CHAPTER 8

Factorization of Measures on Locally Compact Spaces Induced by the Action of a Group, with Help of a Global Cross Section: Theory

From this chapter on, the notation will revert to that of Chapter 1. That is, unlike the notation in Chapters 2–7, spaces will be denoted by script symbols such as $\mathcal{X}$, $\mathcal{Y}$, etc., and random variables by capital symbols such as X, Y, etc. Free use will be made of concepts and definitions in Chapter 2–7 without always giving a reference. All spaces will be locally compact (l.c.), and a measure on such a space will always be understood to be in the Bourbaki sense (Chapter 6); in particular, a measure is regular and finite on compacta.

Suppose that a statistical problem leads to a random variable X with values in a l.c. space $\mathcal{X}$ and having a distribution P that is a member of some family of distributions. We shall assume that this family is absolutely continuous with respect to a measure λ on $\mathcal{X}$, and write $P(dx) = p(x)\lambda(dx)$. Suppose the statistical problem is invariant under the left action of a l.c. group G (what that means exactly depends on the type of problem and is irrelevant for the present discussion) and suppose we would like to obtain a factorization of λ induced by G as described in Chapter 1. The main result of such a factorization is that it leads to the distribution of a maximal invariant.

Recall that the orbits in $\mathcal{X}$ under G are the subsets of $\mathcal{X}$ of the form Gx, $x \in \mathcal{X}$, and that $t : \mathcal{X} \to \mathcal{T}$ (conditions on $\mathcal{T}$ later) is a maximal invariant if it is constant on orbits and distinguishes orbits. The random variable $T = t(X)$ is also called a maximal invariant. The space $\mathcal{T}$ is usually chosen to be a subset of some Euclidean space, if possible. Closely related to the choice of the function t is the choice of a **(global) cross section** in $\mathcal{X}$ under the action of G. This is a set $\mathcal{Z} \subset \mathcal{X}$ which has exactly one point in common with each orbit. Denote by $z(x)$ the point of $\mathcal{Z}$ on the orbit of x:

$$z(x) = \mathcal{Z} \cap Gx. \tag{8.1}$$

The choice of $\mathcal{Z}$ often precedes the choice of t and proceeds by choosing on each orbit a point in a more or less "canonical" way.

8.1. EXAMPLE. Let $\mathcal{X}$ be the space of all $n \times n$ positive definite matrices S with distinct characteristic roots and $G = O(n)$ as described in Example 7.7.8. Then the orbit of a given S consists of all matrices with the same set of characteristic roots as S, say $\lambda_1 > \cdots > \lambda_n > 0$, and an obvious choice of $z(x) = z(S)$ is $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We can write

$$S = \Gamma\Lambda\Gamma', \quad \Gamma \in O(n), \tag{8.2}$$

and letting Γ run through all of $O(n)$ traces the entire orbit of Λ. An obvious choice for $\mathcal{T}$ is $\{\lambda = (\lambda_1, \dots, \lambda_n) : \lambda_1 > \cdots > \lambda_n > 0\} \subset R^n$ and the function t is $t(x) = t(S) = \lambda \in \mathcal{T}$. □

The function $z : \mathcal{X} \to \mathcal{Z}$ given by (8.1) is also a maximal invariant and there is of course for every $x \in \mathcal{X}$ a 1-1 correspondence between $z(x)$ and $t(x)$. In principle we could dispense with t and $\mathcal{T}$ and do everything with z and $\mathcal{Z}$. However, in practice the space $\mathcal{T}$ is usually easier to work with.

