SUFFICIENCY

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Introduction

We undertake with this title a brief survey of various definitions of *sufficiency*, with some of their properties and relationship between them.

Works on this theme are found in a sequence, if not so much as a stream, of developments from the sixties through eighties. We consider such works as attempts at mathematical conceptualization of the statistical notion of sufficiency, and try to examine how far they have been successful in capturing the intuitive and logical content of the notion. Emphasis has been naturally put on the more recent developments, but some earlier results had to be touched upon as long as they make a part of historical or logical background.

A reason for this choice of a theme is that sufficiency today is not as prolific a subject as in early days, making it difficult to draw a *recent trend* out of the publications in last few years. Only a few titles with the word *sufficiency* appear each year in Current Index to Statistics, mostly with their main interest in neighboring though closely related subjects, e.g., ancillarity, information and comparison of experiments. They will be better treated separately under the respective titles, rather than thrown together into such a short survey as this one.

Out of the remaining papers in sufficiency proper, being still fewer in number, we could pick out some fairly recent results to form an additional section on *Basu Theorems.*

Neither a monograph nor a bibliography on this subject recently came into our attention. So the early bibliography by Basu & Speed (1975) as well as the survey *Partial sufficiency* (Basu, 1978) is still partially sufficient (at least) to a reader.

Statistical Notion and Mathematical Definitions

Sufficiency as a statistical notion means the property of a statistic retaining all the relevant information contained in the whole sample. As is well known, it first appeared in Fisher (1920) (see Stigler, 1973, for historical background) which pointed out that an estimate of a parameter can be regarded to *sum up the whole of the information respecting the parameter which a sample provides* if, for any of its given value, the conditional distribution of any other estimate is independent of the parameter. This idea of expressing the notion by means of conditional probability developed into Fisher's (1922) first definition of sufficiency. A statistic T is called sufficient if:

(A) The conditional distribution of the sample when given T does not depend on the parameter.

Subsequently various aspects of the same notion concerning with specific class of inference and decision problems found different expressions in the following definitions. A statistic T is called sufficient if:

(B) The distribution of the sample can be reconstructed from that of T through randomization, or, mathematically, a stochastic kernel (*Blackwell sufficiency*, Blackwell, 1951).

(C) For every decision problem, given a decision function based on the sample, there exists a decision function based on T which is at least as good as the former (*Decision sufficiency* attributed to Bohnenblust, Shapley and Sherman. See Blackwell, 1951).

(D) For any prior distribution of the parameter, the posterior is a function of the sample through T (*Bayesian sufficiency*, Kolmogorov, 1942).

Meanwhile the Definition (A) underwent measure-theoretic sophistications through Halmos & Savage (1949) and Bahadur (1954) giving rise to the following:

Definition 1

Let $E = (X, \mathcal{A}, P)$ be a statistical experiment and $\mathfrak{B}$ be a subfield (more precisely, a sub-σ-field of $\mathcal{A}$). $\mathfrak{B}$ is called sufficient if for every A in $\mathcal{A}$ there exists a $\mathfrak{B}$-measurable function $P(A/\mathfrak{B})(x)$ which satisfies, for all B in $\mathfrak{B}$ and p in P,

$$p(A \cap B) = \int_B P(A/\mathfrak{B})(x)\, dp.$$

A statistic is called sufficient if the subfield induced by it is sufficient.

Notice that this is more general than (A), as it applies to subfields in general, including in its scope those subfields which are not induced by a statistic. Also, it allows the cases where $P(A/\mathfrak{B})(x)$ is not a measure on $\mathcal{A}$. Though $P(A/\mathfrak{B})(x)$ is called conditional *probability*, it is not guaranteed to be a measure by the Radon-Nikodym Theorem, on which this definition is based.

This is the standard definition of sufficiency, most commonly used at present. We also will adopt it here, but will refer to it as *Sufficiency*, with the initial capital S, so as to avoid confusion. Subfield versions are available also for all other definitions. They are to be understood whenever references are made to the definitions.

The very general and measure-theoretical way in which Sufficiency is defined made it possible to prove many useful results with full rigour and under the widest possible conditions. In particular, it implies the conditions (B) and (D) without any restrictions, while (A) and (C) easily follow in the cases where regular conditional probabilities exist.

Dominated Case

The success of Sufficiency was especially remarkable in the dominated case. E is called dominated if there exists a σ-finite measure m on $\mathcal{A}$ wrt. which each p in P has a density dp/dm. In this case, it follows that:

1) X is covered by a countable family of mutually disjoint subsets, called kernels, of the supports $S(p) = \{x;\ dp/dm > 0\}$ of measures p in P. Those measures constitute a countable subfamily P' of P, which is equivalent to P.

