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REMARKS ON A BACKWARD PARABOLIC PROBLEM *

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1. Introduction. After spending a quarter at the UCLA in the early 1998, I got to know Professor Stanley Osher better. He always seems to have insights and nice ideas to do computations for complicated problems. Thus at those times when I encounter problems for which analysis seem either practically impossible or extremely difficult and, for which some reliable computations may give either a reasonable solution or some hints, I often turn to experts like Stanley to see if they can do anything about them. The present article is of such nature, and I would like to dedicate it to Stanley on the occasion of his 60th birthday.

Let $A(x,t)$ be a $n\times n$ matrix-valued measurable function on $R^n\times R^+$ such that
(i) A is periodic in both x and t with period 1,
(ii) A is bounded and positive definite for all $(x,t)\in R^n\times R^+$.
We consider the following problem:

$$(*)\qquad \begin{cases} \frac{\partial u^\varepsilon}{\partial t}-\operatorname{div}\left(A\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)\nabla u^\varepsilon\right)=0 & \text{in}\quad \Omega\times(0,T)\,,\\ u^\varepsilon(t)\equiv 0 & \text{on}\quad \partial\Omega\times[0,T]\,. \end{cases}$$

Here $\varepsilon>0$ is assumed to be very small, and Ω is a bounded smooth domain in R^n. Suppose $f(x)$ is an observed data at the time $t=T$, i.e., $f(x)$ is close to some true solution u^ε of $(*)$ at the time $t=T$. Then our problem is how to recover (one of) such u^ε, a solution of $(*)$, so that $\|u^\varepsilon(\cdot,T)-f(\cdot)\|$ is small for a suitable given norm $\|\cdot\|$.

The above problem contains several issues which need to be cleared up before one can answer the problem precisely. First of all, how do we know (a priori) a given function $f(x)$ (observed data presumably) is close to $u^\varepsilon(x,T)$ for some true solution of $(*)$? Secondly, suppose we know the answer to the above question, then how can one actually construct an approximate solution, $\bar u^\varepsilon(x,t)$ of $(*)$ so that $\|\bar u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t)\|$ will remain small for all $0<t_0\le t\le T$ whenever $\|f-u^\varepsilon(\cdot,T)\|$ is known to be small, where $u^\varepsilon(x,t)$ is a true solution of $(*)$, and here $t_0>0$ is given. (Note, it is, in general, impossible, even for the standard heat equation, that one could construct such approximations which valid on an interval containing $[0,T]$.)

In order for the second question above to make sense, one has to know first that for a given function, suppose that $f(x)$ is close to $u^\varepsilon(x,T)$, for some true solution $u^\varepsilon(x,t)$ of $(*)$, then such $u^\varepsilon(x,t)$ is essentially unique in the sense that if $\tilde u^\varepsilon(x,t)$ is another such solution of $(*)$, then $\|\tilde u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t)\|$ will (remain small) be controlled by $\|u^\varepsilon(\cdot,T)-f\|$, for $0<t_0\le t\le T$. We refer the last issue as the uniqueness question, or more precisely the stability question in the numerical computations.

I had some ideas of handling the problem $(*)$ about one year ago, then I learned from P. Lax some earlier works (nearly 50 years ago!) of F. John, see [1]. Though the approach I had is apparently rather different from that of F. John, they seem to have some deep connections. I shall explain some notions (which I found rather amusing) introduced in John's work in the next section. A solution to the problem $(*)$ (or a somewhat more general problem) will be explained in section 3. In the final section, I shall describe a few issues which may be of interest from both theory and

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computations. I have no intention here to make various statements or estimates more refined. The goal here is to present the problems and certain point views on such problems. I wish to thank P. Lax for bringing John's work to my attention and for several interesting discussions.

