A MULTIPLE CRITERIA OPTIMAL SELECTION PROBLEM

Stephen M. Samuels

Purdue University

and

Bay Chotlos

Butler University

For each $m \geq 2$ and for stopping rules, τ,

$$E\min_{\tau \leq n} \sum_{j=1}^{m} X_{\tau}^{(j)} \approx n^{1-1/m}[(m+1)!/m]^{1/m}$$

if either the $X_i^{(j)}$'s are i.i.d., uniform on $(0,n)$; or $\{X_i^{(1)}\},\ldots,\{X_i^{(m)}\}$ are m independent random permutations of 1 to n and the τ's are based only on relative ranks. This equivalence fails when $m=1$.

1. Introduction.

Chow et al (1964) solved an optimal stopping problem which Lindley (1961) had earlier considered. Lindley tried an approximation which (as he himself noted) was not successful. This article presents an extension of that problem, in which Lindley's approximation does succeed, as well as an extreme value problem for sampling without replacement which is a companion to the optimal stopping problem.

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2. <u>The Original Problem</u>.

Here is a brief description of the problem considered by Lindley (1961) and by Chow et al (1964):

For each $n=1,2,3,\ldots$, there is a vector $\mathbf{X} = \langle X_1,X_2,\ldots,X_n\rangle$ which is a random permutation of the integers 1 to n. One may think of $\mathbf{X}$ as the ranks of n successive "applicants"; rank 1 = best and rank n = worst. $\mathbf{X}$ then determines a vector $\mathbf{Y} = \langle Y_1,Y_2,\ldots,Y_n\rangle$ of relative ranks. ($Y_i = k$ if X_i is the k-th smallest among X_1 to X_i.) Only the Y_i's are observed; i.e. each successive applicant can be compared only with its predecessors. The object is to find a stopping rule, τ, adapted to the Y_i's (i.e. based only on relative ranks) which minimizes EX_τ.

The Y_i's are independent, each Y_i is uniform on the integers 1 to i, and $E(X_i|Y_i) = ((n+1)/(i+1))Y_i$. From these well-known facts it is almost immediate that an equivalent formulation of the problem is: "For such Y_i's find a stopping rule τ, to minimize $E((n+1)/(\tau+1))Y_\tau$."

For each n, it not hard to solve the problem by backward induction as in Chow, Robbins and Siegmund (1971). If we let $c_i^{(n)}$ denote the minimum expected rank achievable with stopping rules $\tau > i$, and s_i be the integer part of $((i+1)/(n+1))c_i$, then, as shown in Chow et al (1964),

$$c_{n-1}^{(n)} = (n+1)/2 \tag{2.1}$$

$$c_{i=1}^{(n)} = \frac{1}{i}\,\frac{n+1}{i+1}\,\frac{s_i(s_i+1)}{2} + \left(1-\frac{s_i}{i}\right) c_i^{(n)} \qquad i=n-1,n-2,\ldots,1.$$

Then

$$c_o^{(n)} = \min_\tau EX_\tau$$

which is attained for τ = the first i for which $Y_i < s_i$.

However, it is not so easy to see what happens as n becomes infinite. Lindley, in effect, used the approximations

$$s_i \approx s_{i+1} \approx \frac{i}{n} c_i; \quad \frac{n+1}{i+1} \approx \frac{n}{i}$$

which transforms (2.1) to

$$\tilde{c}^{(n)}_{n-1} = n/2 \tag{2.2}$$

$$\tilde{c}^{(n)}_{i-1} = \frac{\tilde{c}^{(n)}_i}{n} \frac{\tilde{c}^{(n)}_i}{2} + \left(\frac{1-\tilde{c}^{(n)}_i}{n}\right) \tilde{c}^{(n)}_i$$

