APPENDIX

We include here some results relevant to the weak convergence of processes in $\mathbb{D}[0, 1]$ and $\mathbb{C}[0, 1]$ for the sake of easy reference and without proofs. Our source is the book by Billingsley (1968) (B) on ***Convergence of Probability Measures.***

To begin with, let $\xi_1, \ldots, \xi_m$ be r.v.'s, not necessarily independent and define

$$S_k := \sum_{j=1}^{k} \xi_j, \quad 1 \le k \le m; \qquad M_m := \max_{1 \le k \le m} |S_k|.$$

The following lemma is obtained by combining (12.5), (12.10) and Theorem 12.1 from pp 87–89 of (B).

Lemma A.1. *Suppose there exist nonnegative numbers* $u_1, u_2, \ldots, u_m$, *a* $\gamma \ge 0$ *and an* $\alpha > 0$ *such that*

$$E\{|S_k - S_j|^{\gamma}|S_j - S_i|^{\gamma}\} \le (\sum_{r=i+1}^{k} u_r)^{2\alpha}, \qquad 0 \le i \le j \le k \le m.$$

Then, $\forall\ \lambda > 0$,

$$P(M_m \ge \lambda) \le K_{\gamma,\alpha} \cdot \lambda^{-2\gamma}(\sum_{r=1}^{m} u_r)^{2\alpha} + P(|S_m| \ge \frac{\lambda}{2}),$$

where $K_{\gamma,\alpha}$ *is a constant depending only on* γ *and* α.

The following inequality is given as Corollary 8.3 in (B).

Lemma A.2. *Let* $\{\zeta(t), 0 \le t \le 1\}$ *be a stochastic process on some probability space. Let* $\delta > 0$, $0 = t_0 < t_1 < \ldots < t_r = 1$ *with* $t_i - t_{i-1} \ge \delta$, $2 \le i \le r-1$ *be a partition of* $[0, 1]$. *Then,* $\forall\ \epsilon > 0, \forall\ 0 < \delta \le 1$,

$$P(\sup_{|t-s|<\delta} |\zeta(t) - \zeta(s)| \ge 3\epsilon) \le \sum_{i=1}^{r} P(\sup_{t_{i-1} \le t \le t_i} |\zeta(t) - \zeta(t_{i-1})| \ge \epsilon).$$

Definition: A sequence of stochastic processes $\{\zeta_n\}$ in $\mathbb{D}[0, 1]$ is said to converge weakly to a stochastic process $\zeta \in \mathbb{C}[0, 1]$ if every finite dimensional distribution of $\{\zeta_n\}$ converges weakly to that of ζ and if $\{\zeta_n\}$ is tight with respect to the uniform metric.

The following theorem gives sufficient conditions for the weak convergence of a sequence of stochastic processes in $\mathbb{D}[0, 1]$ to a limit in $\mathbb{C}[0, 1]$. It is essentially Theorem 15.5, p 127 of (B).

Theorem A.1. *Let* $\{\zeta_n(t),\ 0 \le t \le 1\}$ *be a sequence of stochastic processes in* $\mathbb{D}[0, 1]$. *Suppose that* $|\zeta_n(0)| = O_p(1)$ *and that* $\forall\ \epsilon > 0$,

$$\lim_{\eta \to 0} \limsup_n P\Big(\sup_{|s-t|<\eta} |\zeta_n(s) - \zeta_n(t)| \ge \epsilon\Big) = 0.$$

Then the sequence $\{\zeta_n(t),\ 0 \le t \le 1\}$ *is tight, and of* ζ *is the weak limit of a subsequence* $\{\zeta_{n'}(t),\ 0 \le t \le 1\}$, *then* $P(\zeta \in \mathbb{C}[0, 1]) = 1$.

The following theorem gives sufficient conditions for the weak convergence of a sequence of stochastic processes in $\mathbb{C}[0, 1]$ to a limit in $\mathbb{C}[0, 1]$. It is essentially Theorem 12.3, p 95 of (B).

Theorem A.2. *Let* $\{\zeta_n(t),\ 0 \le t \le 1\}$ *be a sequence of stochastic processes in* $\mathbb{C}[0, 1]$. *Suppose that* $|\zeta_n(0)| = O_p(1)$ *and that there exist a* $\gamma \ge 0$, $\alpha > 1$ *and a nondecreasing continuous function* F *on* $[0, 1]$ *such that,*

$$P(|\zeta_n(t) - \zeta_n(s)| \ge \lambda) \le \lambda^{-\gamma} |F(t) - F(s)|^{\alpha}$$

holds for all s, t *in* [0,1] *and for all* $\lambda > 0$.

Then the sequence $\{\zeta_n(t),\ 0 \le t \le 1\}$ *is tight, and if* ζ *is the weak limit of a subsequence* $\{\zeta_{m_n}(t),\ 0 \le t \le 1\}$, *then* $P(\zeta \in \mathbb{C}[0, 1]) = 1$.

We also need a central limit theorem for martingale arrays. Let $(\Omega, \mathcal{F}, P)$ be a probability space; $\{\mathcal{F}_{n,i},\ 1 \le i \le n\}$, be an array of sub σ–fields such that $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$, $1 \le i \le n$; X_{ni} be $\mathcal{F}_{n,i}$ measurable r.v. with $EX_{ni}^2 < \infty$, $E(X_{ni} | \mathcal{F}_{n,i-1}) = 0$, $1 \le i \le n$; and let $S_{nj} = \Sigma_{i \le j} X_{ni}$, $1 \le j \le n$. Then $\{S_{ni}, \mathcal{F}_{n,i};\ 1 \le i \le n,\ n \ge 1\}$ is called *a zero–mean square–integrable martingale array with differences* $\{X_{ni};\ 1 \le i \le n,\ n \ge 1\}$.

The central limit theorem we find useful is Corollary 3.1 of Hall and Heyde (1980) which we state here for an easy reference.

Lemma A.3. *Let* $\{S_{ni}, \mathcal{F}_{n,i};\ 1 \le i \le n,\ n \ge 1\}$ *be a zero–mean square–integrable martingale array with differences* $\{X_{ni}\}$ *satisfying the following conditions.*

(1) $\forall\ \epsilon > 0, \quad \sum_{i=1}^{n} E[X_{ni}^2 I(|X_{ni}| > \epsilon) | \mathcal{F}_{n,i-1}] = o_p(1).$

(2) $\sum_{i=1}^{n} E[X_{ni}^2 | \mathcal{F}_{n,i-1}] \longrightarrow$ a r.v. η^2, in probability.

(3) $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}, \quad 1 \le i \le n, \quad n \ge 1.$

Then S_{nn} *converges in distribution to a r.v.* Z *whose characteristic function at* t *is* $E \exp(-\eta^2 t^2/2)$, $t \in \mathbb{R}$. □