The immediate aim is to bring $\mathcal{X}$ in 1-1 correspondence with a product space $\mathcal{Y} \times \mathcal{Z}$, where $\mathcal{Y}$ is to be a copy of each of the orbits in $\mathcal{X}$. From Section 2.1 we know that the points of an orbit Gx are in 1-1 correspondence with the left cosets of G modulo the isotropy subgroup of G at some arbitrarily chosen point on the orbit. The obvious choice of this point is $z(x)$, but then in order that $\mathcal{Y}$ be in 1-1 correspondence with each $G/G_{z(x)}$ we have to choose $\mathcal{Z}$ in such a way that $G_{z(x)}$ for every x is the same subgroup of G, say G_0. In that case, put $\mathcal{Y} = G/G_0$. For later use we shall need G_0 to be compact in order that a measure (invariant or relatively invariant) on G induces one on $\mathcal{Y}$, by Corollary 7.4.4. The points of $\mathcal{Y}$ will be denoted y, or gG_0, or $[g]$. An arbitrary point $x \in \mathcal{X}$ can be reached by starting at $z(x)$ and then moving along the orbit to x with help of some $g \in G$: $x = gz(x)$, where g is determined only up to the coset $[g]$. Thus, the value of y in this transformation is unique. Therefore, define the function $y : \mathcal{X} \to \mathcal{Y}$ by

$$y(x) = [g] \quad \text{if} \quad x = gz, \quad z \in \mathcal{Z}. \tag{8.3}$$

Then an arbitrary $x \in \mathcal{X}$ has the unique representation

$$x \leftrightarrow (y, z), \quad y = y(x), \quad z = z(x), \tag{8.4}$$

so that (8.4) defines a 1-1 correspondence between $\mathcal{X}$ and $\mathcal{Y} \times \mathcal{Z}$. The group G acts on $\mathcal{Y}$ because it acts on G/G_0 according to (2.1.3) (H there is G_0 here). Furthermore, G acts trivially on $\mathcal{Z}$. Therefore,

$$gx \leftrightarrow (gy, z) \quad \text{if} \quad x \leftrightarrow (y, z). \tag{8.5}$$

Let the 1-1 correspondence between $z(x)$ and $t(x)$ be expressed as a function s:

$$s(t(x)) = z(x). \tag{8.6}$$

Then if $\mathcal{T}$ is the range of the function t, $s : \mathcal{T} \to \mathcal{X}$ is a 1-1 map of $\mathcal{T} \to \mathcal{Z}$ and the 1-1 correspondence between x and (y, z) can be transferred to a 1-1 correspondence between x and $(y, t) = (y, s^{-1}z)$. (Here we have used t to mean a point of $\mathcal{T}$. We shall continue doing so if there can be no confusion with the function t.) For reasons explained in Chapter 1, we impose differentiability conditions on the various spaces and functions. Differentiability will be understood to be in the C^1 sense unless specified otherwise. However, in all examples in this monograph the manifolds and functions will in fact be analytic. We shall assume analyticity explicitly whenever there is special group structure.

8.2. ASSUMPTION. *Let the spaces $\mathfrak{X}$, $\mathfrak{T}$ and the group G be differentiable manifolds with group action $(x,g) \to gx$ of G on $\mathfrak{X}$ differentiable. Let G_0 be a compact subgroup of G, and put $\mathfrak{Y} = G/G_0$. Suppose $s : \mathfrak{T} \to \mathfrak{X}$ is differentiable and maps 1-1 onto a cross section $\mathfrak{Z}$ such that $G_z = G_0$ for all $z \in \mathfrak{Z}$. Define $\varphi : \mathfrak{Y} \times \mathfrak{T} \to \mathfrak{X}$ by*

$$\varphi(y,t) = gs(t) \quad \textit{if} \quad [g] = y, \tag{8.7}$$

so that φ is differentiable and 1-1 onto. Assume that φ has a positive Jacobian at every point $([e],t)$ of $\mathfrak{Y} \times \mathfrak{T}$, where e is the identity element of G.

8.3. ASSUMPTION. *In Assumption 8.2 replace "differentiable" by "analytic."*

8.4. PROPOSITION. *If Assumption 8.2 is satisfied, then φ of (8.7) establishes a diffeomorphism between the three spaces $\mathfrak{X}$, $\mathfrak{Y} \times \mathfrak{T}$, and $\mathfrak{Y} \times \mathfrak{Z}$, and $\mathfrak{Z}$ is a submanifold of $\mathfrak{X}$. If Assumption 8.3 is satisfied, the diffeomorphism is analytic.*