Rewriting this in "differential" form as

$$\frac{\tilde{c}^{(n)}_i - \tilde{c}^{(n)}_{i-1}}{1/n} = \frac{1}{2}(\tilde{c}^{(n)}_i)^2$$

suggests that

$$i/n \to t \Rightarrow \tilde{c}^{(n)}_i \to C(t)$$

with

$$C'(t) = \frac{1}{2}C^2(t); \quad C(1) = \infty;$$

so

$$C(t) = \frac{2}{1-t}; \quad C(0) = 2.$$

But

$$\lim_{n\to\infty} c^{(n)}_0 \neq 2.$$

Indeed, there's really no way that this approximation can work. (In Lindley's words "the approximation is suspect".) The correct answer, in Chow et al (1964), is

$$\lim_{n\to\infty} c_0^{(n)} = \prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{1/(j+1)} \tag{2.3}$$

$$= 3.8695$$

It is noteworthy that (2.2) is precisely the backward induction algorithm for the problem: "For $\tilde{Y}_i$'s which are independent and uniform on the interval $(0,n)$, find the stopping rule, τ, to minimize $E\tilde{Y}_\tau$". In other words, Lindley's approximation is tantamount to replacing the independent sequence of Y_i's uniform on the sets

$$\left\{\frac{1}{i+1}(n+1), \frac{2}{i+1}(n+1), \ldots, \frac{i}{i+1}(n+1)\right\}, \quad i=1,2,\ldots,n$$

by the i.i.d. sequence of $\tilde{Y}_i$'s, all uniform on the interval $(0,n)$.

3. A Two-Criteria Problem.

We now suppose that each of n applicants can be independently ranked according to each of two criteria. We model this by having $\langle X_1^{(1)}, X_2^{(1)}, \ldots, X_n^{(1)}\rangle$ and $\langle X_1^{(2)}, X_2^{(2)}, \ldots, X_n^{(2)}\rangle$ be two independent permutations of the integers 1 to n; and $\langle Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_n^{(1)}\rangle$ and $\langle Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_n^{(2)}\rangle$ be the corresponding sequences of relative ranks. The object is to find a stopping rule, τ, adapted to the Y's, to minimize $E(X_\tau^{(1)} + X_\tau^{(2)})$.

As in the original problem, this is equivalent to finding a stopping rule, τ, to minimize $E\frac{n+1}{\tau+1}(Y_\tau^{(1)} + Y_\tau^{(2)})$, when $Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_n^{(1)}$, $Y_1^{(2)}, Y_2^{(2)}, \ldots, Y_n^{(2)}$ are independent, with $Y_i^{(j)}$ uniform on the integers 1 to i.

Let us call this the Permutation Problem, and also introduce the following Continuous Uniforms Problem: Let $\tilde{Y}_1^{(1)}, \tilde{Y}_2^{(1)}, \ldots, \tilde{Y}_n^{(1)}$, $\tilde{Y}_1^{(2)}, \tilde{Y}_2^{(2)}, \ldots, \tilde{Y}_n^{(2)}$ be i.i.d., each uniform on the interval $(0,n)$. Find a stopping rule, τ, to minimize $E(\tilde{Y}_\tau^{(1)} + \tilde{Y}_\tau^{(2)})$.

Using the same definition of $c_i^{(n)}$ and s_i as in the original problem, the backward induction algorithms are

$$(3.1)\quad c_{n-1}^{(n)} = n + 1$$

$$c_{i-1}^{(n)} = \frac{1}{i^2}\left(\frac{n+1}{i+1}\right)\frac{(s_i-1)s_i(s_i+1)}{3} + \left(1 - \frac{s_i(s_i-1)}{2i^2}\right)c_i^{(n)} \quad i=n-1,n-2,\ldots,1$$

for the Permutation Problem, and

$$(3.2)\qquad \tilde{c}_{n-1}^{(n)} = n$$

$$\tilde{c}_{i-1}^{(n)} = \frac{(\tilde{c}_i^{(n)})^3}{3n^2} + \left(1 - \frac{(\tilde{c}_i^{(n)})^2}{2n^2}\right)\tilde{c}_i^{(n)}$$

for the Continuous Uniforms Problems. Rewriting (3.2) in "differential" form as

$$\frac{\tilde{c}_i^{(n)}/\sqrt{n} - \tilde{c}_{i-1}^{(n)}/\sqrt{n}}{1/n} = \frac{(\tilde{c}_i^{(n)}/\sqrt{n})^3}{6}$$

