Chapter 10

Large deviations for diffusion processes in conuclear spaces and for SPDEs

In Chapter 8, we studied diffusion processes in conuclear spaces governed by stochastic differential equations. In these models, the drift term describes the deterministic evolution of the voltage potentials of the neuron while the diffusion term is added when random stimuli by electric impulses are present.

In this chapter, we derive a large deviation principle (LDP) for such processes when the diffusion term depends on a small parameter which tends to zero. The lower bounds are established by making use of the Girsanov formula in abstract Wiener space. The upper bounds are obtained by Gaussian approximation of the diffusion processes and by taking advantage of the nuclear structure of the state space to pass from compact sets to closed sets.

This chapter is organized as follows: We study the LDP for a class of random variables taking values in Banach spaces in Section 1. Then in Section 2, we apply our basic results to stochastic differential equations in the conuclear spaces investigated in Chapter 8. The material of this section comes from Xiong [60]. In Section 3, we present our results obtained in [32] for LDP of random field solution of SPDEs studied in Section 4.3. Finally, in Section 4, we specialize the results to stochastic reaction-diffusion equations.

10.1 LDP for a class of random variables

Stochastic differential equations or stochastic integrals can usually be regarded as random transformations of some Wiener processes. In this section, we consider a family of Banach space valued random variables which comes from such transformations. As the map is not pointwise, the usual contraction principle for large deviation is not applicable in this case.

Let $(i, \mathcal{H}, \Omega)$ be an abstract Wiener space and let P be the standard Wiener measure on $(\Omega, \mathcal{B}(\Omega))$. Suppose that $\mathcal{X} \subset \mathcal{Y}$ are two separable Banach spaces and $\mathcal{A}_\mathcal{X}$ (resp. $\mathcal{A}_\mathcal{Y}$) is a class of $\mathcal{X}$-valued (resp. $\mathcal{Y}$-valued) random variables on Ω.

Definition 10.1.1 *Let $\ell \in L(\mathcal{H}, \mathcal{Y})$, where $L(\mathcal{H}, \mathcal{Y})$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{Y}$. A $\mathcal{Y}$-valued random variable $\tilde{\ell}$ is called the* **lifting** *of ℓ if for any $\ell_n \in L(\Omega, \mathcal{Y})$ which tends to ℓ in $L(\mathcal{H}, \mathcal{Y})$ we have that $\ell_n(\cdot)$, regarded as a sequence of $\mathcal{Y}$-valued random variables, converges to $\tilde{\ell}$ in probability.*

Let $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{A}_\mathcal{X} \to \mathcal{A}_\mathcal{Y}$ be two mappings and $\{X^\epsilon : \epsilon > 0\}$ be a family of $\mathcal{X}$-valued random variables on Ω with the following properties:
(A1) i) $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}_\mathcal{Y}$ are two linear spaces.
ii) $\mathcal{X} \subset \mathcal{A}_\mathcal{X}$ in the sense that for any $x \in \mathcal{X}$ fixed, the constant mapping given by $X(\omega) \equiv x$, $\forall \omega \in \Omega$, is in $\mathcal{A}_\mathcal{X}$. Similarly, $\mathcal{Y} \subset \mathcal{A}_\mathcal{Y}$.
iii) For any $h \in \mathcal{H}$ and $X \in \mathcal{A}_\mathcal{X}$, we have $T_h X \in \mathcal{A}_\mathcal{X}$, where $(T_h X)(\omega) = X(\omega - h)$.
(A2) There exists a constant K such that

$$\|A(x_1) - A(x_2)\|_\mathcal{Y} \leq K\|x_1 - x_2\|_\mathcal{X}, \quad \forall x_1, x_2 \in \mathcal{X}. \tag{10.1.1}$$

(A3) There exists a continuous map $\hat{B} : \mathcal{X} \times \mathcal{H} \to \mathcal{Y}$ with the following properties
i) For each $x \in \mathcal{X}$, $\hat{B}(x, \cdot) : \mathcal{H} \to \mathcal{Y}$ is linear and the lifting $\tilde{B}(x, \cdot) : \Omega \to \mathcal{Y}$ is an element of $\mathcal{A}_\mathcal{Y}$.
ii) There exists a constant K such that, for any $x_1, x_2 \in \mathcal{X}$, $h \in \mathcal{H}$, we have

$$\|\hat{B}(x_1, h) - \hat{B}(x_2, h)\|_\mathcal{Y} \leq K\|h\|_\mathcal{H}\|x_1 - x_2\|_\mathcal{X}. \tag{10.1.2}$$

For each $x \in \mathcal{X}$, as the constant map $X(\omega) \equiv x$ is in $\mathcal{A}_\mathcal{X}$, we have that $B(x) \equiv B(X)$ is an element of $\mathcal{A}_\mathcal{Y}$ and hence, $B(x)$ is a $\mathcal{Y}$-valued random variable. On the other hand, by Definition 10.1.1, the lifting $\tilde{B}(x, \cdot)$ is also a $\mathcal{Y}$-valued random variable. We make the following further assumptions:
iii) For any $x \in X$, we have $B(x) = \tilde{B}(x, \cdot)$.
iv) For any $h \in \mathcal{H}$, $X \in \mathcal{A}_\mathcal{X}$, the map $B_h(X) : \Omega \to \mathcal{Y}$ given by $B_h(X)(\omega) = \hat{B}(X(\omega), h)$ is in $\mathcal{A}_\mathcal{Y}$ and

$$B(X) - B_h(X) = T_h\left(B(T_{-h}X)\right). \tag{10.1.3}$$

(v) B is exponentially continuous in the following sense: For any $L > 0$, there exists $\delta > 0$ such that for any $\{X_1(\epsilon)\}, \{X_2(\epsilon)\} \subset \mathcal{A}_{\mathcal{X}}$, we have

$$\begin{aligned}&\limsup_{\epsilon\to 0} \epsilon \log P(\sqrt{\epsilon}\|B(X_1(\epsilon)) - B(X_2(\epsilon))\|_{\mathcal{Y}} > \sqrt{\delta}, \\ &\qquad \|X_1(\epsilon) - X_2(\epsilon)\|_{\mathcal{X}} < \delta) \\ \leq\ & -L. \end{aligned} \tag{10.1.4}$$

(A4) i) $\{X^\epsilon\}$ is exponentially tight, i.e., for any $L > 0$, there exists a compact subset C_L of $\mathcal{X}$ such that

$$\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \notin C_L) \leq -L; \tag{10.1.5}$$

ii) For any $\epsilon > 0$, $X^\epsilon \in \mathcal{A}_{\mathcal{X}}$ and satisfies the following equation in $\mathcal{Y}$:

$$X^\epsilon = A(X^\epsilon) + \sqrt{\epsilon} B(X^\epsilon), \quad a.s. \tag{10.1.6}$$

where $A(X^\epsilon)$ is understood as $A(X^\epsilon)(\omega) \equiv A(X^\epsilon(\omega))$, $\forall \omega \in \Omega$.

Let P^ϵ be the probability measure on $\mathcal{X}$ induced by X^ϵ. We proceed to derive large deviation results for $\{P^\epsilon\}$ as $\epsilon \to 0$.

First, we study the LDP for the Gaussian random variables obtained by freezing the right hand side of (10.1.6), i.e. $\forall x \in \mathcal{X}$ fixed, we consider the following family of $\mathcal{Y}$-valued Gaussian random variables $\{X^{\epsilon,x} : \epsilon > 0\}$ given by

$$X^{\epsilon,x} = A(x) + \sqrt{\epsilon} B(x). \tag{10.1.7}$$

We shall need the following theorem due to Kallianpur and Odaira [25] (see also Stroock [52]).

Theorem 10.1.1 *Let μ be a centered Gaussian measure on a separable Banach space E. Let S be a continuous linear map from a separable Hilbert space H to E such that*

$$\|S'\ell\|_H^2 = \int_E (\ell[x])^2 \mu(dx), \ \forall \ell \in E', \tag{10.1.8}$$

where $S' : E' \to H' = H$ is the dual of S. For $\epsilon > 0$, let μ_ϵ be a probability measure on E given by

$$\mu_\epsilon(C) = \mu\left\{x \in E : \sqrt{\epsilon} x \in C\right\}, \ \forall C \in \mathcal{B}(E).$$

Then $\{\mu_\epsilon : \epsilon > 0\}$ satisfies LDP with rate function

$$I_\mu(x) = \inf\left\{\frac{1}{2}\|h\|_H^2 : h \in H \text{ such that } S(h) = x\right\}. \tag{10.1.9}$$

Proof: If the linear map S is injective, the conclusion of the theorem follows from Theorem 3.45 and Theorem 3.48 of [52]. If S is not injective, let $H_0 = \{h \in H : S(h) = 0\}$. Then H_0 is a closed subspace of H. Let $\tilde{H}$ be the orthogonal complement of H_0. $S|_{\tilde{H}}$ is then an injective continuous linear map from $\tilde{H}$ to E. It is easy to see that $S'\ell \in \tilde{H}$ and hence, $S'\ell = (S|_{\tilde{H}})'\ell$, $\forall \ell \in E'$, i.e., (10.1.8) holds for $S|_{\tilde{H}}$. Therefore $\{\mu_\epsilon : \epsilon > 0\}$ satisfies LDP with rate function

$$I_\mu(x) = \begin{cases} \frac{1}{2}\|h\|_H^2 & \exists h \in \tilde{H} \ s.t. \ S(h) = x \\ \infty & \text{otherwise.} \end{cases} \tag{10.1.10}$$

It is clear that $I_\mu(x)$ given by (10.1.10) coincides with the function defined by (10.1.9). ∎

Theorem 10.1.2 $\{\sqrt{\epsilon}B(x) : \epsilon > 0\}$ *satisfies the LDP with rate function*

$$I^{x,0}(y) = \inf\left\{\frac{1}{2}\|h\|_{\mathcal{H}}^2 : h \in \mathcal{H} \ \textit{such that} \ y = \hat{B}(x,h)\right\}.$$

Proof: As $\{\sqrt{\epsilon}B(x)\}$ is a family of centered Gaussian random variables, it follows from Theorem 10.1.1 that we only need to prove the following equality:

$$E\left(|y'[B(x)]|^2\right) = \|\hat{B}(x,\cdot)'y'\|_{\mathcal{H}}^2, \quad \text{for any } y' \in \mathcal{Y}',$$

where $\hat{B}(x,\cdot)' : \mathcal{Y}' \to \mathcal{H}$ is the dual of the linear operator $\hat{B}(x,\cdot) : \mathcal{H} \to \mathcal{Y}$.