PROOF. Take $t \in \mathfrak{T}$ arbitrary. By (8.7), $\varphi([e],t) = s(t) = z$, say, where $z \in \mathfrak{Z}$. Consider z as a point in $\mathfrak{X}$, then by Theorem 3.1.1 and Assumption 8.2 there is a neighborhood of z in $\mathfrak{X}$ on which φ^{-1} is differentiable. The same is then true for a point of the form gz, $g \in G$, since $x \to gx$ and $(y,t) \to (gy,t)$ are diffeomorphic transformations of $\mathfrak{X}$ with itself and of $\mathfrak{Y} \times \mathfrak{T}$ with itself. Since every $x \in \mathfrak{X}$ is of the form gz, $g \in G$, $z \in \mathfrak{Z}$, it follows that φ^{-1} is differentiable on all of $\mathfrak{X}$. Now let $z = s(t)$, $t \in \mathfrak{T}$, be an arbitrary point of $\mathfrak{Z}$. At $([e],t) \in \mathfrak{Y} \times \mathfrak{T}$ take a chart with local coordinates y_i, t_j. Transfer this chart to $z \in \mathfrak{X}$, which is justified by the diffeomorphism between $\mathfrak{Y} \times \mathfrak{T}$ and $\mathfrak{X}$. That is, the y_i and t_j may be taken as local coordinates in a neighborhood, say U, of z. In U, the points of $\mathfrak{Z}$ are parametrized by the t_j, since the y_i are 0 on $\mathfrak{Z}$. This establishes $\mathfrak{Z}$ as a submanifold of $\mathfrak{X}$ (Section 3.4). If Assumption 8.3 is satisfied, then in the preceding argument "differentiable" may be replaced by "analytic." □

8.5. DEFINITION. *A (global) cross section $\mathfrak{Z}$ will be called* **differentiable** *if Assumption 8.2 is satisfied, and* **analytic** *if Assumption 8.3 is satisfied.*

After having established the aspects of the structure of $\mathcal{X}$ under the action of G that are of an algebraic and analytic nature we turn our attention to distributional aspects. Suppose on $\mathcal{X}$ is given a measure λ that is relatively invariant (under G) with multiplier χ. This was defined in Section 7.3, equation (7.3.1), for the right action of a group. Here we have left action of G on $\mathcal{X}$, and the definition is then

$$\lambda(gf) = \chi(g)\lambda(f), \quad g \in G, \ f \in \mathcal{K}(\mathcal{X}), \tag{8.8}$$

or, equivalently,

$$\lambda(gB) = \chi(g)\lambda(B), \quad B \text{ compact } \subset \mathcal{X}. \tag{8.9}$$

Recall that χ is a continuous homomorphism $G \to R_+^*$ and that $\chi \equiv 1$ on G_0 since G_0 is compact (Corollary 7.1.8). This implies, as in Section 7.4, that χ depends on G only through $[g] = y$, so that we shall often write $\chi(y)$ (as in (7.4.5)). The diffeomorphism φ of Proposition 8.4 is certainly a homeomorphism and therefore φ^{-1} transforms λ into a measure on $\mathcal{Y} \times \mathcal{T}$, say λ^*, which is also relatively invariant with multiplier χ under the action $(y,t) \to (gy,t)$ of G on $\mathcal{Y} \times \mathcal{T}$. If $\mu_{\mathcal{Y}}$ is a measure on $\mathcal{Y}$, then $\chi\mu_{\mathcal{Y}}$ will stand for the measure $\chi(y)\mu_{\mathcal{Y}}(dy)$.

8.6. THEOREM. *Let Assumption 8.2 be satisfied and let λ be a relatively invariant measure on $\mathcal{X}$ with multiplier χ. Let μ_G be a version of left Haar measure on G and $\mu_{\mathcal{Y}} = \pi(\mu_G)$ the corresponding invariant measure on $\mathcal{Y}$, where π is the coset projection $G \to G/G_0$. Then there is a measure $\mu_{\mathcal{T}}$ on $\mathcal{T}$ such that*

$$\lambda^* \equiv \varphi^{-1}(\lambda) = \chi\mu_{\mathcal{Y}} \otimes \mu_{\mathcal{T}}. \tag{8.10}$$