2. John's Notion of Well-Behaved Problems. Given a P.D.E. problem, it is said to be well-posed in the sense of Hadamard if one can find a well-defined data space X with the norm $\|\cdot\|_X$ and a well defined solution space Y with the norm $\|\cdot\|_Y$ such that

(i) $\forall f \in X$, $\exists$ unique $u \in Y$ solves the given P.D.E. problem with the given data f, and one denotes such u by Tf;

(ii) $\forall f_1, f_2 \in X$, one has $\|Tf_1 - Tf_2\|_Y \le \triangle(\delta)$ whenever $\|f_1 - f_2\|_X \le \delta$. Here $\triangle(\delta)$ is a monotone increasing continuous function of $\delta > 0$, with $\triangle(0^+) = 0$.

When the problem is linear, one often has $\triangle(\delta) \le C_0\delta$ though it is not necessary. F. John introduced the notions of well-behaved problems if $\triangle(\delta) \le C_0\delta^\alpha$ for some positive constant C_0 and α, and badly-behaved problems if $\triangle(\delta) \ge C_0\left(\log\frac{1}{\delta}\right)^{-1}$ for a positive constant C_0, see [1](p.411–424).

Some ill-posed problems can be well-behaved and some well-posed problems can be badly-behaved. Let us first consider the following example.

Example A. (Cauchy's problem for the Laplacian operator)
Consider

$$\triangle u = 0 \text{ in } B_2^n \subset R^n, \tag{2.1}$$

and let $X = Y = \{u\colon u \text{ harmonic in } B_2^n \text{ and } |u| \le 1\}$. If $u|_{B^n_{\frac{1}{2}}}$ is given, then u is uniquely determined in B_2^n. Moreover, for u_1, $u_2 \in X$ with

$$\|u_1 - u_2\|_X \equiv \|u_1 - u_2\|_{L^\infty\left(B_{\frac{1}{2}}\right)} \le \delta,$$

one has

$$\|u_1 - u_2\|_Y \equiv \|u_1 - u_2\|_{L^\infty(B_1)} \le \delta^\alpha$$

for $\alpha = \alpha_0(n) > 0$ and for all $\delta \le \delta_0(n)$. The last conclusion follows easily from the Hadamard three circles theorem (or its generalization) which is valid for a much larger class of elliptic operators. This is an example of an (classically) ill-posed problem which is well-behaved.

Example B. (Sidewise wave equations)
Next one considers

$$u_{tt} - u_{xx} - u_{yy} = 0 \text{ in } B_1^2(0) \times R. \tag{2.2}$$

Suppose u is given on the infinity cylinder $B^2_{\frac{1}{4}}(0) \times R$. Then it can be easily shown there is a unique u solves the wave equation in $B_1^2(0) \times R$ with $u|_{B^2_{\frac{1}{4}}(0)\times R} \equiv f(x, y, t)$ (say in C^2, which is given). Indeed, after a change of variables, the problem becomes

$$u_{tt} - e^{-2s}(u_{ss} + u_{\theta\theta}) = 0$$

with $\theta \in [0, 2\pi]$ (u periodic in θ with period 2π) and $s \leq 0$ with u is given on $s \leq -s_0$ ($s_0 = \log 4$). One writes solution u as

$$u(t, s, \theta) = \sum_{n=0}^{\infty} \left(a_n(t, s) \cos n\theta + b_n(t, s) \sin n\theta\right),$$