In fact, let $\{e_j\}$ be a CONS of $\mathcal{H}$. Then

$$\begin{aligned} \|\hat{B}(x,\cdot)'y'\|_{\mathcal{H}}^2 &= \sum_j \left\langle \hat{B}(x,\cdot)'y', e_j\right\rangle_{\mathcal{H}}^2 = \sum_j (y'[\hat{B}(x,e_j)])^2 \\ &= \|y' \circ \hat{B}(x,\cdot)\|_{\mathcal{H}}^2 = E\left(|y'[B(x)]|^2\right). \end{aligned}$$ ∎

Theorem 10.1.3 $\{X^{\epsilon,x} : \epsilon > 0\}$ *satisfies the LDP on* $\mathcal{Y}$ *with rate function*

$$I^x(y) = \inf\left\{\frac{1}{2}\|h\|_{\mathcal{H}}^2 : h \in \mathcal{H} \quad s.t. \quad y = A(x) + \hat{B}(x,h)\right\}.$$

Proof: Define a map $\pi : \mathcal{Y} \to \mathcal{Y}$ by $\pi y = A(x) + y$. The result follows easily from Theorem 10.1.2. ∎

Now we define a "rate function" $I(x)$ on $\mathcal{X}$.

Definition 10.1.2 *Let*

$$I(x) = \inf\left\{\frac{1}{2}\|h\|_{\mathcal{H}}^2 : h \in \mathcal{D}(\gamma) \text{ s.t. } x = \gamma(h)\right\} \tag{10.1.11}$$

where $\mathcal{D}(\gamma)$ *is the collection of* $h \in \mathcal{H}$ *such that the following equation*

$$x = A(x) + \hat{B}(x, h) \tag{10.1.12}$$

has a solution x, *denoted by* $\gamma(h)$, *in* $\mathcal{X}$.

Remark 10.1.1 *In general,* γ *can be a multivalued map. Nevertheless, in many applications,* γ *is a single-valued injection with* $\mathcal{D}(\gamma) = \mathcal{H}$.

We state the following Girsanov's formula in abstract Wiener space the proof of which can be found in Kuo [35].

Lemma 10.1.1 *For* $h \in \mathcal{H}$, *we define a linear transformation* T_h *on* Ω *by* $T_h\omega = \omega - h$. *Then the probability measure* $\tilde{P} = P \circ (T_h)^{-1}$ *is equivalent to* P *and*

$$\frac{d\tilde{P}}{dP}(\omega) = exp\left(- < h, \omega >_{\mathcal{H}} - \frac{1}{2}\|h\|_{\mathcal{H}}^2\right).$$

To derive the large deviation lower bound, we need the following assumption

(A5) Let $h \in \mathcal{D}(\gamma)$, $x = \gamma(h) \in \mathcal{X}$ and $Z^\epsilon \in \mathcal{A}_{\mathcal{X}}$ such that

$$Z^\epsilon = A(Z^\epsilon + x) - A(x) + B_h(Z^\epsilon + x) - B_h(x) + \sqrt{\epsilon}B(Z^\epsilon + x). \tag{10.1.13}$$

Then, for any $\delta > 0$

$$P(\omega : \|Z^\epsilon\|_{\mathcal{X}} < \delta) \to 1, \qquad \text{as } \epsilon \to 0.$$

Theorem 10.1.4 *For any open set* G *of* $\mathcal{X}$, *we have*

$$\liminf_{\epsilon \to 0} \epsilon \log P(X^\epsilon \in G) \geq -\inf\{I(x) : x \in G\}. \tag{10.1.14}$$

Proof: Without loss of generality we may assume that the right hand side of (10.1.14) is finite. Then, for any $\delta_1 > 0$, there exists $x \in G$ such that

$$I(x) \leq \inf\{I(y) : y \in G\} + \delta_1 < \infty.$$

For any $\eta > 0$, there exists $h \in \mathcal{D}(\gamma)$ such that $x = \gamma(h)$ and

$$\frac{1}{2}\|h\|_{\mathcal{H}}^2 \leq I(x) + \eta. \tag{10.1.15}$$

Let $Y^\epsilon = X^\epsilon - x$. It follows from (10.1.3), (10.1.6) and (10.1.12) that

$$\begin{aligned}
Y^\epsilon &= A(X^\epsilon) - A(x) + \sqrt{\epsilon}B(X^\epsilon) - \hat{B}(x,h) \\
&= A(Y^\epsilon + x) - A(x) + B_h(Y^\epsilon + x) - B_h(x) \\
&\quad + \sqrt{\epsilon}\left\{B(Y^\epsilon + x) - B_{h/\sqrt{\epsilon}}(Y^\epsilon + x)\right\} \\
&= A(Y^\epsilon + x) - A(x) + B_h(Y^\epsilon + x) - B_h(x) \\
&\quad + \sqrt{\epsilon}T_{h/\sqrt{\epsilon}}\left\{B(T_{-h/\sqrt{\epsilon}}(Y^\epsilon + x))\right\}.
\end{aligned}$$

Let $Z^\epsilon = T_{-h/\sqrt{\epsilon}}Y^\epsilon$. Then

$$\begin{aligned}
Z^\epsilon &= T_{-h/\sqrt{\epsilon}}\{A(Y^\epsilon + x) - A(x) + B_h(Y^\epsilon + x) - B_h(x)\} \\
&\quad + \sqrt{\epsilon}B(Z^\epsilon + x) \qquad\qquad (10.1.16) \\
&= A(Z^\epsilon + x) - A(x) + B_h(Z^\epsilon + x) - B_h(x) + \sqrt{\epsilon}B(Z^\epsilon + x).
\end{aligned}$$

Finally, let $\delta > 0$ be such that $\{y : \|y - x\|_{\mathcal{X}} < \delta\} \subset G$. Let $\tilde{P}$ be given by Lemma 10.1.1 with h replaced by $h/\sqrt{\epsilon}$. Then by (10.1.15) and Lemma 10.1.1, we have

$$\begin{aligned}
&\epsilon \log P(X^\epsilon \in G) \geq \epsilon \log P(\|X^\epsilon - x\|_{\mathcal{X}} < \delta) = \epsilon \log P(\|Y^\epsilon\|_{\mathcal{X}} < \delta) \\
= \;& \epsilon \log \tilde{E}\left(\exp\left(\frac{1}{\sqrt{\epsilon}} < h, \tilde{\omega} >_{\mathcal{H}} - \frac{1}{2\epsilon}\|h\|_{\mathcal{H}}^2\right); \|Y^\epsilon(T_{-h/\sqrt{\epsilon}}\tilde{\omega})\|_{\mathcal{X}} < \delta\right) \\
\geq \;& -I(x) - \eta + \epsilon \log \tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta) \\
& + \epsilon \log \frac{\tilde{E}\left(\exp\left(\frac{1}{\sqrt{\epsilon}} < h, \tilde{\omega} >_{\mathcal{H}}\right); \|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta\right)}{\tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta)}.
\end{aligned}$$

It follows from Jensen's inequality, Hölder's inequality, (10.1.16) and Assumption (A5) that

$$\begin{aligned}
& \epsilon \log P(X^\epsilon \in G) \\
\geq \;& -I(x) - \eta + \epsilon \log \tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta) \\
& + \epsilon \tilde{E}\left(\frac{1}{\sqrt{\epsilon}} < h, \tilde{\omega} >_{\mathcal{H}}; \|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta\right) \Big/ \tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta) \\
\geq \;& -I(x) - \eta + \epsilon \log \tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta) \\
& - \sqrt{\epsilon}\left(\tilde{E}| < h, \tilde{\omega} >_{\mathcal{H}} |^2\right)^{1/2}\left(\tilde{P}(\|Z^\epsilon(\tilde{\omega})\|_{\mathcal{X}} < \delta)\right)^{-1/2} \\
= \;& -I(x) - \eta + \epsilon \log P(\|Z^\epsilon(\omega)\|_{\mathcal{X}} < \delta) \\
& - \sqrt{\epsilon}\|h\|_{\mathcal{H}}\left(P(\|Z^\epsilon(\omega)\|_{\mathcal{X}} < \delta)\right)^{-1/2} \\
\to \;& -I(x) - \eta \geq -\inf\{I(y) : y \in G\} - \delta_1 - \eta.
\end{aligned}$$

(10.1.14) follows as δ_1 and η are arbitrary. ∎

Now we consider the upper bound. The idea is to approximate the non-Gaussian random variable by Gaussian random variables. We first obtain the upper bound for compact sets and then extend to closed sets.