If X is a random variable with values in $\mathcal{X}$ and distribution

$$P(dx) = p(x)\lambda(dx), \tag{8.11}$$

then the distribution of the maximal invariant $T = t(X)$ with values in $\mathcal{T}$ is

$$P^T(dt) = \mu_{\mathcal{T}}(dt) \int p(gs(t))\chi(g)\mu_G(dg). \tag{8.12}$$

PROOF. The equation (8.10) follows immediately from Theorem 7.5.1 by taking in that theorem $X_1 = \mathcal{Y}$ (note that by Proposition 2.3.11 G acts transitively and properly on $\mathcal{Y}$ since G_0 is compact), $X_2 = \mathcal{T}$, $\mu = \lambda^*$, $\mu_1 = \chi\mu_{\mathcal{Y}}$ (as in (7.4.5)) and $\mu_2 = \mu_{\mathcal{T}}$. Now let $(Y,T) = \varphi^{-1}(X)$ and $p^*(y,t) = p(\varphi(y,t))$, then the distribution of (Y,T) is $p^* d\lambda^*$. Using (8.10) this distribution can be written

$$P^{Y,T}(dy,dt) = p^*(y,t)\chi(y)\mu_{\mathcal{Y}}(dy)\mu_{\mathcal{T}}(dt). \tag{8.13}$$

Integration over y provides the marginal distribution of T. The integration over y can be carried out, instead, over the group G (see (7.4.6) and (7.4.7)). For this purpose write p^* as a function of g and t, say $p_1^*(g,t) = p^*(\pi(g),t) = p^*([g],t)$. The latter expression equals $p(x)$ with $x = \varphi([g],t) = gs(t)$ by (8.7). Therefore, the integration over y of the right-hand side of (8.13) yields $\int p^*(y,t)\chi(y)\mu_{\mathcal{Y}}(dy) = \int p(gs(t))\chi(g)\mu_G(dg)$, so that (8.12) follows. □

Equation (8.10) can be written more explicitly as

$$\lambda(dx) = \lambda^*(dy,dt) = \chi(y)\mu_{\mathcal{Y}}(dy)\mu_{\mathcal{T}}(dt). \tag{8.14}$$

An explicit expression for $\mu_{\mathcal{T}}$ is not given by the theorem, but follows from computation of the Jacobian of φ in the points $([e],t)$, $t \in \mathcal{T}$ (this will always be carried out by computing a wedge product of differentials). This leads to an expression $\lambda(dx) = \mu_{\mathcal{Y}}(dy)\nu(dt)$ at these points, with some measure ν on $\mathcal{T}$. Comparison with (8.14) at $y = [e]$ (where $\chi(y) = 1$), i.e, with

$$\lambda(dx) = \mu_{\mathcal{Y}}(dy)\mu_{\mathcal{T}}(dt) \quad \text{at} \quad y = [e], \tag{8.15}$$

shows then that $\mu_{\mathcal{T}} = \nu$.

8.7. EXAMPLE (continuation of Example 8.1). With the choice of $\mathcal{Z}$ and $\mathcal{T}$ of Example 8.1 the isotropy subgroup G_0 consists of all matrices $\operatorname{diag}(\pm 1, \ldots, \pm 1)$ (see Example 7.7.8). Take μ_G as in (7.7.7), then $\mu_{\mathcal{Y}}$ at $y = [e]$ is given by (7.7.16) so that

$$(d\Gamma) = 2^{-n}\mu_{\mathcal{Y}}(dy) \quad \text{at} \quad \Gamma = e. \tag{8.16}$$

Now differentiate (8.2) at $g = e$: $dS = (d\Gamma)\Lambda + d\Lambda + \Lambda d\Gamma' = (d\Gamma)\Lambda - \Lambda d\Gamma + d\Lambda$ (using the skew symmetry of $d\Gamma$ at e, see Section 7.7). Then $ds_{ii} = d\lambda_i$, and for $i < j$, $ds_{ij} = (\lambda_j - \lambda_i)d\gamma_{ij}$ which we may also write as $(\lambda_i - \lambda_j)d\gamma_{ij}$ by the irrelevance of the sign in a wedge product when used to define a measure. Take the wedge product of the above differentials, then $(dS) = (d\Lambda)\prod(\lambda_i - \lambda_j)(d\Gamma)$, where the product is over all $i < j$. With help of (8.16) this becomes

$$(dS) = 2^{-n}\prod_{i<j}(\lambda_i - \lambda_j)\mu_{\mathcal{Y}}(dy)(d\Lambda) \quad \text{at} \quad S = \Lambda. \tag{8.17}$$