2) There exists a *pivotal measure* n, a convex combination of the measures in P'. Each p in P has a density wrt. n.

3) A subfield $\mathfrak{B}$ is Sufficient if and only if the density dp/dn is $\mathfrak{B}$-measurable for each p in P (Neyman Factorization Theorem).

4) If a subfield includes another subfield which is Sufficient, then the former is also Sufficient.

5) There exists the minimal Sufficient subfield, the smallest subfield wrt. which all the densities dp/dn, $p \in P$, are measurable.

The existence of the minimal Sufficient statistic is also proved under a slight additional restriction that P is separable wrt. the total variation distance (Lehmann & Scheffé, 1950).

The term *minimal Sufficient* requires slightly technical clarifications. Burkholder (1961) proved that the following two properties of a Sufficient subfield $\mathfrak{B}$ are equivalent to each other:

i) $\mathfrak{B} \subset \mathfrak{C}$ $[P]$ for every Sufficient subfield $\mathfrak{C}$, and

ii) $\mathfrak{B} \subset \mathfrak{C}$ $[P]$ for every Sufficient subfield $\mathfrak{C}$ such that $\mathfrak{C} \subset \mathfrak{B}$ $[P]$.

$\mathfrak{B}$ is called minimal Sufficient when it has these properties. On the other hand, a Sufficient statistic is called minimal if it is a function of every other Sufficient statistic except on a P-null set which may depend upon the latter statistic.

This minimality does not coincide with the minimality of the Sufficient subfield which the statistic induces. As a result, the minimal Sufficient statistic and subfield may not coincide with each other even when both exist. In this connection, all logically possible kinds of counter examples are actually available (see Bahadur, 1955; and Landers & Rogge, 1972).

In case Sufficiency is replaced by pairwise Sufficiency in i) and ii), then i) does not follow from ii), so that *smallest pairwise Sufficiency* and *minimal pairwise Sufficiency* have to be differentiated.

Undominated Cases

Thus in the dominated case Sufficiency exhibits all the good features to qualify itself for a mathematical embodiment of the statistical notion of sufficiency. However, it came to be known already around 1960 that some of the features are not carried over to the general case. Notably, general validity of 4) and 5) were disproved by the counter examples given by Burkholder (1961, for 4) and Pitcher (1957, for 5), respectively. The phenomena of the failure of 4) and 5) are accordingly called Burkholder and Pitcher pathologies.

Various intermediate conditions more general than domination have been devised in order to avoid these pathologies. Here we present two such conditions, namely, majorization and weak domination. Reader is referred to Luschgy & Mussmann (1985) for details of these and other conditions.

An experiment E is called majorized if there exists a *majorizing measure* m on $\mathcal{A}$ wrt. which each p in P has a density dp/dm.

E is called weakly dominated if the majorizing measure m is further assumed to be localizable (for the definition of localizability see Diepenbrock, 1971, or Ghosh et al., 1981).

The majorized case is more or less the most general case in which positive results are being obtained at present. The non-majorized cases are the places mainly for counter examples, but for some early, universal type of theorems by Bahadur (1954, 1955b), Burkholder (1961) and others.

Weak domination is more general than domination, as localizability of a measure follows from σ-finiteness, and is equivalent to some other conditions which appeared in literature, such as *compactness* (Pitcher, 1965), *coherence* (Hasegawa & Perlman, 1974), etc.

There is a simple but suggestive special case of weak domination, called the discrete case. E is called discrete if X is an uncountable space, $\mathcal{A}$ is the power set, each p in P is a discrete probability and the only P-null set is the empty set. It is Professor D. Basu himself who pointed out with J.K. Ghosh (1967) that the problem of sampling from finite populations falls in this category and thus became one of the pioneers of the study of sufficiency in the undominated cases.

These conditions have been only partly successful in removing the pathologies, insofar as the minimal Sufficient subfield was proved to exist in the weakly dominated case, but not in the majorized case in general (Pitcher, 1965, and Hasegawa and Perlman, 1974). Burkholder pathology persists even in the weakly dominated case.

The reason for this difference between dominated and undominated cases becomes apparent if a parallelism to the passage from 1) through 5) is tried out for the majorized case. It follows that:

1′) X is now covered by an uncountable family of almost disjoint *kernels*. This family is called a maximal decomposition (Diepenbrock, 1971). As before, the kernels are subsets of the supports of measures p in P. Those measures constitute an uncountable equivalent subfamily P' of P.