Lemma 10.1.2 *For $x \in \mathcal{X}$ and $L > 0$, there exists $\delta > 0$ such that*

$$\limsup_{\epsilon \to 0} \epsilon \log P\left(\|X^\epsilon - x\|_{\mathcal{X}} < \delta, \|X^{\epsilon,x} - x\|_{\mathcal{Y}} > (1+K)\delta + \sqrt{\delta}\right) \leq -L.$$

Proof: By (10.1.1), (10.1.6) and (10.1.7), we have

$$\begin{aligned}
& \epsilon \log P\left(\|X^\epsilon - x\|_{\mathcal{X}} < \delta, \|X^{\epsilon,x} - x\|_{\mathcal{Y}} > (1+K)\delta + \sqrt{\delta}\right) \\
\leq\ & \epsilon \log P\left(\|X^\epsilon - x\|_{\mathcal{X}} < \delta, \|X^{\epsilon,x} - X^\epsilon\|_{\mathcal{Y}} > K\delta + \sqrt{\delta}\right) \\
\leq\ & \epsilon \log P\left(\|X^\epsilon - x\|_{\mathcal{X}} < \delta,\right. \\
& \left.\|A(X^\epsilon) - A(x)\|_{\mathcal{Y}} + \sqrt{\epsilon}\|B(X^\epsilon) - B(x)\|_{\mathcal{Y}} > K\delta + \sqrt{\delta}\right) \\
\leq\ & \epsilon \log P\left(\|X^\epsilon - x\|_{\mathcal{X}} < \delta, \sqrt{\epsilon}\|B(X^\epsilon) - B(x)\|_{\mathcal{Y}} > \sqrt{\delta}\right).
\end{aligned}$$

The conclusion of the lemma now follows from (10.1.4). ∎

Theorem 10.1.5 *For any compact subset C of $\mathcal{X}$, we have*

$$\limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \in C) \leq -\inf\{I(x) : x \in C\}.$$

Proof: Let $c < \inf\{I(x) : x \in C\}$. From Theorem 10.1.3, for each $x \in \mathcal{X}$, $I^x(y)$ is lower-semi-continuous in $y \in \mathcal{Y}$. For $x \in C$, as $I^x(x) = I(x) > c$, there exists $\delta(x)'$ such that $I^x(y) > c$ whenever $\|y - x\|_{\mathcal{Y}} < 2\delta(x)'$. Let $\delta(x)$ be determined by $\delta(x)'$ through $\delta' = (1+K)\delta + \sqrt{\delta}$. Since C is compact in $\mathcal{X}$, there exist $x^j \in \mathcal{X}$, $j = 1, 2, \cdots, n$, such that

$$C \subset \cup_{j=1}^n \left\{y \in \mathcal{X} : \|y - x^j\|_{\mathcal{X}} < \delta^j\right\},$$

where $\delta^j = \delta(x^j)$. Hence

$$\begin{aligned}
P(X^\epsilon \in C) &\leq \sum_{j=1}^n P\left(\|X^\epsilon - x^j\|_{\mathcal{X}} < \delta^j\right) \\
&\leq \sum_{j=1}^n P\left(\|X^\epsilon - x^j\|_{\mathcal{X}} < \delta^j, \|X^{\epsilon,x^j} - x^j\|_{\mathcal{Y}} > \delta'^j\right) \\
&\quad + \sum_{j=1}^n P\left(\|X^{\epsilon,x^j} - x^j\|_{\mathcal{Y}} \leq \delta'^j\right), \\
&\equiv \sum_{j=1}^n p_j^{(1)} + \sum_{j=1}^n p_j^{(2)}.
\end{aligned}$$

By Lemma 10.1.2 and Theorem 10.1.3, we have

$$\limsup_{\epsilon\to 0} \epsilon \log p_j^{(1)} \le -L$$

and

$$\limsup_{\epsilon\to 0} \epsilon \log p_j^{(2)} \le -c, \quad j = 1, \cdots, n.$$

Hence

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \in C) \\
\le\ & \limsup_{\epsilon\to 0} \epsilon \log \left((2n) \cdot \max\left\{p_j^{(k)} : 1 \le j \le n; k = 1, 2\right\}\right) \\
\le\ & \max(-L, -c).
\end{aligned}$$

The conclusion of the theorem then follows by letting $L \to \infty$ and $c \to \inf\{I(x) : x \in C\}$. ∎

Theorem 10.1.6 *For any closed subset C of $\mathcal{X}$, we have*

$$\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \in C) \le -\inf\{I(x) : x \in C\}. \tag{10.1.17}$$

Proof: For $L > 0$, let C_L be the compact set given by (10.1.5). Then $C \cap C_L$ is compact and hence, by Theorem 10.1.5

$$\begin{aligned}
\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \in C \cap C_L) &\le -\inf\{I(x) : x \in C \cap C_L\} \\
&\le -\inf\{I(x) : x \in C\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \in C) \\
\le\ & \limsup_{\epsilon\to 0} \epsilon \log \left(2 \max\{P(X^\epsilon \in C \cap C_L), P(X^\epsilon \in C \setminus C_L)\}\right) \\
\le\ & \max\left(\limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \in C \cap C_L), \limsup_{\epsilon\to 0} \epsilon \log P(X^\epsilon \notin C_L)\right) \\
\le\ & \max\left(-\inf\{I(x) : x \in C\}, -L\right).
\end{aligned}$$

Letting $L \to \infty$, we see that (10.1.17) holds. ∎

Finally, we show that the function I defined by (10.1.11) is a rate function in the sense of Donsker and Varadhan.

For each $c > 0$, define the level set

$$L_c = \{x \in \mathcal{X} : I(x) \le c\}.$$

Theorem 10.1.7 *$I(x)$ is a lower-semi-continuous function on $\mathcal{X}$.*

Proof: Let $\{x^n\} \subset L_c$ and $x^n \to x$ in $\mathcal{X}$. We only need to show that $I(x) \leq c$. Let $h^n \in \mathcal{D}(\gamma)$ be such that $x^n = \gamma(h^n)$ and $\|h^n\|_{\mathcal{H}}^2 \leq 2I(x^n) + \frac{2}{n}$. Let

$$y^n = \hat{B}(x, h^n) \quad \text{and} \quad y = x - A(x). \tag{10.1.18}$$

Then by (10.1.1) and (10.1.2)

$$\begin{aligned} & \|y^n - y\|_{\mathcal{Y}} \\ = \ & \left\| \left(\hat{B}(x, h^n) - \hat{B}(x^n, h^n) \right) + (x^n - x) - (A(x^n) - A(x)) \right\|_{\mathcal{Y}} \\ \leq \ & \left(K\|h^n\|_{\mathcal{H}} + 1 + K \right) \|x^n - x\|_{\mathcal{X}} \to 0. \end{aligned}$$

From

$$I^{x,0}(y^n) \leq \frac{1}{2}\|h^n\|_{\mathcal{H}}^2 \leq I(x^n) + \frac{1}{n} \leq c + \frac{1}{n},$$

it follows from Theorem 10.1.2 that $I^{x,0}(y) \leq c$. Hence for any $\eta > 0$, there exists $h \in \mathcal{H}$ such that $y = \hat{B}(x, h)$ and

$$\frac{1}{2}\|h\|_{\mathcal{H}}^2 \leq c + \eta. \tag{10.1.19}$$

By (10.1.18) and (10.1.19) we have $h \in \mathcal{D}(\gamma)$, $x = \gamma(h)$ and $I(x) \leq c$. ∎

Theorem 10.1.8 *For each $c > 0$, L_c is compact in $\mathcal{X}$.*

Proof: Taking $L > c$, we only need to show that $L_c \subset C_L$ where C_L is the compact set appearing in (10.1.5). If this is not true, there exists $x_0 \in L_c \backslash C_L$. As C_L^c, the complement of C_L, is open and $x_0 \in L_c$, it follows from (10.1.5) and (10.1.14) that

$$-c \leq -I(x_0) \leq \liminf_{\epsilon \to 0} \epsilon \log P(X^\epsilon \in C_L^c) \leq \limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \notin C_L) \leq -L.$$

This contradicts the fact that $L > c$. Hence $L_c \subset C_L$ and compact. ∎

We summarize the above results into a theorem.

Theorem 10.1.9 *Under assumptions (A1)-(A5), $\{X^\epsilon\}$ satisfies the LDP on $\mathcal{X}$ with rate function I given by (10.1.11).*

10.2 Application to stochastic differential equations on conuclear spaces

In this section, we consider the LDP of $\{X^\epsilon\}$ governed by the following SDE:

$$X_t^\epsilon = \xi + \int_0^t C(s, X_s^\epsilon)ds + \sqrt{\epsilon}\int_0^t G(s, X_s^\epsilon)dW_s \tag{10.2.1}$$

where $C : \mathbf{R}_+ \times \Phi' \to \Phi'$ and $G : \mathbf{R}_+ \times \Phi' \to L(\Phi', \Phi')$ are two measurable maps, $\xi \in \Phi'$, W is a Φ'-valued Wiener process with covariance Q.

To establish a unique strong solution of (10.2.1) for each $\epsilon > 0$, we assume that (C, G, Q) satisfies the conditions (D1)-(D3) of Chapter 8 and (DM)$'$ (Monotonicity) $\forall t \in [0,T]$ and $v_1, v_2 \in \Phi_{-p}$,

$$2 < C(t, v_1) - C(t, v_2), v_1 - v_2 >_{-q} \leq K\|v_1 - v_2\|_{-q}^2$$

and

$$\|G(t, v_1) - G(t, v_2)\|^2_{L_{(2)}(H_Q, \Phi_{-p})} \leq K\|v_1 - v_2\|^2_{-p}.$$

As ξ is deterministic, the condition (D4) of Chapter 8 is satisfied. Hence, by Theorem 8.3.1, the SDE (10.2.1) has a unique strong solution X^ϵ for each $\epsilon > 0$. It follows from Corollary 3.2.1 that there exists $r_2 > 0$ such that $W_\cdot \in C([0,T], \Phi_{-r_2})$ a.s. As X^ϵ is the strong solution of the SDE (10.2.1), we may assume that the stochastic basis $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ is given by

$$\Omega = C([0,T], \Phi_{-r_2}), \mathcal{F}_t = \mathcal{B}_t(C([0,T], \Phi_{-r_2}))$$

and P is the probability measure induced by W on $C([0,T], \Phi_{-r_2})$.

Theorem 10.2.1 *Let*

$$\mathcal{H} = \left\{\int_0^\bullet \dot{h}_s ds : \dot{h} \in L^2([0,T], H_Q)\right\}.$$

Then $\mathcal{H} \subset \Omega$ and $(i, \mathcal{H}, \Omega)$ is an abstract Wiener space, where i is the canonical injection from $\mathcal{H}$ to Ω. Further, P is the standard Wiener measure on Ω.

Proof: It is clear that $\mathcal{H} \subset \Omega$. We identify H_Q' with H_Q by the Riesz representation theorem and let $\tilde{\Phi}_{-r_2} \subset H_Q$ be the dual of Φ_{-r_2} such that

$$< \ell, v >\tilde{}_{-r_2} =< \ell, v >_{H_Q}, \forall \ell \in \tilde{\Phi}_{-r_2}, v \in H_Q \subset \Phi_{-r_2},$$

where $< \cdot, \cdot >\tilde{}_{-r_2}$ is the pairing between $\tilde{\Phi}_{-r_2}$ and Φ_{-r_2}. We define $\tilde{\Omega} \subset \mathcal{H}$ as the dual of Ω in a similar manner.