Let the relatively invariant measure λ of Theorem 8.6 be (dS) here (this is even invariant since G is compact so that $\chi \equiv 1$). Then by comparing the right-hand sides of (8.15) and (8.17) we see that

$$\mu_{\mathcal{T}}(dt) = 2^{-n}\prod_{i<j}(\lambda_i - \lambda_j)(d\Lambda) \tag{8.18}$$

and (8.12) becomes

$$P^T(d\Lambda) = 2^{-n}\prod_{i<j}(\lambda_i - \lambda_j)(d\Lambda)\int p(\Gamma\Lambda\Gamma')\mu_{O(n)}(d\Gamma) \tag{8.19}$$

with $\mu_{O(n)} = (d\Gamma)$ at $\Gamma = I_n$. □

Further applications of Theorem 8.6 and the use of (8.15) to obtain $\mu_{\mathcal{T}}$ will appear in Chapter 10. Those will be called **Type II** problems. The remainder of this chapter will assume more structure on the statistical model. This will be called **special group structure** and the problems it can handle will be called **Type I**. Several examples of those are treated in Chapter 9.

The further structure alluded to above consists in the presence of an additional group H that acts on $\mathcal{X}$ such that roughly speaking G and H together are transitive over $\mathcal{X}$. The aim is to obtain an orbit in $\mathcal{X}$ under the action of H as a cross section $\mathcal{Z}$. This will require several conditions. First, let K be a Lie group that is transitive over $\mathcal{X}$ such that Assumption 5.9.1 is satisfied. Let x_0 be an arbitrary point of

$\mathcal{X}$ and let G_0, H_0, K_0 be the isotropy subgroups at x_0 of G, H, K, respectively. Then $\mathcal{X}$ is in 1-1 correspondence with K/K_0, but we need this to be at least a homeomorphism. We shall therefore make the following assumption.

8.8. ASSUMPTION. *$\mathcal{X}$ and K/K_0 are homeomorphic. (This will be true, in particular, if the action of K on $\mathcal{X}$ is proper (Corollary 2.3.15) or if K is second countable (Lemma 2.3.17).)*

In our applications K will always be put together of subgroups of the general linear group and of groups of matrices under translation. Then K is a submanifold of some Euclidean space and therefore second countable. Assumption 8.8 together with the conclusion of Theorem 5.9.9, if applicable, sets up a homeomorphism between $\mathcal{X}$ and $G/G_0 \times H/H_0$. Then one can take $\mathcal{Y} = G/G_0$ as before, $\mathcal{T} = H/H_0$, $\mathcal{Z} = Hx_0$, and the function s of (8.6): $s(hH_0) = hx_0$. Before stating this formally it is convenient to recast Assumption 5.9.3 in terms of the actions of G and H on $\mathcal{X}$.

8.9. ASSUMPTION. *For arbitrary $g \in G$, $h \in H$, assume that $gx_0 = hx_0$ implies $gx_0 = hx_0 = x_0$.*

The following lemma establishes the equivalence of Assumptions 5.9.3 and 8.9. Note that no topology is involved, only algebra.

8.10. LEMMA. *Let K be a group with subgroups G and H such that $K = GH$, and let K act transitively on $\mathcal{X}$ to the left. Choose any $x_0 \in \mathcal{X}$ and let G_0, H_0, K_0 be the isotropy subgroups at x_0 of G, H, K, respectively. Then the statements A and B below are equivalent.*

(A) (i) $G \cap H = G_0 \cap H_0$; (ii) $K_0 = G_0 H_0$.
(B) For every $g \in G$, $h \in H$, $gx_0 = hx_0$ implies $gx_0 = hx_0 = x_0$.