then coefficients $a_n(t, s)$, $b_n(t, s)$ would satisfy the equation

$$v_{tt} = e^{-2s}\left(v_{ss} + n^2 v\right).$$

The data is specified on $\{s \leq -s_0\}$. The latter is simply an initial value problem for the one-dimensional linear wave-type operator. Hence the problem is well-posed in the classical sense for each Fourier coefficients. One then check the problem (2.2) is also well-posed. On the other hand, by studying the complex continuation of Bessel functions, John [1](p.419–420) showed that: there are $f_1, f_2 \in X$ such that $\|f_1 - f_2\|_X \leq \delta$, but $\|u_1 - u_2\|_Y \geq \left(\log \frac{1}{\delta}\right)^{-\frac{1}{3}}$. Here $X = Y \equiv \left\{u\colon \Box u = 0 \text{ with } |u| \leq 1 \text{ on } B_1^2(0) \times R\right\}$, and $\|u\|_X = \|u\|_{L^\infty\left(B^2_{\frac{1}{4}}(0)\times R\right)}$, $\|u\|_Y = \|u\|_{L^\infty\left(B_1^2(0)\times R\right)}$. Moreover, the situation will not change if one replaces L^∞-norms with C^k-norms. We note that in order to have $\|u_1 - u_2\| \leq 0.001$, one would require in the data to satisfy $\delta \leq 10^{-100,000,000}$! The latter is practically impossible.

3. Problem $(*)$ is Well-Behaved. Let us consider a slightly more general problem than $(*)$:

$$\left\{\begin{array}{lll} \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}\left(A\left(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\nabla u^\varepsilon\right) = 0 & \text{in} & \Omega \times (0, T), \\ u^\varepsilon(t) \equiv 0 & \text{on} & \partial\Omega \times [0, T]. \end{array}\right. \tag{3.1}$$

Here $A(x, \xi, \eta)$ is a matrix-valued function for $(x, \xi, \eta) \in R^n \times R^n \times R$, and it is periodic in ξ and η variables with period 1. Otherwise we assume A to be measurable and satisfies

$$\Lambda^{-1}|x|^2 \leq A_{ij}(x, \xi, \eta)\lambda_i\lambda_j \leq \Lambda|x|^2,$$

for all $\lambda \in R^n$ and for a constant $\Lambda \in [1, \infty)$.

It is well-known that (see [4]) the equations (3.1) homogenize to a limiting equation of the form:

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial t} - \operatorname{div}\left(\bar{A}(x)\nabla u\right) = 0 & \text{in} & \Omega \times (0, T), \\ u|_{\partial\Omega\times[0,T]} \equiv 0. \end{array}\right. \tag{3.2}$$

What this means is that, if we consider the initial value problem for (3.1) with $u^\varepsilon(x, 0) \rightharpoonup u(x, 0)$ in $L^2(\Omega)$ weakly, then the corresponding solutions $u^\varepsilon(x, t) \in L_t^\infty L_x^2(\Omega) \cap L_t^2 H^1(\Omega) \equiv \mathcal{B}$ converge weakly in $\mathcal{B}$ to u the corresponding solution of (3.2).

Note

$$\mathcal{B} = \left\{v(x, t)\colon \|v(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega |\nabla v|^2(x, \tau)\, dx d\tau < \infty, \text{ for all } 0 \leq t \leq T\right\}.$$

Moreover, if $u^\varepsilon(x, 0) \equiv \varphi(x) \in H^1(\Omega)$, then one can shown, for some $\alpha_0 > 0$ that

$$\|u^\varepsilon - u\|_{L^2(\Omega\times[0,T])} \leq C_0\varepsilon^{\alpha_0} \tag{3.3}$$

as $\varepsilon \to 0^+$. (See [4]).

Thus it is natural to consider similar issues for the problem (3.2) first, we start with a few observations.

Observation I (Backward uniqueness)

Consider a solution u of the problem (3.2) in $\Omega \times (0,T)$. Let

$$H(t) = \int_\Omega u^2(x,t)\,dx,$$

then

$$\frac{d}{dt}H(t) = -2E(t) \text{ for } 0 < t < T,$$
$$E(t) = \int_\Omega \left\langle \bar{A}(x)\nabla u, \nabla u \right\rangle dx.$$

Consider the quantity $N(t) = \frac{E(t)}{H(t)}$. It is easy to check that

$$\frac{d}{dt}N(t) \leq 0 \text{ for } 0 < t < T.$$

In particular, one has $\log H(t)$ is a convex function on $[0,T]$, and hence

$$\log H(t) \leq \frac{t}{T}\log H(T) + \left(1 - \frac{t}{T}\right)\log H(0),$$

or equivalently

$$H(t) \leq H(T)^{\frac{t}{T}}\, H(0)^{\left(1-\frac{t}{T}\right)}, \text{ for } t \in (0,T].$$

We thus conclude that if $H(T) = 0$, then $H(t) \equiv 0$ for $0 < t < T$.