Define $\sqrt{Q_{r_2}}^{(1)} : H_Q \to \tilde{\Phi}_{-r_2}$ as the dual of the isometry $\sqrt{Q_{r_2}}' : \Phi_{-r_2} \to H_Q$. Then, by Lemma 3.2.2, $\sqrt{Q_{r_2}}^{(1)}$ is an isometry from H_Q onto $\tilde{\Phi}_{-r_2}$ and for $e_1, e_2 \in H_Q$,

$$\begin{aligned}
&\left\langle \sqrt{Q_{r_2}}^{(1)} e_1, \sqrt{Q_{r_2}}^{(1)} e_2 \right\rangle_{H_Q} \\
= &\sum_j \left\langle \sqrt{Q_{r_2}}^{(1)} e_1, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{H_Q} \left\langle \sqrt{Q_{r_2}}^{(1)} e_2, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{H_Q} \\
= &\sum_j \left\langle \sqrt{Q_{r_2}}^{(1)} e_1, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{\tilde{-r_2}} \left\langle \sqrt{Q_{r_2}}^{(1)} e_2, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{\tilde{-r_2}} \\
= &\sum_j \left\langle e_1, \sqrt{Q_{r_2}}' \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{H_Q} \left\langle e_2, \sqrt{Q_{r_2}}' \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{H_Q} \\
= &\sum_j \left\langle v_1, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{-r_2} \left\langle v_2, \sqrt{Q_{r_2}}' \phi_j^{-r_2} \right\rangle_{-r_2} \\
= &\sum_j \left\langle e_1, \phi_j^{-r_2} \right\rangle_{-r_2} \left\langle e_2, \phi_j^{-r_2} \right\rangle_{-r_2} =< e_1, e_2 >_{-r_2} \qquad (10.2.2)
\end{aligned}$$

where $v_i \in \Phi_{-r_2}$ such that $e_i = \sqrt{Q_{r_2}}' v_i$, $i = 1, 2$. Let

$$\mathcal{H}_0 = \left\{ f = \int_0^\bullet \ell_s ds : \ell \in C^1([0,T], \tilde{\Phi}_{-r_2}), \ell_0 = \ell_T = 0 \right\}.$$

Then $\mathcal{H}_0 \subset \tilde{\Omega}$ is a dense subset of $\mathcal{H}$. For any $f \in \mathcal{H}_0$, let $h_s \in H_Q$ be such that $\ell_s = \sqrt{Q_{r_2}}^{(1)} h_s$. It is easy to see that

$$< f, W >_{\tilde{\Omega}} = \int_0^T \left\langle \dot{h}_t, \sqrt{Q_{r_2}}' W_t \right\rangle_{H_Q} dt$$

is a Gaussian random variable with mean 0. To show that $(i, \mathcal{H}, \Omega)$ is an abstract Wiener space and P is the standard Wiener measure on Ω we only need to show that

$$E\{< f, W >_{\tilde{\Omega}}\}^2 = \|f\|^2_{\mathcal{H}}.$$

Let $v_t \in \Phi_{-r_2}$ be such that $\dot{h}_t = \sqrt{Q_{r_2}}' v_t$. Then

$$\begin{aligned}
E\{< f, W >_{\tilde{\Omega}}\}^2 &= E\left(\int_0^T < \dot{h}_t, \sqrt{Q_{r_2}}' W_t >_{H_Q} dt \right)^2 \\
&= E\left(\int_0^T < v_t, W_t >_{-r_2} dt \right)^2 \\
&= \int_0^T \int_0^T E\{W_t[\theta_{r_2} v_t] W_s[\theta_{r_2} v_s]\} dt ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_0^T (t \wedge s) \left\langle \sqrt{Q_{r_2}}\theta_{r_2} v_t, \sqrt{Q_{r_2}}\theta_{r_2} v_s \right\rangle_{r_2} dtds \\
&= \int_0^T \int_0^T (t \wedge s) \left\langle \sqrt{Q_{r_2}}' v_t, \sqrt{Q_{r_2}}' v_s \right\rangle_{-r_2} dtds \\
&= \int_0^T \int_0^T (t \wedge s) \left\langle \dot{h}_t, \dot{h}_s \right\rangle_{-r_2} dtds.
\end{aligned}
$$

On the other hand, by (10.2.2)

$$
\begin{aligned}
\|f\|^2_{\mathcal{H}} &= \int_0^T \left\| \int_0^t \sqrt{Q_{r_2}}^{(1)} \dot{h}_s ds \right\|^2_{H_Q} dt \\
&= \int_0^T \int_0^t \int_0^t \left\langle \sqrt{Q_{r_2}}^{(1)} \dot{h}_{s_1}, \sqrt{Q_{r_2}}^{(1)} \dot{h}_{s_2} \right\rangle_{H_Q} ds_1 ds_2 dt \\
&= \int_0^T \int_t^T \int_t^T \left\langle \dot{h}_{s_1}, \dot{h}_{s_2} \right\rangle_{-r_2} ds_1 ds_2 dt \\
&= \int_0^T \int_0^T (s_1 \wedge s_2) \left\langle \dot{h}_{s_1}, \dot{h}_{s_2} \right\rangle_{-r_2} ds_1 ds_2. \qquad \blacksquare
\end{aligned}
$$

Let p_2 be an index such that $p_2 \geq q_1$ and the canonical injection from Φ_{-p_1} to Φ_{-p_2} is Hilbert-Schmidt, where q_1 is determined by p_1 through Assumption (D). Let $p_3 \geq q_2$ be defined similarly. Now we regard X^ϵ as Φ_{-p_2}-valued processes and consider their LDP as $\epsilon \to 0$.

Let $\mathcal{X} = C([0,T], \Phi_{-p_2})$, $\mathcal{Y} = C([0,T], \Phi_{-p_3})$, and let $\mathcal{A}_{\mathcal{X}}$ (resp. $\mathcal{A}_{\mathcal{Y}}$) be the collection of Φ_{-p_2} (resp. Φ_{-p_3}) valued adapted continuous processes. It is clear that Condition (A1) holds.

Let $A : \mathcal{X} \to \mathcal{Y}$ be given by

$$
A(x)_t = \xi + \int_0^t C(s, x_s) ds, \ \forall x \in \mathcal{X}.
$$

To verify the condition (10.1.1) we need the following assumption.
(D5) (Lipschitz) $\forall t \in [0,T]$ and $v_1, v_2 \in \Phi_{-p}$,

$$
\|C(t, v_1) - C(t, v_2)\|^2_{-q} \leq K \|v_1 - v_2\|^2_{-p}.
$$

Condition (A2) follows directly from Assumptions (D1) and (D5).

For $X \in \mathcal{A}_{\mathcal{X}}$, let

$$
B(X)_t = \int_0^t G(s, X_s) dW_s.
$$

It is easy to see that B is a map from $\mathcal{A}_{\mathcal{X}}$ to $\mathcal{A}_{\mathcal{X}}$. Now we verify the condition (A3) for B with $\mathcal{Y}$ replaced by $\mathcal{X}$ (i.e. a stronger condition than (A3)).

We define a map $\hat{B} : \mathcal{X} \times \mathcal{H} \to \mathcal{X}$ as follows

$$
\hat{B}(x, h)_t = \int_0^t G(s, x_s) \dot{h}_s ds \quad \forall x \in \mathcal{X}, h \in \mathcal{H}.
$$

As

$$
\begin{aligned}
\|\hat{B}(x,h)-\hat{B}(y,h)\|_{\mathcal{X}} &= \sup_{0\le t\le T}\left\|\int_0^t (G(s,x_s)-G(s,y_s))\dot{h}_s ds\right\|_{-p_2} \\
&\le \int_0^T \|G(s,x_s)-G(s,y_s)\|_{L_{(2)}(H_Q,\Phi_{-p_2})}\|\dot{h}_s\|_{H_Q} ds \\
&\le \int_0^T \sqrt{K}\|x_s-y_s\|_{-p_2}\|\dot{h}_s\|_{H_Q} ds \\
&\le \sqrt{KT}\|h\|_{\mathcal{H}}\|x-y\|_{\mathcal{X}},
\end{aligned}
$$

and similarly, since

$$\|\hat{B}(x,e_1)-\hat{B}(x,e_2)\|_{\mathcal{X}} \le \sqrt{KT}(1+\|x\|_{\mathcal{X}})\|e_1-e_2\|_{\mathcal{H}},$$

$\hat{B}$ is a continuous map. It is clear that for each $x\in\mathcal{X}$, $\hat{B}(x,\cdot):\mathcal{H}\to\mathcal{X}$ is linear and the lifting map $\tilde{B}(x,\cdot):\Omega\to\mathcal{X}$ is given by

$$\tilde{B}(x,\cdot)_t = \int_0^t C(s,x_s)dW_s.$$

Hence $\tilde{B}(x,\cdot)=B(x)\in\mathcal{A}_{\mathcal{X}}$.

For any $h\in\mathcal{H}$, $X\in\mathcal{A}_{\mathcal{X}}$, we have

$$B_h(X) = \int_0^t G(s,X_s)\dot{h}_s ds$$

in $\mathcal{A}_{\mathcal{X}}$. Further it follows from

$$
\begin{aligned}
B(X)_t - B_h(X)_t &= \int_0^t G(s,X_s)dW_s - \int_0^t G(s,X_s)\dot{h}_s ds \\
&= \int_0^t G(s,X_s)d(W_s-h_s)
\end{aligned}
$$

and

$$
\begin{aligned}
T_h\left(B(T_{-h}X)\right)_t &= T_h\left(\int_0^t G(s,X_s(\omega+h))dW_s\right) \\
&= \int_0^t G(s,X_s)d(W_s-h_s),
\end{aligned}
$$

that (10.1.3) holds. Therefore we have proved (i)-(iv) of Assumption (A3). To verify the last condition of (A3) we need the following lemmas.