suggests that

$$1/n \to t \Rightarrow \tilde{c}_i^{(n)}/\sqrt{n} \to D(t)$$

with

$$D'(t) = \frac{1}{6}D^3(t); \qquad D(1) = \infty$$

so

$$D(t) = \sqrt{\frac{3}{1-t}}\ ; \qquad D(0) = \sqrt{3}$$

These heuristics can be validated, for the Continuous Uniforms Problem, by establishing the inequalities

$$(3.3) \qquad \left(\frac{3}{1-\frac{i-2\sqrt{n-i}}{n}}\right)^{1/2} < \frac{\tilde{c}_i^{(n)}}{\sqrt{n}} < \left(\frac{3}{1-\frac{i-2}{n}}\right)^{1/2}$$

which are a special case of (8.8), to be proved later.

From (3.1) and (3.2), it can be shown that, for each n, the Permutation Problem risk is greater than the Continuous Uniforms Problem risk, i.e.,

$$(3.4) \qquad c_0^{(n)} > \tilde{c}_0^{(n)} \qquad n=2,3,\ldots$$

Indeed

$$(3.4.1) \qquad c_{n-1}^{(n)} = n+1 = \frac{n+1}{n}\,\tilde{c}_{n-1}^{(n)}$$

and

$$(3.4.2) \qquad c_i^{(n)} > \frac{n+1}{n}\,\tilde{c}_i^{(n)} \Rightarrow c_{i-1}^{(n)} > \frac{n+1}{n}\,\tilde{c}_{i-1}^{(n)} \quad i=n-1,n-2,\ldots,1$$

as will be proved in Section 6.

But, asymptotically, the difference seems to disappear, at least to first order, according to the following numerical results:

n	$c_0^{(n)}/\sqrt{n}$
500	1.9215
1,000	1.8818
2,500	1.8397
10,000	1.7957
100,000	1.7574
	$\sqrt{3}$ = 1.7321

The limit of $c_0^{(n)}/\sqrt{n}$ is indeed $\sqrt{3}$, as a special case of Theorem 2 to follow.

Unlike the original problem, where the risk never exceeds four, in the two-criteria problem the risk goes to infinity like $\sqrt{n}$. This is inevitable, by the following extreme-value results:

$$(3.5) \qquad E[\min_{1\le i\le n}(\tilde{Y}_i^{(1)} + \tilde{Y}_i^{(2)})] \approx \sqrt{\pi/2}\,\sqrt{n}$$

which is a special case of (4.1.1) derived in Section 7; and

$$(3.6) \qquad E[\min_{1\le i\le n}(X_i^{(1)} + X_i^{(2)})] \approx \sqrt{\pi/2}\,\sqrt{n}$$

which can be proved by first noting that the left side of (3.6) is equal to

$$(3.7) \qquad E[\min_{1\le i\le n}(i + X_i^{(2)})] = 1 + \frac{1}{n!}\sum_{j=0}^{n-1}(n-j)^j(n-j)!$$

Analysis of the right side of (3.7) shows that $\min_{1\le i\le n}(X_i^{(1)} + X_i^{(2)})/\sqrt{n}$ converges in distribution and in mean to the square root of an exponential random variable with parameter 1/2.

From (3.5) and (3.6) we see that, to first order at least, the extreme-value means are asymptotically identical for both the Permutation Problem and the Continuous Uniforms Problem.

4. An m-Criteria Problem.

For the extension to an arbitrary number of criteria, both the Permutation Problem and the Continuous Uniforms Problem results agree asymptotically,to the first order (i.e. Lindley's approximation is successful). Here are the main results:

THEOREM 1. Extreme-value results for $m > 2$. As $n \to \infty$

$$(4.1) \qquad \frac{1}{n^{1-1/m}} E \min_{1\le i\le n} \sum_{j=1}^{m} X_i^{(j)} \to (m!)^{1/m}\,\Gamma(1+1/m)$$

if either

(4.1.1) $X_i^{(j)}$'s are i.i.d., uniform on the interval (0,n)

or

(4.1.2) $$\langle X_1^{(1)},\ldots,X_n^{(1)}\rangle,\ldots,\langle X_1^{(m)},\ldots,X_n^{(m)}\rangle$$

are m independent random permutations of the integers 1 to n.