PROOF. Suppose B holds. Let $f \in G \cap H$, then $fx_0 = fx_0$ is of the form $gx_0 = hx_0$ ($g \in G$, $h \in H$) so that $f \in G_0 \cap H_0$. Hence, $G \cap H \subset G_0 \cap H_0$ so that A(i) has been shown. Next, let $k_0 \in K_0$ so that k_0 (as any member of K) can be written $k_0 = gh$ ($g \in G$, $h \in H$). Then $ghx_0 = x_0$ so that $g^{-1}x_0 = hx_0$ which, by B, implies $g \in G_0$, $h \in H_0$. It follows that $K_0 \subset G_0 H_0$, proving A(ii).

Now suppose A holds and suppose $gx_0 = hx_0$ for some $g \in G$, $h \in H$. It is to be shown that $g \in G_0$, $h \in H_0$. Since $g^{-1}hx_0 = x_0$ we have $g^{-1}h \in K_0$ so that $g^{-1}h = g_0h_0$ for some $g_0 \in G_0$, $h_0 \in H_0$, using A(ii). This can be written $gg_0 = hh_0^{-1}$. Let the common value be f, then $f \in G \cap H = G_0 \cap H_0$ by A(i), so that $g = fg_0^{-1} \in G_0$, $h = fh_0 \in H_0$. □

The conditions needed for the next theorem are collected in the following assumption.

8.11. ASSUMPTION. *Let the Lie group K act continuously and transitively on the l.c. space $\mathfrak{X}$ and let G, H be two closed Lie subgroups of K such that $K = GH$. For arbitrary $x_0 \in \mathfrak{X}$ let G_0, H_0, K_0 be the isotropy subgroups at x_0 of G, H, K, respectively. Let Assumptions 5.9.2 and 8.8 be satisfied (true, for instance, if K is second countable). Furthermore assume*

(i) for every $g \in G$, $h \in H$, $gx_0 = hx_0$ implies $gx_0 = hx_0 = x_0$;
(ii) $hG_0h^{-1} = G_0$ for every $h \in H$.

8.12. THEOREM. *Let Assumption 8.11 be satisfied and define φ : $G/G_0 \times H/H_0 \to \mathfrak{X}$ by*

$$\varphi(gG_0, hH_0) = ghx_0. \tag{8.20}$$

Then φ is a homeomorphism and is the function φ of (8.7) if we put $\mathcal{Y} = G/G_0$, $\mathcal{T} = H/H_0$, $\mathcal{Z} = Hx_0$, and $s(t) = s(hH_0) = hx_0 \in \mathcal{Z}$.

PROOF. Using Lemma 8.10, Assumption 8.11 is the union of Assumption 8.8 and Assumptions 5.9.1–5.9.4. Therefore, Theorem 5.9.9 applies. The function ϕ of (5.9.6) and φ of (8.20) differ only in that the range of the former is K/K_0, of the latter $\mathfrak{X}$. Let $\psi : K/K_0 \to \mathfrak{X}$ be the homeomorphism of Assumption 8.8. Then $\varphi = \psi \circ \phi$, which is the composition of two homeomorphisms. Comparison of (8.20) and (8.7) establishes the remaining claims. □

8.13. REMARK. In 8.8 it is not assumed that the homeomorphism is bi-analytic, nor even a diffeomorphism. However, in all applications in this monograph the space $\mathfrak{X}$ is in fact an analytic manifold with the action of K on $\mathfrak{X}$ analytic and the homeomorphism ψ of 8.8 an analytic diffeomorphism. The same is true of φ, and $\mathfrak{Z}$ is then an analytic cross section by Definition 8.5. $\square$

From now on we shall assume G_0 and H_0 to be compact, which implies $K_0 = G_0 H_0$ to be compact. This makes it possible to derive invariant measures on the cosets from invariant measures on the group. Given a relatively invariant measure λ on $\mathfrak{X}$ we would like to factor it in the manner of (8.14) by using the homeomorphism of Theorem 8.12. Now $\mathfrak{T} = H/H_0$ is also a homogeneous space, and $\mu_{\mathfrak{T}}$ can be derived from left Haar measure on H. The precise form of the factorization follows from the formula (7.6.5) derived in Section 7.6.