Lemma 10.2.1 (Garsia) *Let $(\mathcal{Z},d)$ be a metric space and let ψ be a continuous map from $[0,T]$ to $\mathcal{Z}$. Suppose that Ψ and p are increasing functions in $x\ge 0$ such that $\Psi(0)=p(0)=0$ and Ψ is convex. Let*

$$\eta = \int_0^T\int_0^T \Psi\left(\frac{d(\psi(t),\psi(s))}{p(|t-s|)}\right) dtds.$$

Then, for any $t, s \in [0,T]$, we have

$$d(\psi(t), \psi(s)) \leq 8 \int_0^{|t-s|} \Psi^{-1}(\eta u^{-2}) dp(u),$$

where Ψ^{-1} denotes the inverse function.

Proof: The case of $\mathcal{Z} = \mathbf{R}$ has been proved by Garsia [11]. For general metric space, the lemma follows exactly from the same arguments. ∎

Lemma 10.2.2 *Let $f : [0,T] \times \Omega \to L_{(2)}(H_Q, \Phi_{-p_2})$ be a measurable map such that*

$$\sup_{0 \leq t \leq T} \|f(t,\omega)\|^2_{L_{(2)}(H_Q, \Phi_{-p_2})} \leq K_1 \quad a.s.$$

where K_1 is a constant. Let

$$\psi(t,\omega) = \int_0^t f(s,\omega) dW_s \quad and \quad 0 < \alpha < \frac{1}{2}.$$

Then, for any $L > 0$, there exists a constant $M > 0$ such that

$$\limsup_{\epsilon \to 0} \epsilon \log P\left([\psi]_{-p_2,\alpha} > \frac{M}{\sqrt{\epsilon}}\right) \leq -L,$$

where

$$[\psi]_{-p_2,\alpha} = \sup_{0 \leq s < t \leq T} \frac{\|\psi(t) - \psi(s)\|_{-p_2}}{|t-s|^\alpha}.$$

Proof: Let

$$g(r,\omega) = f(r,\omega)|t-s|^{-\alpha} 1_{[s,t]}(r) \quad \text{and} \quad \gamma_u = \int_0^u g(r) dW_r.$$

Then

$$(\psi(t) - \psi(s))|t-s|^{-\alpha} = \gamma_T.$$

It follows from Itô's formula that

$$\begin{aligned} \|\gamma_u\|^2_{-p_2} &= \sum_k \int_0^u 2 \left\langle g(r)'\theta_{p_2}\gamma_r, \phi_k^{p_2} \right\rangle_{p_2} dW_r[\phi_k^{p_2}] \\ &\quad + \int_0^u \|g(r)\|^2_{L_{(2)}(H_Q, \Phi_{-p_2})} dr. \end{aligned}$$

By Itô's formula again, we have

$$\begin{aligned} &\sqrt{1 + \|\gamma_T\|^2_{-p_2}} - 1 \\ = &\sum_k \int_0^T \left\langle g(r)'\theta_{p_2}\gamma_r, \phi_k^{p_2} \right\rangle_{p_2} (1 + \|\gamma_r\|^2_{-p_2})^{-1/2} dW_r[\phi_k^{p_2}] \\ &+ \frac{1}{2} \int_0^T \|g(r)\|^2_{L_{(2)}(H_Q, \Phi_{-p_2})} (1 + \|\gamma_r\|^2_{-p_2})^{-1/2} dr \\ &- \frac{1}{8} \int_0^T Q\left(g(r)'\theta_{p_2}\gamma_r, g(r)'\theta_{p_2}\gamma_r\right) (1 + \|\gamma_r\|^2_{-p_2})^{-3/2} dr. \quad (10.2.3) \end{aligned}$$

As the first term on the right hand side of (10.2.3) is a real-valued continuous square integrable martingale, by Theorem 7.3 in Ikeda and Watanabe ([18], p.86), it can be represented as $\hat{W}_\tau$, where $\hat{W}$ is a one-dimensional Brownian motion and

$$\begin{aligned}
\tau &= \int_0^T Q(g(r)'\theta_{p_2}\gamma_r, g(r)'\theta_{p_2}\gamma_r)(1+\|\gamma_r\|_{-p_2}^2)^{-1}dr \\
&\leq \int_0^T \|g(r)\|_{L_{(2)}(H_Q,\Phi_{-p_2})}^2 |\gamma_r|_{-p_2}^2 (1+\|\gamma_r\|_{-p_2}^2)^{-1}dr \\
&\leq \int_s^t \|f(r)\|_{L_{(2)}(H_Q,\Phi_{-p_2})}^2 |t-s|^{-2\alpha}dr \leq K_1T.
\end{aligned}$$

Hence

$$\sqrt{1+\|\gamma_T\|_{-p_2}^2} - 1 \leq \sup_{t\in[0,K_1T]} |\hat{W}_t| + K_1T,$$

i.e.

$$\|\gamma_T\|_{-p_2}^2 \leq \left(1+\sup_{t\in[0,K_1T]} |\hat{W}_t| + K_1T\right)^2 - 1.$$

It follows from Fernique's theorem (see [35] p.159) that there exist constants $K_2, K_3 > 0$ such that

$$E\exp\left(K_2\|\gamma_T\|_{-p_2}^2\right) \leq K_3. \tag{10.2.4}$$

Let

$$\eta = \int_0^T\int_0^T \exp\left(K_2\frac{\|\psi(t)-\psi(s)\|_{-p_2}^2}{|t-s|^\alpha}\right)dtds.$$

Then

$$E\eta \leq \int_0^T\int_0^T E\exp\left(K_2\|\gamma_T\|_{-p_2}^2\right)dtds \leq K_3T^2.$$

Let $\Psi(x) = e^{K_2x^2}-1$ and $p(x) = x^\alpha$ for $x \geq 0$. Then Ψ and p satisfy the conditions of Lemma 10.2.1 and

$$\eta - T^2 = \int_0^T\int_0^T \Psi\left(\frac{\|\psi(t)-\psi(s)\|_{-p_2}}{p(|t-s|)}\right)dtds.$$

Hence

$$\begin{aligned}
\|\psi(t)-\psi(s)\|_{-p_2} &\leq 8\int_0^{t-s}\Psi^{-1}\left(\frac{\eta-T^2}{u^2}\right)p(du) \\
&\leq 8\int_0^{t-s}\sqrt{\left\{\log\left(1+\frac{\eta}{u^2}\right)\right\}\Big/ K_2}du^\alpha \\
&= \frac{8}{\sqrt{K_2}}\int_0^{|t-s|^\alpha}\sqrt{\log\left(1+\frac{\eta}{v^{2/\alpha}}\right)}dv \\
&\leq \frac{8}{\sqrt{K_2}}\int_0^{|t-s|^\alpha}\left(\sqrt{|\log\eta|}+\sqrt{\log 2}+\sqrt{2|\log v|/\alpha}\right)dv.
\end{aligned}$$

It is easy to see that, for any $\alpha' < \alpha$, we have

$$\lim_{a\to 0} a^{-\alpha'/\alpha} \int_0^a \sqrt{|\log u|}du = 0$$

and hence, there exists a constant K_4 such that

$$\|\psi(t) - \psi(s)\|_{-p_2} \le K_4 \left(\sqrt{|\log \eta|} + 1\right) |t-s|^{\alpha'}. \tag{10.2.5}$$

As α is any number in $(0, \frac{1}{2})$, we may replace α' in (10.2.5) by α. Hence

$$\begin{aligned} & \limsup_{\epsilon\to 0} \epsilon \log P\left([\psi]_{-p_2,\alpha} > \frac{M}{\sqrt{\epsilon}}\right) \qquad (10.2.6)\\ \le\ & \limsup_{\epsilon\to 0} \epsilon \log P\left(K_4\left(1+\sqrt{|\log\eta|}\right) > \frac{M}{\sqrt{\epsilon}}\right)\\ =\ & \limsup_{\epsilon\to 0} \epsilon \log P\left(\eta > \exp\left[\left(\frac{M}{\sqrt{\epsilon}K_4} - 1\right)^2\right]\right)\\ \le\ & \limsup_{\epsilon\to 0} \epsilon \log \left\{\exp\left[-\left(\frac{M}{\sqrt{\epsilon}K_4} - 1\right)^2\right] E\eta\right\} = -\left(\frac{M}{K_4}\right)^2. \end{aligned}$$

Taking $M = K_4\sqrt{L}$, the lemma is proved. ∎

Now we verify the last condition of (A3).

Theorem 10.2.2 *B is exponentially continuous.*

Proof: Let $X, Y \in \mathcal{A}_{\mathcal{X}}$ and ψ be given by Lemma 10.2.2 with

$$f(t) = \delta^{-1}(G(t,X_t) - G(t,Y_t))1_{\|X_t - Y_t\|_{-p_2}<\delta}.$$

Then from $(DM)'$,

$$\|f(t)\|^2_{L_{(2)}(H_Q,\Phi_{-p_2})} \le \delta^{-2}K\|X_t - Y_t\|^2_{-p_2} 1_{\|X_t-Y_t\|_{-p_2}<\delta} \le K,$$

and hence

$$\begin{aligned} & P\left(\sqrt{\epsilon}\|B(X) - B(Y)\|_{\mathcal{X}} > \sqrt{\delta}, \|X - Y\|_{\mathcal{X}} < \delta\right)\\ \le\ & P\left(\|\psi\|_{\mathcal{X}} > \frac{1}{\sqrt{\delta\epsilon}}\right) \le P\left(T^\alpha[\psi]_{-p_2,\alpha} > \frac{1}{\sqrt{\delta\epsilon}}\right). \end{aligned}$$

Our result follows from Lemma 10.2.2. ∎

It is clear that the second condition of (A4) holds. The next two theorems verify the first condition of (A4). For any $M > 0$ and $\alpha \in (0, 1/2)$, let

$$C^1_M \equiv \left\{x \in C([0,T], \Phi_{-p_2}) : \sup_{0\le t\le T} \|x_t\|_{-p_1} \le M\right\}$$

and

$$C^2_{M,\alpha} \equiv \{x \in C([0,T], \Phi_{-p_2}) : [x]_{-p_2,\alpha} \leq M\}.$$