Observation II (Compactness)

Let $u \geq 0$ be a solution of (3.2), then one has Moser type Harnack inequality: (See [2])

$$\sup_{x\in\Omega} u(x,t) \leq C\left(n, \Lambda, \frac{t}{T}\right)\sup_{x\in\Omega} u(x,T).$$

Obviously such estimate is valid for much more general class of parabolic operators. Moreover, one can even calculate $C\left(n,\Lambda,\frac{t}{T}\right)$ from estimates of Fundamental solutions of such operators. However, we should certainly not elaborate any more on such points.

If we let

$$\mathcal{S} = \{u \geq 0,\ u \text{ is a bounded solution of } (3.2)\}$$

and let $\mathcal{S}_t = \{u(\cdot,t)\colon u \in \mathcal{S}\}$, for $0 \leq t \leq T$. Then $\forall f \in \mathcal{S}_T$, there is a unique $g \in \mathcal{S}_{t_0}$, $0 < t_0 < T$, such that one may find a $u \in \mathcal{S}$ with $u|_{t=t_0} = g$ and $u|_{t=T} = f$. The map $f \to g$ is, in fact, compact in the sense that

$$\|f\|_{L^\infty(\Omega)} \leq a, \text{ then } \|g\|_{L^\infty(\Omega)} + \|g\|_{C^\alpha(\bar{\Omega})} \leq C\left(\frac{t_0}{T}, \Lambda, a, n\right). \tag{3.4}$$

On the other hand, the map $g \to f$ is obviously compact. Both follows Nash-Moser's Hölder estimate for solutions of such solutions. (See [3])

In general, we may combine Observation I with such Hölder estimate to show the statement (3.4) without assumption that u is nonnegative but rather one considers bounded solutions.

Observation III (Construction)

We consider a solution u of (3.4) in $\Omega \times (0,T)$. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ such that

$$\begin{cases} \operatorname{div}\left(\bar{A}(x)\nabla\phi_j\right) + \lambda_j\phi_j = 0 & \text{in} \quad \Omega, \\ \phi_j \equiv 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.5}$$

for $j = 1, 2, \cdots$. We write $u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t}\phi_j(x)$. Suppose$\|u(\cdot,t)\|_{L^2(\Omega)} \le M$ for all $t \in [0,T]$. Then one has, in particular that

$$\sum_{j=1}^{\infty} c_j^2 \le M^2.$$

Since $u(x,T) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j T}\phi_j(x)$, one has

$$\left|\int_\Omega u(x,T)\cdot\phi_j(x)\,dx\right| = |c_j|\,e^{-\lambda_j T} \le Me^{-\lambda_j T}. \tag{3.6}$$

Similar estimates as (3.6) are valid also for nonnegative solutions u of (3.2). However, in this case, since the Harnack's estimate can not go all the way to $t = 0$, one would replace the right hand of (3.6) by $C\left(n, \Lambda, \frac{\varepsilon_0}{T}\right) Me^{-\lambda_j(T-\varepsilon_0)}$, for any $0 < \varepsilon_0 < T$.

