

in a space

$$\mathcal{H}_{\eta_0} := \{z\in\mathcal{H}, \sup_{t\in\mathbb{R}}\|z(t)\|_{H^2_\eta}\le Ce^{\eta_0 t}\}.$$

That (4.15) has a unique solution in $\mathcal{H}_{\eta_0}$, for any $0<\omega<\eta_0<\beta$, follows from the above bounds (4.11) and the contraction mapping theorem. Indeed, by Corollary 4.5, the bounds on the integrands of (4.15) are identical to those of the finite-dimensional case, up to the harmless integrable factor $(1+t^{-1/2})$ in the unstable term.

The more delicate proof of smoothness relies on these same bounds. It is thus carried as in the finite-dimensional case.

One checks that the center manifold contains all bounded solutions exactly as in the finite-dimensional case. By definition of $\Phi_\alpha$, these solutions correspond to solutions of the original differential equation (1.2) remaining close to a translate of $\bar u^0$. The assertion concerning stability is proved as in the finite-dimensional case (Theorem 3.6 and Proposition 3.7). □

*Proof.* [Proof of Theorem 1.2.] The above theorem states that the reduced equations (4.10) possess a local center manifold. It has dimension 3, that is 2 for the neutral eigenvalues $\pm i\tau(0)$ of $\Pi_1L(0)$ and 1 for the $\epsilon$ direction. The spectrum of $L_0$ in restriction to the sections $\epsilon=$ constant in the center manifold satisfies the assumption of Theorem 2.2. Hence for all $a>0$ small enough, there exists $\epsilon(a)$ and a periodic orbit $t\mapsto v(t)$ on the center manifold that solves (4.10) for the value $\epsilon(a)$ of the parameter. The associated shift $\alpha$ has the form described in Theorem 3.8, that is $\alpha(t)=\sigma^a t+\theta^a(t)$, where $\sigma^a$ is a constant drift and $\theta^a$ is periodic. With (4.2), the corresponding solution of (4.3) is

$$v(x+\alpha^a(t))+\bar u^0(x+\alpha(t))-\bar u^0(x),$$

and the corresponding solution of the original equations (1.6) is

$$u^a(x,t)=\mathbf{u}^a(x+\alpha(t),t):=v(x+\alpha^a(t))+\bar u^0(x+\alpha(t)).$$

Uniqueness of solutions of (1.6) in this form follows from uniqueness of the periodic orbits on the reduced center manifold. The assertion regarding stability is proved as in the finite-dimensional case (Theorem 3.8 and Proposition 3.9). ▯

**4.3. Examples.** Consider a reaction-diffusion-convection system (1.2), with $u \in \mathbb{R}^n$, and a traveling wave $\bar{u}$ that solves (1.2):

$$\bar{u}(x - st), \qquad \lim_{z \to \pm\infty} \bar{u}(z) = u_\pm. \tag{4.16}$$

After the change of variable $x \to x + st$, the linearized operator about (4.16) is

$$L(\bar{u}, \partial_x)u := \partial_x^2 u + (\partial_2 f(\bar{u}, \partial_x \bar{u}) + s)\partial_x u + \partial_1 f(\bar{u}, \partial_x \bar{u})u.$$

The associated constant-coefficient, conjugated operators $M_\pm$ are

$$M_\pm(u_\pm, \partial_x) := e^{\pm\eta x} L(u_\pm, \partial_x) e^{\mp\eta x} = L(u_\pm, \mp\eta + \partial_x).$$

Let

$$\partial_j f_\pm := \lim_{x \to \pm\infty} \partial_j f(\bar{u}(x), \partial_x \bar{u}(x)), \quad j = 1, 2.$$

With these notations,

$$M_\pm(u_\pm, i\kappa) = (\eta^2 \mp 2i\kappa\eta - \kappa^2)\mathrm{Id} + (\partial_2 f_\pm + s)(\mp\eta + i\kappa) + \partial_1 f_\pm.$$

A complex number $\lambda$ lies in the $L^2$ spectrum of $M_\pm$ if and only if

$$\det\,(M_\pm(i\kappa) - \lambda) = \det\,(L(u_\pm, \mp\eta + i\kappa) - \lambda) = 0, \tag{4.17}$$

for some $\kappa \in \mathbb{R}$. Hence, (1.2)-(4.16) satisfy the weighted norm condition if and only if there exists $\eta \geq 0$ such that for all $\kappa \in \mathbb{R}$, the roots of the algebraic equations (4.17) all have strictly negative real part.