THEOREM 2. Stopping rule results for $m \geq 2$. As $n \to \infty$

(4.2) $$\frac{1}{n^{1-1/m}} E\,[\min_{\tau \leq n} \sum_{j=1}^{m} X_\tau^{(j)}] \to (\frac{(m+1)!}{m})^{1/m}$$

if either

(4.2.1) $X_i^{(j)}$'s are i.i.d, uniform on the interval (0,n)

or

(4.2.2) $$\langle X_1^{(1)},\ldots,X_n^{(1)}\rangle,\ldots,\langle X_1^{(m)},\ldots,X_n^{(m)}\rangle$$

are m independent random permutations of the integers 1 to n, and τ's are based only on relative ranks.

The proofs of these theorems are in sections 7 and 8.

5. The Prophet's Advantage.

It is interesting to compare the right sides of (4.1) and (4.2). The latter represents the best that can be done, asymptotically, with stopping rules, while the former is the corresponding best for a "prophet": one who can foresee all n applicants and choose the best one.

The ratio of the right sides of (4.2) to (4.1), after simplification, becomes

$$(1+1/m)^{1/m}/\Gamma(1+1/m). \tag{5.1}$$

If we define $H(x) = \Gamma(1+x)/(1+x)^x$, then (5.1) is $1/H(1/m)$. By a standard integral formula, we have

$$H(x) = (1+x) \int_0^1 u^x(-\log u)^x \, du, \tag{5.2}$$

from which it is not hard to verify that the derivative of H is negative, and, by the dominated convergence theorem, $H(x) \to 1$ as $x \to 0$. It follows that (5.1) decreases to 1 as m increases to infinity. So, to first order, the prophet's advantage steadily disappears as the number of criteria increases.

Here are some values of the ratio (5.1):

TABLE 1
RATIO OF ASYMPTOTIC OPTIMAL STOPPING RULE RISK
TO OPTIMAL SEE-ALL RISK

m	Value of (5.1)
1	3.8695
2	1.3820
3	1.2326
4	1.1666
5	1.1296
10	1.0612
20	1.0297
30	1.0196
40	1.0146
50	1.0117
70	1.0083
100	1.0058

6. Proof of (3.4): Perm. Prob. Risk > Cont. Uniforms Prob. Risk.

It is convenient to normalize by letting $d_i = c_i/\sqrt{3}(n+1)$ and

$\tilde{d}_i = \tilde{c}_i/\sqrt{3}n$. Then $d_{n-1} = \tilde{d}_{n-1} = 1/\sqrt{3}$ and, for $i=n-1,n-2,\ldots,1$,

$$\tilde{d}_{i-1} = \tilde{d}_i - \frac{1}{2}\tilde{d}_i^3$$

Now $f(x) = x - \frac{1}{2}x^3$ is increasing in x for $x < \sqrt{2/3}$, and $d_i \leq d_{n-1} < \sqrt{2/3}$ for all i, so it suffices to show that, for $i=n-1,n-2,\ldots,1$,

$$d_{i-1} \geq d_i - \frac{1}{2}d_i^3.$$

This is clearly true whenever s_i is 0 or 1; for then $d_{i-1} = d_i$. Now $s_i \leq 1$ if and only if $d_i < 2/\sqrt{3}(i+1)$. For the remainder of the proof, we rewrite (3.1) in terms of

$$\alpha_i = \frac{i+1}{n+1}c_i - s_i = \sqrt{3}\,(i+1)\,d_i - s_i$$

which yields

$$(6.1)\quad d_{i-1} = d_i - \frac{1}{2}d_i^3 - (i+\tfrac{1}{2})\,d_i^3 + \frac{\sqrt{3}}{2}(i+1)\,d_i^2 - \frac{2+3\alpha_i(1-\alpha_i)}{6}\,d_i + \frac{\alpha_i(1-\alpha_i^2)}{3\sqrt{3}(i+1)}\,.$$

After neglecting the last term on the right side, then minimizing the right side with respect to α (which is constrained to be between 0 and 1), we have

$$(6.2)\qquad d_{i-1} - (d_i - \frac{1}{2}d_i^3) \geq d_i\left[\frac{\sqrt{3}}{2}(i+1)d_i - (i+\tfrac{1}{2})d_i^2 - \frac{11}{24}\right]$$

The right side of (6.2) is positive for all $i \geq 1$ at the values $d_i = 2/\sqrt{3}(i+1)$ and $d_i = 1/\sqrt{3}$; hence for all d_i between these limits.