2′) A *pivotal measure* n can be defined as the sum of the measures in P' restricted to the respective kernels.

3′) A subfield $\mathfrak{B}$ is pairwise Sufficient and contains the supports $S(p)$ for all p in P (*pairwise Sufficiency with supports*, abbreviated as PSS), if and only if the density dp/dn is $\mathfrak{B}$-measurable for each p in P (Analogue of Neyman Factorization Theorem, Ramamoorthi & Yamada, 1982).

4′) If a subfield includes another subfield which is PSS, then the former is also PSS.

5′) There exists the smallest subfield which is PSS (Ghosh et al., 1981).

Thus, instead of Sufficiency in the dominated case, here we arrive at PSS, a property in between Sufficiency and pairwise Sufficiency. Notice further that the likelihood ratios are seen in 3′) to be functions of the sample through PSS rather than Sufficiency, which coincides with the former in the dominated case.

On the other hand, if we insist upon retaining all the nice properties of Sufficiency, i.e. (B) through (D) as well as 4) and 5), we have to take to something even more restrictive than domination, as it would require a type of sample space with regular conditional probabilities. Barndorff-Nielsen (1978) points out it and puts forward one such framework: An Euclidean sample space with the Borel field and a dominated P, in which only *B-sufficiency* (defined in terms of the existence of regular conditional probability $P(A/T)$ common to all p in P) of statistics rather than subfields is treated. This would restrict us almost within the purview of Definition (A) and would mean little more than a return to Fisher's old setup.

Relationship Between the Definitions

Much attention has been directed to the relationship between various definitions of sufficiency, especially on the question as to whether Sufficiency follows from other definitions. It is quite rightly so, as Sufficiency is defined solely in measure-theoretic terms and, unlike other definitions, is not directly concerned with specific statistical problems, though it was also originated in an estimation problem. It is relevant to ask whether the requirement for Sufficiency is just appropriately strong, or actually stronger than the requirements for other definitions.

We take up this question as regards decision, Bayes and, in addition, *test sufficiency*, as it is often called in literature. Blackwell sufficiency would require some preliminaries from the comparison of experiments which is beyond our scope.

A subfield $\mathfrak{B}$ is called test sufficient if for any test function there exists a $\mathfrak{B}$-measurable test function whose expectation is identical with the former for all p in P.

It was proved in a series of classical results in Bahadur (1955a, b), Blackwell (1951), Kudo (1967) and others that each of the four concepts including Blackwell sufficiency implies pairwise Sufficiency. In the case of decision sufficiency we need some clarification on the precise definition of a decision problem, but we will not go into the details. In the dominated case as pairwise Sufficiency implies Sufficiency, each of the four concepts implies Sufficiency.

Things are again very different in the undominated case. First, pairwise Sufficiency does not imply Sufficiency in general. Secondly, the implication, e.g. *Bayes sufficiency implies Sufficiency* obviously fails in the face of Burkholder pathology, as Sufficiency implies Bayes sufficiency and a subfield including Bayes sufficient subfield is Bayes sufficient. So the implication needs to be modified to a weaker statement: *A Bayes sufficient subfield includes a Sufficient subfield.* The same modifications are made also in regard to decision, test and Blackwell sufficiency.

This modified statement is proved by Ramamoorthi (1980) for decision sufficiency: A decision sufficient subfield includes at least one Sufficient subfield in it.

The statement *a test sufficient subfield includes a Sufficient subfield* is obviously more difficult to follow, as test sufficiency is weaker than decision sufficiency. Indeed, since the paper of Brown (1975) which says that it holds true for the discrete case, little progress has been seen, but for a recent proof of *PSS does not imply test sufficiency* for the weakly dominated case by Kusama & Fujii (1987). Even this statement, not at all surprising, cannot be readily proved for more general cases.

The questions concerning Bayes sufficiency are even more technical, as Bayes sufficiency involves a measurable structure on P, and appears to be weaker than all other definitions. In the extreme case, it is no more than pairwise Sufficiency if P has the discrete σ-field. It follows from test sufficiency if both X and P have countably generated σ-fields, and from decision sufficiency in the general case (Ramamoorthi, 1980. Incidentally Blackwell sufficiency also follows from decision sufficiency). On the other hand, a rather natural *counter example*, in which P is a standard Borel space, is available to show that Bayes sufficiency does not imply Sufficiency (Blackwell and Ramamoorthi, 1982).