In the scalar case, $n = 1$, equation (4.17) simplifies to

$$\lambda = (\eta^2 \mp 2i\kappa\eta - \kappa^2) + (\partial_2 f_\pm + s)(\mp\eta + i\kappa) + \partial_1 f_\pm, \quad \kappa \in \mathbb{R}, \tag{4.18}$$

two parabolae opening to the left in the complex plane. The solutions of (4.18) have strictly negative real parts if

$$\eta^2 \mp (\partial_2 f_\pm + s)\eta + \partial_1 f_\pm < 0. \tag{4.19}$$

When $\partial_1 f_\pm \neq 0$, equation (4.19) has solutions if and only if

$$\partial_1 f_\pm < \frac{(\partial_2 f_\pm + s)^2}{4}, \tag{4.20}$$

in which case

$$\eta = \frac{\pm(\partial_2 f_\pm + s)}{2} \geq 0,$$

is a good choice.

**4.3.1. Scalar convection-diffusion equations.** When (1.2) does not include any reaction term, that is $f(u, \partial_x u) = -f_0(u)\partial_x u$, then, using the limit $\partial_x \bar{u} \to 0$ as $x \to \pm\infty$, one has $\partial_1 f_\pm = 0$. Besides, examination of the traveling-wave ODE (first-order scalar, after an integration in $x$) reveals that existence of a profile connecting hyperbolic rest points implies *Lax's condition*,

$$\mp(\partial_2 f_\pm + s) < 0.$$

Hence one can find a small $\eta > 0$ such that (4.19) holds.

REMARKS 4.7. 1. Lax's condition states that all the characteristic enter the shock. If for instance $s = 0$, then signals convected by $\partial_2 f_+ < 0$ can be expected to decay in a weighted norm with a growing weight $e^{\eta x}$. Likewise, signals convected by $\partial_2 f_- > 0$ can be expected to decay in a weighted norm with a decaying weight $e^{-\eta x}$. Hence one can expect stability in $\hat{W}^{1,\infty}_{\pm\eta}$. This is the classical "convection-enhanced" stability phenomenon mentioned at the beginning of the introduction; see also a similar remark in section 1.1.4, [ZH].

2. Similar considerations in the system case $n > 1$ yield that the weighted norm condition is satisfied if and only if the profile is "totally compressive" in the sense that all characteristics enter the shock. This is not satisfied for any physical example that we know of; in particular, shock waves for gas dynamics or magnetohydrodynamics (MHD) always possess at least one characteristic mode outgoing from the shock. However, an interesting class of planar ($n = 2$) examples arises in certain simplified models for MHD, as discussed in [Fre, FreL]. See [ZH] for further discussion.

3. In the scalar case, Sturm–Liouville theory gives that the spectrum of the linearized operator is real, hence Poincaré–Hopf bifurcation cannot occur. Moreover, traveling waves are stable if and only if monotone in $x$ [S]. In particular, in the conservative case, $f(u, \partial_x u) = g(u)_x$, for which the traveling-wave ODE may be integrated to obtain a scalar first-order system, traveling waves are always stable. However, our center manifold results may be used to explore the simpler, transcritical, pitchfork, etc. bifurcations that may occur through passage of an eigenvalue through the origin in the nonconservative case.

**4.3.2. Reaction-diffusion equations.** Consider the above system (1.2), where $f = f(u)$, that is the equation contains no convection term, and a traveling wave (4.16). Equation (4.17) simplifies to

$$\det\left(f'(u_\pm) + s(\mp\eta + i\kappa) + (\eta^2 \mp 2i\kappa\eta - \kappa^2 - \lambda)\mathrm{Id}\right) = 0. \tag{4.21}$$

If

$$\Re\sigma(f'(u_\pm)) < 0, \tag{4.22}$$

then the weighted norm condition is satisfied for $\eta = 0$. In the scalar case $n = 1$, with speed $s = 0$, the traveling-wave ODE is Hamiltonian– the nonlinear oscillator $u'' = -f(u)$– and $\Re\sigma(f'(u_\pm)) \le 0$ is necessary in order that a connection between $u_\pm$ exist, the alternative being that one or the other of $u_\pm$ be a nonlinear center. Thus, (4.22) holds generically.