7. <u>Proof of Theorem 1: Extreme-Value Results</u>.

In the Continuous Uniforms case

(7.1) $$P\left(\min_{1<i<n} \sum_{j=1}^{m} X_i^{(j)} > nx\right) = \left(1 - \frac{x^m}{m!}\right)^n \qquad 0 < x < 1$$

$$< \left(1 - \frac{x^m}{m^m}\right)^n \qquad 0 < x < m$$

so

$$E\left[\frac{1}{n^{1-1/m}} \min_{1<i<n} \sum_{j=1}^{m} X_i^{(j)}\right] = \left(\int_0^1 + \int_1^m\right) P\left(\min_{1<i<n} \sum_{j=1}^{m} X_i^{(j)} > nx\right) n^{1/m} dx.$$

The first integral is

$$\int_0^{n^{1/m}} \left(1 - \frac{y^m}{m!n}\right)^n dy$$

which converges to the right side of (4.1), while the second integral is less than

$$\int_{n^{1/m}}^{\infty} e^{-y^m/m^m} dy = m^m e^{-n/m^m} \to 0 \quad \text{as } n \to \infty.$$

This proves (4.1.1).

The key to proving (4.1.2) is the Kolmogorov-Smirnov bound. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d., uniform on the interval $(0,n)$; let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the order of statistics, and X_i be the rank of Y_i (so the X_i's are a random permutation of 1 to n). Then

$$\max_{1<i<n} |X_i - Y_i| = \max_{1<k<n} |Y_{(k)} - k| = \max_{1<k<n} |Y_{(k)} - n\hat{F}(Y_{(k)})|$$

where $\hat{F}$ is the empirical distribution function of the Y_i's. Hence

$$E\left[\max_{1<i<n} |X_i - Y_i|\right] < \sqrt{n}E\left(\sqrt{n} \sup_{0<t<1} (\hat{F}(t) - t/n)\right) = 0(n^{1/2}) = o(n^{1-1/m}) \quad \text{if } m > 3$$

which follows from results of Dvoretzky, Kiefer, and Wolfowitz (1956). Applying this to the Permutations Case establishes (4.1.2) for all $m \geq 3$.

Fortunately, the case of $m = 2$ was handled directly, in Section 3.

8. Proof of Theorem 2: Optimal Stopping Results.

In the Continuous Uniforms case, the backward induction algorithm can be partially derived, using (7.1)

(8.1.1) $$\tilde{c}_{n-1}^{(n)} = \frac{m}{2} n$$

(8.1.2) $$\tilde{c}_i^{(n)} < n \Rightarrow \tilde{c}_{i-1}^{(n)} = \frac{m}{(m+1)!n^m}(\tilde{c}_i^{(n)})^{m+1} + \tilde{c}_i^{(n)}\left(1 - \frac{(\tilde{c}_i^{(n)})^m}{m!n^m}\right)$$

The goal of proving that $\tilde{c}_0^{(n)}/n^{1-1/m}$ converges to the right side of (4.2) can be motivated by rewriting (8.1.2) in "differential" form

$$\frac{\tilde{c}_i^{(n)}/n^{1-1/m} - \tilde{c}_{i-1}^{(n)}/n^{1-1/m}}{1/n} = \frac{(\tilde{c}_i^{(n)}/n^{1-1/m})^{m+1}}{(m+1)!}$$

which suggests the differential equation

$$D'(t) = D^{m+1}(t)/(m+1)! \text{ on } [0,1); \ D(1) = \infty$$

whose solution

$$D(t) = \left(\frac{(m+1)!}{m(1-t)}\right)^{1/m}$$

gives the right side of (4.2) when evaluated at $t = 0$.