Then what does Bayes sufficiency at all imply? Suppose $\mathcal{A}$ and a subfield $\mathfrak{B}$ are countably generated. Then $\mathfrak{B}$ is Bayes sufficient if and only if it is Sufficient for almost all p in P wrt. every prior measure on P (Ramamoorthi, 1980).

LeCam's Framework of *L*-Space and *M*-Space

An entirely different approach has been proposed by LeCam (1964) to bypass the difficulties discussed above by means of function spaces and further developed towards various directions (by e.g. Littaye-Petit, Piednoir & Van Cutsem, 1969; Siebert, 1979; Luschgy & Mussmann, 1985; and recently LeCam himself, 1986) through the seventies and eighties.

Let $E = (X, \mathcal{A}, P)$ be an experiment. The band $L(E)$ generated by P in the space of bounded signed measures on $\mathcal{A}$ is called the *L*-space of E. If E is majorized and n is a majorizing measure equivalent to P (shown to exist by Diepenbrock, 1971), then $L(E)$ coincides with $\{f.n;\ f \in L_1(X, \mathcal{A}, n)\}$, where $f.n$ denotes the bounded signed measure having f as the density wrt. n. Assign the total variation topology to $L(E)$, denote by $M(E)$ its topological dual and call it the *M*-space of E. Sufficiency is now defined for a sublattice of $M(E)$ as follows: A sublattice H is sufficient if there exists a positive linear projection π of $M(E)$ onto H such that $< p, \pi f > \; = \; < p, f >$ for all f in $M(E)$ and p in P.

It then follows that a sublattice including a sufficient sublattice is sufficient, and the smallest sufficient sublattice exists. Thus this *sufficiency* appears to be free from both Burkholder and Pitcher pathologies.

Two concepts of *transition* and *deficiency* play important parts in the theory. A transition is defined as a positive linear mapping from the *L*-space of an experiment F to that of another experiment E which preserves the norms of the positive elements. The deficiency of F to E is devised for measuring how much less informative F is than E when they share a same parameter space. Write $E = (X, \mathcal{A}, P)$, $F = (Y, \mathfrak{B}, Q)$ with $P = \{p_\theta;\ \theta \in \Theta\}$ and $Q = \{q_\theta;\ \theta \in \Theta\}$ where Θ is the common parameter space. The deficiency of F to E is defined by

$$d(F, E) = Inf\left\{ [\underset{\theta \in \Theta}{Sup} \| \tau \cdot q_\theta - p_\theta \|];\ \tau \text{ is a transition from F to E} \right\}.$$

Now take a sublattice H of $M(E)$. There is an experiment F whose *M*-space is H, provided H is closed. It is proved that H is sufficient if and only if the deficiency of F to the original experiment E is 0.

Specialize these concepts to the case of $E = (X, \mathcal{A}, P)$ and $F = (X, \mathfrak{B}, P)$ where $\mathfrak{B}$ is a subfield of $\mathcal{A}$. A transition from $L(F)$ to $L(E)$ is then a generalization of a stochastic kernel from $(X, \mathfrak{B})$ to $(X, \mathcal{A})$ and "$d(F, E) = 0$" is a generalization of Blackwell sufficiency. Hence the foregoing paragraph is interpreted as *sufficiency implies Blackwell sufficiency* in the present context.

Instead of starting from $E = (X, \mathcal{A}, P)$ and going to $M(E)$ via $L(E)$, it is also possible to take an abstract *L*-space as L and construct the whole theory directly based on it. Thereby P appears as a set of positive elements p with the norms 1 in L, but not X or $\mathcal{A}$. This is more like what LeCam (1964) actually did. Here we have followed the way Siebert (1979) presented the theory.

Torgersen (1979) undertakes a further generalization by including unbounded functions into M, and develops an estimation theory which has a theorem: Every estimable function admits a UMVU if and only if a quadratically complete sufficient statistic exists.

Such abstract developments render highly refined appearance to the theory, though the departure from the basis of the sample space invites critical comments.

It is not very easy to compare this theory to the measure theoretic treatment, as the concepts do not necessarily correspond to each other. When we try to locate a counterpart of a subfield $\mathfrak{B}$, it is found in $M(E)$ in the form of the sublattice $H(\mathfrak{B})$, the totality of the $\mathfrak{B}$-measurable functions. Whether $H(\mathfrak{B})$ is a sufficient sublattice or not can be decided only when it happens to be closed, so as to admit the projection used to define the sufficiency of a sublattice. And in that event, the sublattice $H(\mathfrak{B})$ is sufficient in $M(E)$ if and only if the subfield $\mathfrak{B}$ is pairwise sufficient in E (Littaye-Petit et al., 1969).