**4.3.3. Majda's model.** Consider Majda's model of reacting flow (M), as given in the introduction. It is proved in [M, LyZ.1] for $D = 0$ and [La] for $D > 0$ (the relevant case here) that, for a given choice of $u_+, s$, for $s$ large enough and $u_+ > u^i$, there exists a unique (up to translation) *strong detonation profile*, that is a traveling wave

$$\bar{u}(x-st), \bar{z}(x-st), \tag{4.23}$$

solution of (1.21), with endstates

$$(u_-, 0), \quad (u_+, 1), \quad u_i < u_- < u^i < u_+, \tag{4.24}$$

satisfying Lax's condition

$$f'(u_+) < s < f'(u_-). \tag{4.25}$$

The detonation wave decays exponentially to its endstates $(u_\pm, z_\pm)$, as $x \to \pm\infty$. The detonation wave corresponds to a gas-dynamical shock followed by a chemical reaction. The gas is for instance compressed, which increases the temperature and triggers the reaction (see [FD], page 14). The values of $z$ at the endstates, namely $z_- = 0$ and $z_+ = 1$, mean that the gas is completely burnt in the course of the reaction.

The linear operator in the equation for $(u, z) := (\tilde{u} - \bar{u}, \tilde{z} - \bar{z})$ is, after the change of variable $x \to x - st$,

$$L \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} (s - f'(\bar{u}))\partial_x u - (f''(\bar{u})\partial_x \bar{u})u + B\partial_x^2 u + kq\varphi(\bar{u})z + kq\bar{z}\varphi'(\bar{u})u \\ s\partial_x z + D\partial_x^2 z - k\varphi(\bar{u})z - k\bar{z}\varphi'(\bar{u})u \end{pmatrix}.$$

The associated, constant-coefficient operators are

$$L_\pm(\partial_x) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} (s - f'(u_\pm))\partial_x u + B\partial_x^2 u + kq\varphi(u_\pm)z \\ s\partial_x z + D\partial_x^2 z - k\varphi(u_\pm)z \end{pmatrix}.$$

Here, we used $z_\pm\varphi'(u_\pm) = 0$ and $\lim_{x\to\pm\infty} \partial_x \bar{u}(x) = 0$. A complex number $\lambda$ lies in the $L^2$ spectrum of the conjugated operators $e^{\pm\eta x} L_\pm e^{\mp\eta x}$ if and only if

$$\det\, (L_\pm(i\kappa \mp \eta) - \lambda) = 0,$$

for some $\kappa \in \mathbb{R}$. This condition decouples into a gas-dynamical equation

$$\lambda = (s - f'(u_\pm))(\mp\eta + i\kappa) + B(\eta^2 \mp 2i\eta\kappa - \kappa^2), \quad \kappa \in \mathbb{R}, \tag{4.26}$$

and a reaction equation

$$\lambda = s(\mp\eta + i\kappa) + D(\eta^2 \mp 2i\eta\kappa - \kappa^2) - k\varphi(u_\pm), \quad \kappa \in \mathbb{R}. \tag{4.27}$$

Both (4.26) and (4.27) are parabolae opening to the left in the complex plane. Using (4.25), $s > 0$, $\varphi(u_+) = 0$ and $\varphi(u_-) > 0$, one can find $\eta > 0$ such that

$$\mp(s - f'(u_\pm))\eta + B\eta^2 < 0, \quad \mp s\eta + D\eta^2 - k\varphi(u_\pm) < 0. \tag{4.28}$$

With this choice of $\eta$, $\sigma_\eta(L_+)$ and $\sigma_{-\eta}(L_-)$ both lie strictly in the stable complex half-plane. Moreover, (4.28) is a continuous condition on the bifurcation parameters $k, u_-$. In other words, for this choice of $\eta$, the weighted norm condition is satisfied by system (1.21).

**Appendix A. Center Manifold proofs.**

*Proof.* [Proof of Proposition 2.1] We sketch the proof of [Bre]. We work on the truncated equations (2.3). As a consequence, the following construction is only local in a neighborhood of the origin in $\mathbb{R}^n$. We use projectors $\pi_s, \pi_u$, $\pi_c$ satisfying (2.4) and the bounds (2.5)-(2.6), choosing for instance spectral projectors of $A$, such that $\mathrm{Range}\,\pi_c = \Sigma_c$.