8.14. THEOREM. *Let Assumption 8.11 be satisfied and in addition suppose that G_0 and H_0 (and therefore K_0) are compact. Choose versions μ_G and μ_H of left Haar measure on G, H, respectively, and let $\mu_{\mathfrak{Y}}$, $\mu_{\mathfrak{T}}$ be the corresponding measures on $\mathfrak{Y} = G/G_0$ and $\mathfrak{T} = H/H_0$, respectively. Let λ be a relatively invariant measure on $\mathfrak{X}$ under the action of K with multiplier χ and define*

$$\beta(h) = \chi(h)\Delta^K(h)\Delta^H(h^{-1}), \quad h \in H. \tag{8.21}$$

Then under the homeomorphism φ of Theorem 8.12, λ factors into a product measure on $\mathfrak{Y} \times \mathfrak{T}$ as follows:

$$\lambda(dx) = c\chi(y)\mu_{\mathfrak{Y}}(dy)\beta(t)\mu_{\mathfrak{T}}(dt), \tag{8.22}$$

where the positive constant c depends on the chosen versions of μ_G and μ_H. Consequently, if X is a random variable with values in $\mathfrak{X}$ and distribution $P(dx) = p(x)\lambda(dx)$, and if X corresponds to (Y, T) with values in $\mathfrak{Y} \times \mathfrak{T}$, then the distribution of T is

$$P^T(dt) = c\beta(t)\mu_{\mathfrak{T}}(dt) \int p(ghx_0)\chi(g)\mu_G(dg), \quad [h] = t, \tag{8.23}$$

in which the integrand on the right-hand side depends on h only through $[h] = t$. If H is normal in K, then $\beta(t) = \chi(t)$, and if G is normal in K, then $\beta(t) = \chi(t)\delta(t)$ with δ defined in Corollary 7.6.3. In particular, if G and H commute, then (8.23) holds with $\beta(t) = \chi(t)$.

PROOF. We shall identify $\mathcal{X}$ and K/K_0. The situation is the same as in Section 7.4, with K and $\mathcal{X}$ here taking the place of G and Y there, and λ here taking the place of ν of Corollary 7.4.4. Equating the right-hand sides of (7.4.6) and (7.4.7) while taking (7.4.5) into account, yields, after appropriate change of notation,

$$\int f(x)\lambda(dx) = c \int f(kx_0)\chi(k)\mu_K(dk) \tag{8.24}$$

for some $c > 0$, where $\mu_K(dk)$ is any choice of left Haar measure on K and f is λ-integrable. Take 7.6.5 and replace $f(k)$ by $f(kx_0)\chi(k)$. Combine this with (8.24) to obtain

$$\int f(x)\lambda(dx) = c \iint f(ghx_0)\chi(gh)\Delta^K(h)\Delta^H(h^{-1}) \cdot \mu_G(dg)\mu_H(dh). \tag{8.25}$$

On the right-hand side of (8.25) the integrand depends on g and h only through $[g] = y$ and $[h] = t$ so that the integration can be carried out on $\mathcal{Y} \times \mathcal{T}$ with respect to $\mu_{\mathcal{Y}}(dy)\mu_{\mathcal{T}}(dt)$. Then observe that $\chi(gh) = \chi(g)\chi(h) = \chi(y)\chi(t)$ and, after taking account of (8.21), (8.22) results. To obtain (8.23) multiply (8.22) on both sides by $p(x) = p(ghx_0)$ and integrate over $\mathcal{Y}$. Then replace the integral over $\mathcal{Y}$ by one over G. The statements about H or G normal in K follow from Corollary 7.6.2 and Corollary 7.6.3, respectively. □

The constant c in (8.22) and (8.23) may be evaluated in either of two different ways. The first way consists of choosing a density p sufficiently simple that the integral with respect to t of the right-hand side of (8.23) can be carried out explicitly. Setting the result equal to 1 determines c. The second way, which we shall always follow in this monograph because it requires usually less computation, consists of writing (8.22) at $x = x_0$, so $y = [e]$, $t = [e]$, and consequently $\chi(y) = \beta(t) = 1$:

$$\lambda(dx) = c\mu_{\mathcal{Y}}(dy)\mu_{\mathcal{T}}(dt), \quad \text{at} \quad x = x_0. \tag{8.26}$$

Then we shall express both sides in terms of differential forms, and comparison of the forms yields c. It is assumed here, of course, that $\mathcal{X}$ is a differentiable manifold.