In this correspondence between $\mathfrak{B}$ and $H(\mathfrak{B})$, no criterion inherent in $M(E)$ is readily available to distinguish Sufficiency, PSS and pairwise Sufficiency of $\mathfrak{B}$ on the basis of the properties of $H(\mathfrak{B})$ as a sublattice. So the sufficient sublattices correspond to these three kinds of subfields altogether.

This suggests significance of pairwise Sufficiency, and in particular PSS, as something more than a mathematical tool. Remember that the role played by PSS in the majorized case is very similar to, if not as important as, that of Sufficiency in the dominated case.

In the weakly dominated case, PSS possesses some more properties almost parallel to those of Sufficiency (Yamada, 1980). Suppose that $\mathfrak{B}$ is PSS and f is an integrable function. Then there exists a function g which satisfies $g = E_p[f/\mathfrak{B}]$ a.e. for each p in P, and falls only a little short of being $\mathfrak{B}$-measurable. In precise terms, g is measurable wrt. the strong completion of $\mathfrak{B}$, i.e. $\mathfrak{B} \vee \mathcal{N}(P)$ on the support of each p in P, though on the whole space it is measurable only wrt. the weak completion $\cap \{\mathfrak{B} \vee \mathcal{N}(p);\ p \in P\}$ ($\mathcal{N}(P)$ and $\mathcal{N}(p)$ mean the families of P-null and p-null sets, respectively).

This property can then be used to prove analogues of test sufficiency and Rao-Blackwell property for PSS, by providing improved test and estimator which are close to being $\mathfrak{B}$-measurable.

Further attempts have been made at extending these properties to the majorized experiments (Yamada, 1988).

Basu Theorems

This means the two renowned theorems of Basu on independence of sufficient and ancillary statistics (see Basu, 1982). Because of their nature of connecting such basic concepts as sufficiency, ancillarity, completeness and independence, related works still appear in literature. We first state the theorems. Assume until otherwise noticed that T is a sufficient statistic.

I. Suppose that T is boundedly complete. Then an ancillary statistic S is independent of T (for all p in P).

II. Assume that there is no splitting set. Then a statistic S which is independent of T is ancillary.

A splitting set is defined as "a set A such that $p(A) = 1$ for some p's and 0 for all other p's in P" by Koehn & Thomas (1975). A slightly different condition to be assumed in II and some remarks on the conditions are found in Basu (1982) and Basu & Cheng (1981). Bayesian versions of these and related theorems are given in Basu & Pereira (1983).

Recently Goossen (1986), while working on *conditional completeness*, made a remark that the assumption of sufficiency of T in I and II can be replaced by sufficiency of T for (S, T).

Lehmann (1981) gives two theorems as *adaptations of Basu's theorems*, aiming at characterizations of (bounded) completeness. Basu theorems as such are not exactly a characterization, as the independence of all the ancillaries from T does not imply bounded completeness of T unconditionally. The reason for this gap, Lehmann considers, lies in the difference between ancillarity and completeness in their nature, one being concerned with the whole distribution while the other only with the expectations. Notice the modifications accordingly made on each concept to bridge the gap in the theorems thus constructed:

III. T is boundedly complete if and only if every bounded function of T is uncorrelated with every bounded *first order ancillary* (a statistic whose expectation is independent of p).

IV. T is F_1-complete if and only if every ancillary is independent of T (F_1 means the class of all functions $f(T)$ such that $f(T) = E[g/T]$ for some two valued function g. T is called F_1-complete if $f \in F_1$ and $E_p(f(T)) = 0$ for all p together imply $f(T) = 0$).

Basu theorems are closely related to invariance theory, where conditions for sufficiency, ancillarity and mutual independence of an invariant and an equivariant statistic S are studied.

A recent work of this kind is Eberl (1983), which deals with the n-dimensional location model. Let S be the maximal invariant and T be an equivariant statistic in this model. Neither sufficiency of T as such nor its bounded completeness is assumed. It follows that:

V. T is independent of S if and only if T is *invariantly sufficient* (i.e. $p(C/T)$ is independent of p for all invariant sets C).

Similar questions are asked and considerable amount of results have been obtained with remarkable applications in more general invariant models like compact or locally compact spaces and/or groups. As they cannot be detailed here, readers are referred to, e.g., Dasgupta (1979) and Ramamoorthi (1990) for such results and remarks on their connection with Basu theorems.

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