If we divide the random variables of (4.2.1) by n, they become i.i.d., uniform on (0,1) for all n. It follows that

(8.2) $$\frac{1}{n}\tilde{c}_i^{(n)} = \frac{1}{n-i}\tilde{c}_0^{(n-i)}$$

So the double sequence is in reality a single sequence, (8.1.2) becomes

(8.3) $$\frac{1}{n}\tilde{c}_0^{(n)} < 1 \Rightarrow \frac{\tilde{c}_0^{(n+1)}}{n+1} = \frac{\tilde{c}_0^{(n)}}{n} - \frac{1}{(m+1)!}\left(\frac{\tilde{c}_0^{(n)}}{n}\right)^{m+1}$$

and our goal can be restated as: Is

(8.4) $$\frac{\tilde{c}_0^{(n)}}{n} \approx \left(\frac{(m+1)!}{mn}\right)^{1/m} ?$$

First we need values k(m) for which

(8.5) $$n > k(m) \Rightarrow \frac{1}{n}\tilde{c}_0^{(n)} < 1.$$

Let

(8.6) $$k(m) = 1 + 2^m m!\ \ell n\ (m-1).$$

To show that (8.5) holds, it suffices to exhibit stopping rules with risks less than n whenever $n > k(m)$. Let

$$\tau_n = \text{first } i < n \text{ such that } \frac{1}{n}\sum_{j=1}^{m} X_i^{(i)} < 1/2$$
$$= n \text{ if no such } i.$$

Then

(8.7) $$E\left[\frac{1}{n}\sum_{j=1}^{m} X_\tau^{(j)}\right] < \frac{1}{2}P(\tau<n) + \frac{m}{2}P(\tau=n)$$
$$= \frac{1}{2}[1-(1-p)^{n-1}] + \frac{m}{2}(1-p)^{n-1}$$

where, from (7.1), $p = 1/m!2^m$. The right side of (8.7) is less than one if and only if $n-1 > [\ell n(m-1)]/[-\ell n(1-p)]$ hence less than one if n is at least the right side of (8.6).

(Note: Although the bound given by (8.6) is a crude one for large m, $k(2) = 1$ is best possible for $m = 2$.)

Now we are ready to validate (8.4) by deriving the inequalities

$$\left(\frac{(m+1)!}{m(n+\beta\sqrt{n})}\right)^{1/m} \leq \frac{\tilde{c}_0^{(n)}}{n} \leq \left(\frac{(m+1)!}{m(n-\alpha)}\right)^{1/m} \text{ for all } n \geq k(m) \tag{8.8}$$

if $k(m) - \frac{(m+1)!}{m} \leq \alpha < k(m)$ and $\beta = \beta(m)$ sufficiently large.
First we note that the two inequalities hold for $n = k(m)$ since the right side is at least one. Next we proceed by induction, using (8.3) which reduces the problem to showing that

$$A^{-1/m} - B^{-1/m} + \frac{1}{m} A^{-(1+1/m)} \tag{8.9}$$

is

(a) Positive if $A = n - \alpha$ and $B = n + 1 - \alpha$;

(b) Negative if $A = n + \beta\sqrt{n}$ and $B = n + 1 + \beta\sqrt{n}$.

Multiplying (8.9) through by the positive quantity

$$A^{1/m}B^{1/m} \sum_{k=0}^{m-1} A^{k/m}B^{(m-k-1)/m}$$

transforms it to

$$A - B + \frac{1}{m} \sum_{k=1}^{m} (B/A)^{k/m} \tag{8.10}$$

for which (a) becomes trivial. To establish (b), write B/A as $1 + \delta$, and use the familiar inequality

$$(1 + \delta)^r < 1 + r\delta \quad \text{if } 0 < r < 1$$

to bound (8.10) above by

$$1 - \delta A + \frac{m+1}{2m}\delta$$

which, in case (b), is easily seen to be negative for all $r \geq 2$ if β is at least 2.

(Note: Taking (8.2) into acount, the inequalities (3.3) are the special case of (8.8) with $m = 2$, $\alpha = -2$, and $\beta = 2$.)