Theorem 10.2.3 *For any $L > 0$, there exists $M > 0$ such that*

$$\limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \notin C^1_M) \leq -L. \tag{10.2.7}$$

Proof: We have by Itô's formula,

$$\begin{aligned}
&(1 + \|X^\epsilon_t\|^2_{-p_1})^{\frac{1}{\epsilon}} - (1 + \|\xi\|^2_{-p_1})^{\frac{1}{\epsilon}} \qquad (10.2.8)\\
= \;& \frac{1}{\epsilon}\int_0^t 2C(s, X^\epsilon_s)[\theta_{p_1} X^\epsilon_s](1 + \|X^\epsilon_s\|^2_{-p_1})^{\frac{1}{\epsilon}-1} ds \\
&+ \int_0^t \|G(s, X^\epsilon_s)\|^2_{L_{(2)}(H_Q,\Phi_{-p_1})}(1 + \|X^\epsilon_s\|^2_{-p_1})^{\frac{1}{\epsilon}-1} ds \\
&+ \frac{2}{\sqrt{\epsilon}} \sum_k \int_0^t X^\epsilon_s[G(s, X^\epsilon_s)'\phi^{p_1}_k](1 + \|X^\epsilon_s\|^2_{-p_1})^{\frac{1}{\epsilon}-1} dW_s[\phi^{p_1}_k] \\
&+ \frac{2(1-\epsilon)}{\epsilon} \int_0^t Q\left(G(s, X^\epsilon_s)'\theta_{p_1} X^\epsilon_s, G(s, X^\epsilon_s)'\theta_{p_1} X^\epsilon_s\right) \\
&\qquad\qquad (1 + \|X^\epsilon_s\|^2_{-p_1})^{\frac{1}{\epsilon}-2} ds.
\end{aligned}$$

Let

$$\tau = \inf\{t \geq 0 : \|X^\epsilon_t\|_{-p_1} > M\}$$

and $\tau = \infty$ if the set $\{t \geq 0 : \|X^\epsilon_t\|_{-p_1} > M\}$ is empty. It follows from (D3) and (10.2.8) that

$$\begin{aligned}
E(1 + \|X^\epsilon_{t\wedge\tau}\|^2_{-p_1})^{\frac{1}{\epsilon}} \;\leq\;& (1 + \|\xi\|^2_{-p_1})^{\frac{1}{\epsilon}} \\
&+ K\left(\frac{3}{\epsilon} - 1\right) \int_0^t E(1 + \|X^\epsilon_{s\wedge\tau}\|^2_{-p_1})^{\frac{1}{\epsilon}} ds.
\end{aligned}$$

From Gronwall's inequality, we have

$$E(1 + \|X^\epsilon_{T\wedge\tau}\|^2_{-p_1})^{\frac{1}{\epsilon}} \leq (1 + \|\xi\|^2_{-p_1})^{\frac{1}{\epsilon}} e^{K(3/\epsilon - 1)T}.$$

Then

$$\begin{aligned}
(1 + M^2)^{\frac{1}{\epsilon}} P\left(\sup_{0 \leq t \leq T} \|X^\epsilon_t\|_{-p_1} > M\right) \;\leq\;& E\left(1 + \|X^\epsilon_{T\wedge\tau}\|^2_{-p_1}\right)^{\frac{1}{\epsilon}} \\
\leq\;& (1 + \|\xi\|^2_{-p_1})^{\frac{1}{\epsilon}} e^{(3K/\epsilon - K)T}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\epsilon \log P\left(\sup_{0 \leq t \leq T} \|X^\epsilon_t\|_{-p_1} > M\right) \\
\leq\;& -\log(1 + M^2) + \log(1 + \|\xi\|^2_{-p_1}) + (3K - \epsilon K)T.
\end{aligned}$$

Choosing M such that

$$L = \log(1+M^2) - \log(1+\|\xi\|_{-p_1}^2) - 3KT,$$

we see that (10.2.7) holds. ∎

Theorem 10.2.4 *For any $\alpha \in (0, 1/2)$ and $L > 0$, there exist two constants M and M' such that*

$$\limsup_{\epsilon \to 0} \epsilon \log P\left(X^\epsilon \in C_M^1 - C_{M',\alpha}^2\right) \le -L.$$

Proof: Let M be given by Theorem 10.2.3 and ψ by Lemma 10.2.2 with

$$f(t) = G(t, X_t^\epsilon) 1_{\|X_t^\epsilon\|_{-p_1} \le M}.$$

Then

$$\|f(t)\|^2_{L_{(2)}(H_Q, \Phi_{-p_2})} \le K(1 + \|X_t^\epsilon\|^2_{-p_2}) 1_{\|X_t^\epsilon\|_{-p_2} \le M} \le K(1+M^2).$$

Hence

$$\begin{aligned} & P(X^\epsilon \in C_M^1 - C_{M',\alpha}^2) = P(X^\epsilon \in C_M^1, [X^\epsilon]_{-p_2,\alpha} > M') \\ \le\ & P\Big(X^\epsilon \in C_M^1, \\ & \sup_{0 \le s < t \le T} \frac{\int_s^t \|C(r, X_r^\epsilon)\|_{-p_2} dr + \sqrt{\epsilon}\|\psi(t) - \psi(s)\|_{-p_2}}{|t-s|^{-\alpha}} > M'\Big) \\ \le\ & P\left(\sqrt{\epsilon}[\psi]_{-p_2,\alpha} > M' - \sqrt{K(1+M^2)}T^{1-\alpha}\right). \end{aligned}$$

The assertion of the theorem then follows from Lemma 10.2.2. ∎

Corollary 10.2.1 *Let $C_L \equiv C_M^1 \cap C_{M',\alpha}^2$. Then C_L is a compact subset of $\mathcal{X}$ and*

$$\limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \notin C_L) \le -L.$$

Proof: As $\{v \in \Phi_{-p_2} : \|v\|_{-p_1} \le M\}$ is compact in Φ_{-p_2}, we see that C_L is compact in $\mathcal{X}$. Further,

$$\begin{aligned} & \limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \notin C_L) \\ \le\ & \limsup_{\epsilon \to 0} \epsilon \log \left\{P(X^\epsilon \notin C_M^1) + P(X^\epsilon \in C_M^1 - C_{M',\alpha}^2)\right\} \\ \le\ & -L. \end{aligned}$$

∎

Next we verify Assumption (A5).

Theorem 10.2.5 *Let Z^ϵ be given by (10.1.13). Then Z^ϵ converges to 0 in P.*

Proof: As in (10.2.3), we have

$$\begin{aligned}\|Z_t^\epsilon\|_{-p_2}^2 &= 2\sqrt{\epsilon}\sum_k \int_0^t \langle G(s, Z_s^\epsilon + x_s)'\theta_{p_2} Z_s^\epsilon, \phi_k^{p_2}\rangle_{p_2} dW_s[\phi_k^{p_2}] \\ &\quad +\epsilon\int_0^t \|G(s, Z_s^\epsilon + x_s)\|^2_{L_{(2)}(H_Q,\Phi_{-p_2})} ds \\ &\quad +\int_0^t 2\langle Z_s^\epsilon, \{C(s, Z_s^\epsilon + x_s) - C(s, x_s)\} \\ &\qquad\qquad +\{G(s, Z_s^\epsilon + x_s) - G(s, x_s)\}\dot{h}_s\rangle_{-p_2} ds.\end{aligned}$$

Hence, by the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}&\rho(r) \equiv E\sup_{0\le t\le r}\|Z_t^\epsilon\|_{-p_2}^2 \\ \le\ & 8\sqrt{\epsilon}E\left(\int_0^r Q(G(s, Z_s^\epsilon + x_s)'\theta_{p_2} Z_s^\epsilon, G(s, Z_s^\epsilon + x_s)'\theta_{p_2} Z_s^\epsilon) ds\right)^{1/2} \\ & +E\int_0^r \left(K + 2\sqrt{K}\|\dot{h}_s\|_{H_Q}\right)\|Z_s^\epsilon\|_{-p_2}^2 ds \\ & +E\int_0^r \epsilon K\left(1 + \|Z_s^\epsilon + x_s\|_{-p_2}^2\right) ds \\ \le\ & 8\sqrt{\epsilon}E\sqrt{\int_0^r K\left(1 + \|Z_s^\epsilon + x_s\|_{-p_2}^2\right)\|Z_s^\epsilon\|_{-p_2}^2 ds} \\ & +\sqrt{\int_0^r \left(K + 2\sqrt{K}\|\dot{h}_s\|_{H_Q}\right)^2 ds}\sqrt{\int_0^r \rho(s)^2 ds} \\ & +\int_0^r \epsilon K\left(1 + 2\rho(s) + 2\|x_s\|_{-p_2}^2\right) ds.\end{aligned}$$

Hence, there exist two constants K_5 and K_6 such that

$$\rho(r)^2 \le \sqrt{\epsilon}K_5 + K_6\int_0^r \rho(s)^2 ds.$$

From the Chebyshev and Gronwall inequalities, we have $\forall\delta > 0$,

$$P(\|Z^\epsilon\|_{\mathcal{X}} \ge \delta) \le \delta^{-2}\rho(T) \to 0 \quad \text{as } \epsilon \to 0.$$

∎

We summarize our results in the following theorem.

Theorem 10.2.6 *Under Assumptions (D1)-(D3), (DM)' and (D5), $\{X^\epsilon\}$ satisfies the LDP on $C([0,T], \Phi_{-p_2})$ with rate function*

$$I(x) = \inf\left\{\frac{1}{2}\|h\|_{\mathcal{H}}^2 : h \in \mathcal{D}(\gamma)\ s.t.\ x = \gamma(h)\right\}$$

where $\mathcal{D}(\gamma)$ is the collection of $h \in \mathcal{H}$ such that the following equation

$$x_t = \xi + \int_0^t (C(s,x_s) + G(s,x_s)\dot{h}_s)ds. \tag{10.2.9}$$

has at least one solution (denoted by $x = \gamma(h)$) in $\mathcal{X}$. ∎

Let us now make the following additional assumption.
(D6): $\forall t \in [0,T]$ and $v \in \Phi_{-p_2}$, $G(t,v) : H_Q \to \Phi_{-p_2}$ is invertible.