(i) The center manifold is the graph of the map $\phi$ defined on $\Sigma_c$ by $\phi(X_c) = Y_{X_c}(0)$ where $Y_{X_c}$ is the unique solution in $\mathcal{S}_\eta$ of

$$\begin{aligned} Y_{X_C}(t) = \Gamma(X_C, Y_{X_c})(t) &:= e^{tA}X_c + \int_0^t e^{(t-t')A}\pi_c G(Y(t'))dt' \\ &+ \int_{-\infty}^t e^{(t-t')A}\pi_s G(Y(t'))dt' - \int_t^\infty e^{(t-t')A}\pi_u G(Y(t'))dt', \end{aligned} \tag{A.1}$$

in the space of slowly growing applications

$$\mathcal{S}_\eta := \{t \mapsto z(t) \in \mathbb{R}^n, \sup_{t\in\mathbb{R}} e^{-\eta|t|}|z(t)| < \infty\}, \qquad \text{for } \omega < \eta < \beta.$$

That (A.1) has a unique solution in $\mathcal{S}_\eta$ follows from the contraction mapping theorem, the bounds (2.5)-(2.6), and the control of the Lipschitz norm of $G^\delta$ :

$$\|G^\delta\|_{Lip} := \sup_x \frac{|G^\delta(x)|}{|x|} \leq C\delta, \tag{A.2}$$

where $C$ depends on $F$, through $G$. (Proof: $\|G^\delta\|_{Lip} \leq |dG^\delta|_{L^\infty} \leq |d\rho^\delta G|_{L^\infty} + |\rho^\delta dG|_{L^\infty} \leq C\delta^{-1}\delta^2|d^2G|_{L^\infty_{Loc}} + C\delta|d^2G|_{L^\infty_{Loc}}$.)

Indeed, one checks that $\|G^\delta\|_{L^\infty} < \infty$ implies that the map $\Gamma(X_c, \cdot)$ maps $\mathcal{S}_\eta$ into $\mathcal{S}_\eta$; moreover its Lipschitz norm is bounded by

$$\|G^\delta\|_{Lip}\Big(\frac{2C_\omega\eta}{(\eta-\omega)(\eta+\omega)} + \frac{4C_\omega\beta}{(\beta-\eta)(\beta+\eta)}\Big). \tag{A.3}$$

The regularity of $\phi$ is more difficult to prove. Indeed, as Bressan points out, the solution operator defined in (A.1) is not regular as a map $\Sigma_c \times \mathcal{S}_\eta \to \mathcal{S}_\eta$; it is however $C^l$ as a map $\Sigma_c \times \mathcal{S}_{\eta'} \to \mathcal{S}_\eta$, for $(l+1)\eta' < \eta$. This fact is proved by the same estimates used in the fixed point argument for the resolution of (A.1). This eventually implies smoothness of $\phi$; see [Bre].

(ii) Let $X$ be a trajectory of (2.1) that is contained in a small ball about the origin. Then $X \in \mathcal{S}_\eta$ and $X$ is a trajectory for the truncated equations (2.3); it can be represented as

$$\begin{aligned} X(t) = \pi_c(e^{(t-t_c)A}X(t_c) &+ \int_{t_c}^t e^{(t-t')A}G^\delta(X(t'))dt') \\ &+ \pi_s(e^{(t-t_s)A}X(t_s) + \int_{t_s}^t e^{(t-t')A}G^\delta(X(t'))dt') \\ &+ \pi_u(e^{(t-t_u)A}X(t_u) + \int_{t_u}^t e^{(t-t')A}G^\delta(X(t'))dt'). \end{aligned}$$

Now using the fact that $X$ belongs to $\mathcal{S}_\eta$, the bounds (2.5), (2.6) and (A.2), in the limit $t_s \to -\infty$, $t_c \to 0$, $t_u \to \infty$, one finds that $X$ satisfies the representation (A.1), hence belongs to the center manifold.

(iii) Let $X_0 \in \mathcal{C}$. By construction of $\mathcal{C}$, the unique trajectory of (2.1) on which $X_0$ lies belongs to $\mathcal{S}_\eta$. By (ii), this trajectory is contained in $\mathcal{C}$.

(iv) See [Bre].

(v) Let $X$ be a solution of (2.1) that is eventually small. For $t_0$ large enough and $t_0 \leq t$, $|X(t)| < 1$. Hence for large times, $X$ coincides with the solution $X_\delta$ of the truncated equations that satisfies $X_\delta(0) = X(t_0)$. The assumption on $X$ implies that $X_\delta$ is slowly growing at infinity: $X_\delta \in \mathcal{S}_0^+$. By Proposition 2.2, as $t \to \infty$, $X_\delta$, hence $X$, is exponentially approximated by a trajectory on the center manifold. □