We note that (3.6) gives a necessary condition (and in fact also sufficient) for a given function $f(x)$ to be equal (or close) to $u(x,T)$, a solution $u(x,t)$ of (3.2). Indeed, if $\|u(\cdot,t) - f\|_{L^\infty(\Omega)} \le \delta M$, then this necessary condition reads:

$$\left|\int_\Omega f(x)\,\phi_j(x)\,dx\right| \le \delta M + Me^{-\lambda_j T},$$

for $j = 1, 2, \cdots$. Moreover, in this case, the function

$$\bar{u}_N(x,t) = \sum_{j=1}^{N} \bar{c}_j e^{-\lambda_j(t-T)}\phi_j(x),$$

where $\bar{c}_j = \int_\Omega f(x)\,\phi_j(x)\,dx$, $j = 1, 2, \cdots$, would be a nice approximation of true solution $u(x,t)$ of (3.2) if one let N be sufficiently large. Indeed, we note that $\bar{u}$ is also a solution of (3.2), and

$$\|u(\cdot,t) - \bar{u}_N(\cdot,t)\|_{L^2(\Omega)}^2 \le \sum_{j=1}^{N} (c_j - \bar{c}_j)^2 e^{-2\lambda_j(t-T)} + \sum_{j=N+1}^{\infty} c_j^2 e^{-2\lambda_j(t-T)}.$$

Using the fact that $|c_j| \le Me^{-\lambda_j T}$ and that $|c_j - \bar{c}_j| \le \delta M$, for $j = 1, 2, \cdots$, one has

$$\|u(\cdot,t) - \bar{u}_N(\cdot,t)\|_{L^2(\Omega)}^2 \le M^2\delta^2 \sum_{j=1}^{N} e^{-2\lambda_j(t-T)} + M^2 \sum_{j=N+1}^{\infty} e^{-2\lambda_j(t-T)}. \tag{3.7}$$

From (3.7) and eigenvalue asymptotic, one may derive, for $0 < t_0 \le t \le T$, that

$$\left\| u(\cdot,t) - \bar{u}_N(\cdot,t) \right\|_{L^2(\Omega)} \le C_0 \delta^{\frac{t_0}{2T}} M. \tag{3.8}$$

Here one may have to choose N properly.

From (3.8) one thus conclude the problem $(*)$ when the operator is replaced by that in (3.2) is, in fact, well-behaved whenever one stay away from $t = 0$. Due to the estimate (3.3), one then may apply the above construction to yield the following:

THEOREM 1. *Problem* $(*)$ *is well-behaved in the sense that: for all sufficiently small* $\delta > 0$, *and* $\varepsilon < \delta$, *and suppose that the observed data* $f(x)$ *is such that* $\left\| f(x) - u(x,T) \right\|_{L^2(\Omega)} \le \delta M$, $M = \|f\|_2$, *where* u *is a true solution of* (3.2), *then there is a true solution of* (3.1) *such that*

$$\left\| u^\varepsilon(\cdot,t) - \bar{u}_N(\cdot,t) \right\|_{L^2(\Omega)} \le C_0 \left(\varepsilon^{\alpha_0} + \delta^{\frac{t_0}{2T}} \right) M, \quad t_0 \le t \le T$$

for a suitably chosen N.

4. Final Remarks. Though in the previous section, we have sketched a construction for approximate solution of $(*)$ or (3.1). It is however not clear that whether such a construction is practical in computation. How to do efficient computations for such inverse parabolic problems involving small scales remains to be interesting. It is particularly so when the homogenized operators are unknown.

A much hard issue would be to do computations for problems of type:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}\left(A\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) \nabla u^\varepsilon \right) = 0 \quad \text{in} \quad \Omega \times (0,T), \\ u^\varepsilon|_{\partial\Omega \times [0,T]} \equiv 0. \end{cases} \tag{4.1}$$

with observed data $f(x)$ at the time $t = T$. One of the key point in dealing with $(*)$ or (3.1) is the fact that homogenized operator is a time-independent parabolic operator. On the other hand, it is always possible to look for the following minimization problem:

$$\inf \left\{ \int_\Omega \left| u_h^\varepsilon(x,T) - f(x) \right|^2 dx \colon\ h \in L^2(\Omega) \right\}. \tag{4.2}$$

Here $u_h^\varepsilon(x,t)$ is the solution of (4.1) with $u^\varepsilon(x,0) = h(x)$, $x \in \Omega$. Whether or not (4.2) is practical to carry out numerically is another issue.