Theorem 10.2.7 *i) Under Assumptions (D1)-(D3), $(DM)'$ and (D5), $\mathcal{D}(\gamma) = \mathcal{H}$ and γ is continuous from $\mathcal{H}$ into $\mathcal{X}$.*
ii) If, in addition, (D6) holds, then γ is injective and $I(x)$ is given by

$$I(x) = \begin{cases} \frac{1}{2}\int_0^T \|G(t,x_t)^{-1}\{\dot{x}_t - C(t,x_t)\}\|_{H_Q}^2 dt & \text{if } \dot{x}_t \text{ exists} \\ \infty & \text{otherwise.} \end{cases}$$

Proof: i) First we consider $h \in \mathcal{H}$ such that

$$\|h\|_\infty \equiv \sup_{0\le t\le T} \|\dot{h}_t\|_{H_Q} < \infty.$$

For any $t \geq 0$ and $u \in \Phi$, let

$$\tilde{C}(t,u) = C(t,u) + G(t,u)\dot{h}_t \quad \text{and} \quad \tilde{G}(t,u) = 0.$$

It is easy to show that $(\tilde{C},\tilde{G})$ satisfies the conditions (D1)-(D3) and $(DM)'$ with $p_0(T)$ and K replaced by $\max\{p_0(T), r_1\}$ and $2K(1+\|h\|_\infty^2)$ respectively. Hence (10.2.9) has a unique solution in $C([0,T],\Phi_{-p_1})$.

For any $h \in \mathcal{H}$, let $h^n \in \mathcal{H}$ such that $\|h^n\|_\infty < \infty$, $\forall n \in \mathbf{N}$, and $h^n \to h$ in $\mathcal{H}$ as $n \to \infty$. Let $x^n = \gamma(h^n) \in C([0,T],\Phi_{-p_1})$. Then

$$x_t^n = \xi + \int_0^t \left(C(s,x_s^n) + G(s,x_s^n)\dot{h}_s^n\right) ds. \tag{10.2.10}$$

For any $u \in \Phi_{-p_1}$, let $\phi_m \in \Phi$ such that $\phi_m \to u$ in Φ_{-p_1}. Then for any $t \in [0,T]$,

$$\begin{aligned} 2\langle C(t,u),u\rangle_{-p_2} &= \lim_{m\to\infty} 2\langle C(t,\phi_m),\phi_m\rangle_{-p_2} \\ &= \lim_{m\to\infty} 2C(t,\phi_m)[\theta_{p_2}\phi_m] \\ &\leq \lim_{m\to\infty} K\left(1+\|\phi_m\|_{-p_2}^2\right) \\ &= K\left(1+\|u\|_{-p_2}^2\right). \end{aligned} \tag{10.2.11}$$

Hence by (D3), (10.2.10) and (10.2.11),

$$\begin{aligned}
\|x_t^n\|_{-p_2}^2 &= \|\xi\|_{-p_2}^2 + \int_0^t 2\langle C(s,x_s^n), x_s^n\rangle_{-p_2} ds \\
&\quad + \int_0^t 2\left\langle x_s^n, G(s,x_s^n)\dot{h}_s^n\right\rangle_{-p_2} ds \\
&\le \|\xi\|_{-p_2}^2 + K\int_0^t \left(1+\|x_s^n\|_{-p_2}^2\right) ds \\
&\quad + 2\int_0^t \|x_s^n\|_{-p_2} \left\|G(s,x_s^n)\sqrt{Q_{p_2}}' v_s^n\right\|_{-p_2} ds \\
&\le \|\xi\|_{-p_2}^2 + K\int_0^t \left(1+\|x_s^n\|_{-p_2}^2\right) ds \\
&\quad + 2\int_0^t \|x_s^n\|_{-p_2} \left\|G(s,x_s^n)\sqrt{Q_{p_2}}'\right\|_{L_{(2)}(\Phi_{-p_2})} \|v_s^n\|_{-p_2} ds \\
&\le \|\xi\|_{-p_2}^2 + K\int_0^t \left(1+\|x_s^n\|_{-p_2}^2\right) ds \\
&\quad + 2\int_0^t \|x_s^n\|_{-p_2} \sqrt{K(1+\|x_s^n\|_{-p_2}^2)}\,\|\dot{h}_s^n\|_{H_Q} ds,
\end{aligned}$$

where $v_s^n \in \Phi_{-p_2}$ such that $h_s^n = \sqrt{Q_{p_2}}' v_s^n$. By Hölder's inequality we have

$$\|x_t^n\|_{-p_2}^4 \le 3\|\xi\|_{-p_2}^4 + 3(K^2T + 4K\|h^n\|_{\mathcal{H}}^2)\int_0^t (1+\|x_s^n\|_{-p_2}^2)^2 ds. \tag{10.2.12}$$

It follows from Gronwall's inequality that for some constant K_7 such that $\|x^n\|_{\mathcal{X}} \le K_7$, $\forall n \ge 1$. Now we prove that x^n converges in $\mathcal{X}$. Making use of (10.2.10) again, it follows from the same arguments as in the derivation of (10.2.12) that

$$\|x^n - x^m\|_{\mathcal{X}} \le K_8\|h^n - h^m\|_{\mathcal{H}} \tag{10.2.13}$$

where K_8 is a constant. Hence, there exists $x \in \mathcal{X}$ such that $x^n \to x$. It follows from (10.2.10) again we see that x is a solution of (10.2.9). The uniqueness of the solution of (10.2.9) and the continuity of γ follows easily from (10.2.13).

ii) It is easy to see that γ is injective and hence $I(x)$ is given by

$$I(x) = \begin{cases} \frac{1}{2}\|h\|_{\mathcal{H}}^2 & \text{if } h \in \mathcal{H} \ s.t.\ x = \gamma(h) \\ \infty & \text{otherwise.} \end{cases}$$

But $x = \gamma(h)$ is equivalent to

$$h = \int_0^\bullet G(t,x_t)^{-1}\{\dot{x}_t - C(t,x_t)\} dt,$$

and hence, ii) follows immediately. ∎

10.3 Application to SPDEs

In this section, we apply the results of Section 1 to random fields $\{X^\epsilon(t,q),\ t \geq 0,\ q \in \mathcal{O}\}$ governed by the SPDE

$$\begin{aligned} dX^\epsilon(t,r) &= (L(t)X^\epsilon(t,r) + R(t,r,X^\epsilon(t,r)))drdt \\ &\quad + \sqrt{\epsilon}F(t,r,X^\epsilon(t,r))W(drdt) \end{aligned} \tag{10.3.1}$$

with initial condition

$$X^\epsilon(0,r) = \xi(r).$$

In addition to Assumptions (RD1)-(RD4) made in Section 4.3, we assume that F satisfies the following condition:
(RD5) There exists a constant K(R,F,T) such that, for all $x \in \mathbf{R}$, $r \in \mathcal{O}$ and $0 \leq t \leq T$,

$$|F(t,r,x)| \leq K(R,F,T). \tag{10.3.2}$$

To derive the large deviation result for random fields $\{X^\epsilon(t,r)\}_{\epsilon>0}$, we prove an analogue of Garsia's theorem for a general bounded open domain $\mathcal{O}$ satisfying the **cone condition**. The latter condition which we assume throughout this section means that there exist two positive constants a and a_0 such that, for any $r \in \mathcal{O}$, there exists a cone C_r with vertex at r with height a and base radius a_0.

For any hypercube Q in $\mathbf{R}^d$, we denote by Q' the hypercube in $\mathbf{R}^d$ such that Q and Q' have the same center with edges parallel to the co-ordinate axes and $e(Q) = 2e(Q')$, where $e(Q)$ is the common length of the edge of Q. For any set $C \subset \mathbf{R}^d$, let $|C|$ be its Lebesgue measure.

Lemma 10.3.1 *1° There exists a constant K_9 such that*

$$|Q \cap \mathcal{O}| \geq K_9|Q|, \tag{10.3.3}$$

for any hypercube Q such that

$$e(Q) \leq a/\sqrt{d} \quad \text{and} \quad Q' \cap \mathcal{O} \neq \emptyset. \tag{10.3.4}$$

2° Let Q be a hypercube satisfying (10.3.4). For any $r \in Q' \cap \mathcal{O}$ and $0 < \delta < e(Q)$, there exists a hypercube $Q_1 \subset Q$ such that $r \in Q_1'$ and $e(Q_1) = \delta$.

Proof: 1° Let $r \in Q' \cap \mathcal{O}$. As $e(Q) \leq a/\sqrt{d}$, C_r is not contained in Q. Otherwise

$$a \geq \text{diameter}(Q) \geq \text{diameter}(C_r) > a.$$

Let C_r' be the maximal cone contained in $C_r \cap Q$ such that its base is parallel to the base of C_r. Then the base of C_r' intersects with the boundary of Q.

Let q be a point in the intersection mentioned above. Let b (resp. ℓ) be the slant edge of C'_r (resp. C_r). Then

$$b = |rq| \geq \text{distance}(Q', Q^c) = \frac{e(Q)}{4}$$

where Q^c denotes the complement of Q.

It is easy to see that the height and base radius of C'_r are $\frac{ab}{\ell}$ and $\frac{a_0 b}{\ell}$ respectively. Hence

$$\frac{|C'_r|}{|C_0|} = \frac{d^{-1}V_{d-1}\left(a_0 b/\ell\right)^{d-1}(ab/\ell)}{d^{-1}V_{d-1}a_0^{d-1}a} = \left(\frac{b}{\ell}\right)^d \geq \left(\frac{e(Q)}{4\ell}\right)^d,$$

where V_{d-1} is the volume of the unit ball in $\mathbf{R}^{d-1}$. Therefore

$$|Q \cap \mathcal{O}| \geq |C'_r| \geq \left\{\frac{e(Q)}{4\ell}\right\}^d |C_0| = \frac{|C_0|}{(4\ell)^d}|Q| \equiv K_1|Q|.$$

2° Extend d segments through r with the following properties:
(i) they are orthogonal to each other and lie in Q;
(ii) each has length δ and parallels to an edge of Q;
(iii) r divides each segment into two parts, the length of each being not less than $\delta/4$.