As in the proof of Theorem 1, The Permutation Problem case, (4.2.2), will be handled indirectly by showing that

$$\lim_{n\to\infty} c_0^{(n)}/\tilde{c}_0^{(n)} = 1$$

and invoking (4.2.1). We do this in two steps:

(1) $\liminf_{n\to\infty} c_0^{(n)}/\tilde{c}_0^{(n)} = 1$

If the optimal stopping rule for the Permutation Problem is used in the Continuous Uniforms Problem, the expected loss is not more than

$$c_0^{(n)} + E\left[\max_{1\leq i\leq n}\left|\sum_{j=1}^{m}(\text{i-th ranks}) - \sum_{j=1}^{m}(\text{i-th uniforms})\right|\right]$$

The latter term is $O(\sqrt{n})$ by the Kolomogorov-Smirnov bound; hence for $m \geq 3$ it is $o(\tilde{c}_0^{(n)})$ since $\tilde{c}_0^{(n)} = O(n^{1-1/m})$ as shown earlier in the proof. Thus $\tilde{c}_0^{(n)} \leq c_0^{(n)} + o(\tilde{c}_0^{(n)})$.

The above argument doesn't work for $m = 2$. But, fortunately, we had another argument, namely (3.4).

(2) $\limsup_{n\to\infty} c_0^{(n)}/\tilde{c}_0^{(n)} \leq 1$

Consider the following <u>randomized rule</u> for the Permutation Problem:

If the i-th relative rank in the j-th criterion is $R_i^{(j)}$, observe an independent random variable, $U_i^{(j)}$, conditionally uniform on the interval $(\frac{R_i^{(j)}-1}{i} n, \frac{R_i^{(j)}}{i} n)$ and use the optimal Continuous Uniforms Problem stopping rule, τ, on these randomized values.

The expected loss for this rule is then

$$(8.11) \qquad \tilde{c}_0^{(n)} + E \sum_{j=1}^{m} (\frac{n+1}{\tau+1} R_\tau^{(j)} - U_\tau^{(j)})$$

The goal is to show that the second term is $o(\tilde{c}_0^{(n)})$. The terms in the sum are at most

$$(\frac{n+1}{\tau+1} R - \frac{R-1}{\tau} n) < \frac{n}{\tau}$$

so the expectation can be bounded above by

$$(8.12) \qquad mn\ P(\tau < n^{1/m+\epsilon}) + mn^{1-1/m-\epsilon} P(\tau > n^{1/m+\epsilon})$$

Now, from our solution to the Continuous Uniforms Problem, we can see that

$$\{\tau < n^{1/m+\epsilon}\} \subset \{\min_{i<n^{1/m+\epsilon}} \sum_{j=1}^{m} \tilde{X}_i^{(j)} < Bn^{1-1/m}\}$$

for some B which doesn't depend on n. (Here we denote continuous uniform r.v.'s by $\tilde{X}$.) Hence, from (7.1),

$$nP(\tau < n^{1/m+\epsilon}) < n[1-(1-B^m/m!n)^{n^{1/m+\epsilon}}] < \frac{B^m}{m!} n^{1/m+\epsilon}$$

which is $o(n^{1-1/m})$ for small ϵ if $m > 3$. Thus both terms of (8.12) are of small enough order when $m > 3$.

For $m = 2$ we need a sharper bound, so we replace (8.12) by

$$(8.13) \qquad 2nP(\tau < \delta n^{1/2})+2\delta^{-1}n^{1/2}P(\delta n^{1/2}<\tau < n^{1/2+\epsilon})+2n^{1/2-\epsilon}P(\tau > n^{1/2+\epsilon})$$

an upper bound for which is of the form

$$(8.14) \qquad D^2\delta n^{1/2}+\delta^{-1}B^2n^{\epsilon}+2n^{1/2-\epsilon}$$

with B and D not depending on n or δ. (D can be taken to be about 2.) Thus, by choosing δ small enough, (8.13) can be made asymptotically smaller than any multiple of $n^{1/2}$, hence small with respect to $\tilde{c}_0^{(n)}$.

9. Conclusion.

The approximation which consists of replacing the discrete uniform random variables by continuous uniforms--herein referred to as "Lindley's approximation"--has been shown to be valid (to first order) for all $m \geq 2$. When $m = 2$ direct calculation is possible. This is indeed fortunate because some of the methods used here for $m \geq 3$ (where the problem is too intractable to permit direct calculation) break down when $m = 2$.

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