*Proof.* [Proof of Proposition 2.2] Let $X \in \mathcal{S}_\eta^+$ be a solution of the truncated equations (2.3). As in [Bre], extend $X$ to $X^*$, defined as $X^*(t) := 0$ for $t < 0$, and $X^*(t) := X(t)$ for $t \geq 0$. Then $X^*$ solves

$$(X^*)' = AX + G^\delta(X) + x(t), \qquad x(t) := \begin{cases} -AX(0) - G^\delta(X(0)), & t < 0, \\ 0, & t \geq 0, \end{cases}$$

and can thus be represented as the sum of its stable, center and unstable components, choosing possibly different initial times for the three components:

$$X^* = X_s^* + X_c^* + X_u^* + \phi, \tag{A.4}$$

where

$$X_j^*(t) := e^{(t-t_j)A} X^*(t_j) + \int_{t_j}^t e^{(t-t')A} \pi_j G^\delta(X^*(t')) dt',$$

and

$$\phi(t) := \int_{t_s}^t e^{(t-t')A} \pi_s x(t') dt' + \int_{t_c}^t e^{(t-t')A} \pi_c x(t') dt' + \int_{t_u}^t e^{(t-t')A} \pi_u x(t') dt'.$$

Now consider a small perturbation $Z$ of $X^*$ such that $X^* + Z$ is a global solution of (2.3). Then $X^* + Z$ can be decomposed as in (A.4), with $\phi = 0$. If $X^* + Z$ is assuming to be slowly growing at infinity, then letting $t_s \to -\infty$, $t_c \to \infty$ and $t_u \to \infty$,

$$e^{(t-t_j)A}(X^* + Z)(t_j) \to 0,$$

with the bounds (2.5) and (2.6), and thus

$$\begin{aligned} Z = & \int_{-\infty}^t e^{(t-t')A} \pi_s (G^\delta(X^* + Z) - G^\delta(X^*)) dt' \\ & - \int_t^\infty e^{(t-t')A} (\pi_c + \pi_u)(G^\delta(X^* + Z) - G^\delta(X^*)) dt' \\ & - \int_{-\infty}^t e^{(t-t')A} \pi_s x(t') dt' + \int_t^\infty e^{(t-t')A} (\pi_c + \pi_u) x(t') dt'. \end{aligned} \tag{A.5}$$

The equation (A.5) has a unique solution in the space of fast decaying functions at $+\infty$ :

$$\mathcal{F}_\eta := \{t \mapsto z(t) \in \mathbb{R}^n, \sup_{t \in \mathbb{R}} e^{\eta t} |z(t)| < \infty\}.$$

One checks indeed that the map $Z \mapsto$ r.h.s. of (A.5), maps $\mathcal{F}_\eta$ into itself, with a Lipschitz norm bounded by

$$\|G^\delta\|_{Lip}\Big(\frac{C_\omega}{\eta-\omega}+\frac{C_\beta}{\beta-\eta}\Big). \tag{A.6}$$

The bound (A.6) is strictly smaller than 1 as soon as (A.3) is. Now consider $X^*+Z$, where $Z$ solves (A.5). By definition of $Z$, $X^*+Z$ is a trajectory of (2.3). Besides, it belongs to $\mathcal{S}_\eta$, hence by (ii) of Proposition 2.1, it is contained in the center manifold. The exponential decay of $Z$ then gives (2.8). Moreover,

$$e^{\eta t}|Z(t)| \le \frac{C_\omega}{\omega(1-K)}|AX(0)+G^\delta(X(0))|, \tag{A.7}$$

where $K$ is the Lipschitz norm of the map in (A.5). □

*Proof.* [Proof of Proposition 2.3] Let $X$ be a stable periodic orbit on the center manifold, contained in a small neigborhood of the origin. Consider a trajectory $\tilde X$ of the truncated equations, such that $\tilde X(0)$ is a small perturbation of $X(0)$. The assumption on $\sigma(dF(0))$ implies that all solutions have a slow growth at infinity. In particular, $\tilde X \in \mathcal{S}^+_\eta$, for all $\eta > 0$, where the projectors in (2.5) and (2.6) can be chosen to be spectral projectors. Proposition 2.2 then implies that $\tilde X$ is exponentially approximated by a trajectory $Y$ on the center manifold, as $t \to \infty$. Moreover, by (2.7), $Y(0)$ is close to $\tilde X(0)$, and $Y$ remains close to $\tilde X$ for all times. In particular, $Y(0)$ is close to $X(0)$. By stability of $X$ on $\mathcal{C}$, it follows that if $\tilde X(0)$ is small enough, then $Y$ remains close to $X$ for all times. Hence $\tilde X$ remains close to $X$ for all times. The case when $X$ is asymptotically stable is handled in the same way. Finally, since $\tilde X(0)$ was assumed sufficiently small, $\tilde X$ and $X$ remain so as well, and so the cutoff has no effect, i.e., the conclusions remain valid for the original, unmodified equations (2.1).

Proof of second assertion: omitted. □

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