Finally we note that, in [1](p.389–402), F. John considered

$$\begin{cases} \frac{\partial u}{\partial t} = u_{xx} & -\infty < x < \infty,\ -T \le t \le 0, \\ u(x,0) \equiv f(x). \end{cases} \tag{4.3}$$

He asked the question: if for a $f(x)$, (4.3) has a solution and if $\tilde{f}(x)$ is close to f, how can one find $\tilde{u}(x,t)$ from $\tilde{f}(x)$ so that $\tilde{u}$ is close to u for all $-T \le t \le 0$?

To guarantee the uniqueness, one assumes that $|u|$ is bounded. As we have seen that $u \ge 0$ works also. John found that a necessary condition for such f is that it is an entire function and that $|f(x+iy)| \le N_u(a) e^{|y|^2/4a}$ for all $(x,y) \in R^2$. Here $0 < a \le T$, and $N_u(a) = \|u(x,-a)\|_{L^\infty(R^2)}$. Thus if we define $M_f(a) = \left\| e^{-y^2/4a} |f(x+iy)| \right\|_{L^\infty(R^2)}$, then $M_f(a) \le N_u(a)$.Note $N_u(a)$ is a monotone increasing function of $a \in (0,T]$. On the other hand, from the representation formula, one can also derive

$$N_u(a) \le \sup_{a \le b < T} M_f(b) \left(1 - \frac{a}{b} \right)^{-1/2}.$$

It is then easy to conclude the following:

For any f with $M_f(a) < \infty$, $0 < a < T$, there is a unique u with $N_u(a) < \infty$, $0 < a < T$, and u depends continuously on f in the following sense: $\forall\ 0 < \alpha < T$, $\beta > 0$, let h be in the set $U_{\alpha,\beta}(f) = \{g\colon M_{g-f}(\alpha) < \beta\}$, then the corresponding solution u_h will be in the set $U_{\alpha,\beta}(u) = \{v\colon N_{v-u}(\alpha) < \beta\}$.

John then considered those $u \geq 0$ solution of (4.3). Suppose f is an entire function with

$$|f(x+iy)| \leq \mu e^{|y|^2/4T}\ ,\ \mu = \|f\|_{L^\infty(R)}\,.$$

Then $M_f(T) = \mu$ and $N_u(a) \leq \mu\left(1-\frac{a}{T}\right)^{-1/2}$. John constructed approximate solutions of the form

$$\bar{u}(x,-t) = \sum_{j=-m}^{m} c_j^m(t)\,\bar{f}(x+jh)\,.$$

Then the total error

$$\begin{aligned} E(x,t) &= |u(x,-t) - \bar{u}(x,-t)| \\ &\leq E_1 + E_2. \end{aligned}$$

Here

$$E_1 = \left| u(x,-t) - \sum_{j=-m}^{m} c_j^m(t)\, f(x+jh) \right|$$

is the truncation error and

$$E_2 = \left| \sum_{j=-m}^{m} c_j^m(t)\left(f(x+jh) - \bar{f}(x+jh)\right) \right|$$

is the data error.

It was proven that (by suitable choices of $c_j^m(t)$)

$$E \leq e_1(t)\,\mu + e_2(t)\,\varepsilon$$

where

$$\begin{aligned} \varepsilon &= \left\|f - \bar{f}\right\|_{L^\infty(R)}, \\ e_1(t) &= \left\|\frac{E_1(\cdot,t)}{\mu}\right\|_{L^\infty(R)}, \\ e_2(t) &= \left\|\frac{E_2(\cdot,t)}{\mu}\right\|_{L^\infty(R)}. \end{aligned}$$

Various more refined estimates then needed in order to show (4.3) is well-behaved.

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