Construct a hypercube Q_1 with edges parallel to those of Q and all the end points of the d segments mentioned above are in the surface of Q_1. It can be easily checked that $r \in Q'_1$ and $e(Q) = \delta$. ∎

Lemma 10.3.2 *Let ψ be a continuous function on $\mathcal{O}$. Let Ψ and p be increasing functions in $x \geq 0$ such that $\Psi(0) = p(0) = 0$ and Ψ convex. Let*

$$\eta = \int_{\mathcal{O}}\int_{\mathcal{O}} \Psi\left(\frac{|\psi(r) - \psi(q)|}{p(|r-q|)}\right) dr dq \leq \infty.$$

Then, for any $r, q \in \mathcal{O}$ with $|r-q| \leq a/2$, we have

$$|\psi(r) - \psi(q)| \leq 8\int_0^{2|r-q|} \Psi^{-1}\left(\frac{\eta}{K_{10}u^{2d}}\right) p(du), \tag{10.3.5}$$

where K_{10} is a constant and Ψ^{-1} denotes the inverse function.

Proof: If $\eta = \infty$ then (10.3.5) is obviously satisfied. Let η be finite. For $r, q \in \mathcal{O}$, if $|r-q| \leq a/2$, we have a hypercube Q'_0 such that $r, q \in Q'_0$ and $e(Q'_0) \leq a/2\sqrt{d}$. Define Q_0 to be a hypercube having the same center as Q'_0 and $e(Q_0) = 2e(Q'_0)$. As $r \in Q'_0 \cap \mathcal{O}$, it follows from Lemma 10.3.1 that $|Q_0 \cap \mathcal{O}| \geq K_9|Q_0|$. From here on we proceed similarly as in [11]. By Lemma 10.3.1

again, there exists a decreasing sequence $\{Q_n\}_{n\geq 0}$ of hypercubes such that $r \in Q'_n$ and $p(x_{n-1}) = 2p(x_n)$ for all $n \geq 1$, where $x_n = \sqrt{d}e(Q_n)$. Let

$$\tilde{Q} = Q \cap \mathcal{O} \quad \text{and} \quad \psi_Q = \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \psi(r)dr.$$

Then by (10.3.3) and Jensen's inequality, we have

$$\begin{aligned}
\Psi\left(\frac{|\psi_{Q_n} - \psi_{Q_{n-1}}|}{p(x_{n-1})}\right) &\leq \Psi\left(\frac{1}{|\tilde{Q}_{n-1}||\tilde{Q}_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n}\frac{|\psi(r)-\psi(q)|}{p(x_{n-1})}drdq\right)\\
&\leq \frac{1}{|\tilde{Q}_{n-1}||\tilde{Q}_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n}\Psi\left(\frac{|\psi(r)-\psi(q)|}{p(x_{n-1})}\right)drdq\\
&\leq \frac{1}{K_9^2|Q_{n-1}||Q_n|}\int_{\tilde{Q}_{n-1}}\int_{\tilde{Q}_n}\Psi\left(\frac{|\psi(r)-\psi(q)|}{p(|r-q|)}\right)drdq\\
&\leq \frac{\eta}{K_{10}x_n^{2d}}
\end{aligned}$$

since

$$|Q_{n-1}||Q_n| = [e(Q_{n-1})e(Q_n)]^d = \left(\frac{x_{n-1}}{\sqrt{d}}\right)^d\left(\frac{x_n}{\sqrt{d}}\right)^d \geq \frac{x_n^{2d}}{d^d}.$$

Here we have that $K_{10} = d^{-d}K_9^2$. Hence, noting that $p(x_{n-1}) = 2p(x_n) = 4p(x_{n+1})$ we have

$$\begin{aligned}
|\psi_{Q_n} - \psi_{Q_{n-1}}| &\leq \Psi^{-1}\left(\frac{\eta}{K_{10}x_n^{2d}}\right)p(x_{n-1})\\
&= 4\Psi^{-1}\left(\frac{\eta}{K_{10}x_n^{2d}}\right)(p(x_n) - p(x_{n+1}))\\
&\leq 4\int_{x_{n+1}}^{x_n}\Psi^{-1}\left(\frac{\eta}{K_{10}u^{2d}}\right)p(du).
\end{aligned}$$

Summing up over n from 0 to ∞, we have

$$|\psi(r) - \psi_{Q_0}| \leq 4\int_0^{x_0}\Psi^{-1}\left(\frac{\eta}{K_{10}u^{2d}}\right)p(du).$$

It is clear that the above procedure applies with r replaced by q, and hence

$$|\psi(r) - \psi(q)| \leq 8\int_0^{x_0}\Psi^{-1}(\frac{\eta}{K_{10}u^{2d}})p(du).$$

Making Q_0 as small as possible, we have x_0 arbitrarily close to $2|r-q|$. This finishes the proof of (10.3.5). ∎

It is easy to check that $[0,T]\times\mathcal{O}$ also satisfies the cone condition. Therefore, Lemma 10.3.2 is applicable to continuous functions defined on $[0,T]\times\mathcal{O}$.

Since only Corollary 10.3.1 below will be used in the rest of this section, we will use the notation a for the height of the new cone although its value has been changed.

Corollary 10.3.1 *Let ψ be a continuous function on $[0,T]\times\mathcal{O}$. Let Ψ and p be increasing functions in $x\ge 0$ such that $\Psi(0)=p(0)=0$ and Ψ convex. Let*

$$\eta=\int_0^t\int_{\mathcal{O}}\int_0^t\int_{\mathcal{O}}\Psi\left(\frac{|\psi(t,r)-\psi(s,q)|}{p(\rho((t,r),(s,q))}\right)dtdrdsdq.$$

Then, for any $(t,r),(s,q)\in[0,T]\times\mathcal{O}$ with $\rho\equiv\rho((t,r),(s,q))\le a/2$, we have

$$|\psi(t,r)-\psi(s,q)|\le 8\int_0^{2\rho}\Psi^{-1}\left(\frac{\eta}{K_{11}u^{2d}}\right)p(du), \tag{10.3.6}$$

where K_{11} is a constant and Ψ^{-1} denotes the inverse function.

As the solution X^ϵ is a function of the Brownian sheet $\{W(t,r):t\in[0,T],r\in\mathcal{O}\}$, we may assume that

$$\Omega=C(\mathcal{O}\times[0,T]),\ \mathcal{F}=\mathcal{B}(C(\mathcal{O}\times[0,T])),$$

P is the probability measure induced by $\{W(t,r):t\in[0,T],r\in\mathcal{O}\}$ and $\mathcal{F}_t$ is the sub-σ-field of $\mathcal{F}$ generated by $\{w(s,r):s\le t\ r\in\mathcal{O}\}$. Let $\mathcal{H}\subset\Omega$ be the space of all $h\in\Omega$ with the following property: There exists $\hat{h}\in L^2([0,T]\times\mathcal{O})$ such that

$$h(t,r)=\int_0^t\int_{\mathcal{O}_r}\hat{h}(s,q)dsdq$$

where $\mathcal{O}_r=\{q\in\mathcal{O}:q_j\le r_j\ j=1,\cdots,d\}$. For $e_1,e_2\in\mathcal{H}$, let

$$<e_1,e_2>_{\mathcal{H}}=\int_{\mathcal{O}}\int_0^T\hat{e}_1(r,t)\hat{e}_2(r,t)drdt.$$

Then $<\cdot,\cdot>_{\mathcal{H}}$ is an inner product on $\mathcal{H}$ under which $\mathcal{H}$ becomes a separable Hilbert space.

Let $0<\alpha<\alpha_2$ and $\mathcal{X}=\mathcal{Y}=C([0,T],\mathbf{B}_\alpha)$. Define

$$\begin{aligned}(A(x))(t,r) &= \int_{\mathcal{O}}G(t,0,r,q)\xi(q)dq \\ &\quad+\int_0^t\int_{\mathcal{O}}G(t,s,r,q)R(s,q,x(s,q))dqds,\quad\forall x\in\mathcal{X}.\end{aligned} \tag{10.3.7}$$

Let $\mathcal{A}_{\mathcal{X}}$ be the class of all $\mathbf{B}_\alpha$-valued adapted continuous processes. For $X\in\mathcal{A}_{\mathcal{X}}$, define

$$(B(X))(t,r)=\int_0^t\int_{\mathcal{O}}G(t,s,r,q)F(s,q,X(s,q))W(dsdq). \tag{10.3.8}$$

The next lemma is useful for the verification of (10.1.4) and (10.1.5).

Lemma 10.3.3 *Let $f(s,q,\omega)$ be an adapted random field such that*

$$\|f\|_\infty \equiv \sup\{|f(s,q,\omega)| : s \in [0,T],\ q \in \mathcal{O} \text{ and } \omega \in \Omega\} < \infty.$$

Let

$$\psi(t,r) = \int_0^t \int_{\mathcal{O}} G(t,s,r,q) f(s,q) W(dsdq).$$

Then for any $0 < \alpha < \alpha_2$, we have $[\psi]_\alpha < \infty$ a.s. and $\forall L > 0\ \exists \delta > 0$ s.t.

$$\limsup_{\epsilon \to 0} \epsilon \log P\left([\psi]_\alpha > \frac{1}{\sqrt{\epsilon\delta}}\right) \le -L$$

where

$$[\psi]_\alpha = \sup\left\{\frac{|\psi(t,r) - \psi(s,q)|}{\rho((t,r),(s,q))^\alpha} : \rho((t,r),(s,q)) \le \frac{a}{2}\right\}.$$

Proof: For any $0 \le t_1 < t_2 \le T$, $r_1, r_2 \in \mathcal{O}$, let

$$M_t = \int_0^t \int_{\mathcal{O}} \frac{G(t_1,s,r_1,q) - G(t_2,s,r_2,q)}{\rho((t_1,r_1),(t_2,r_2))^\alpha} f(s,q,\omega) W(dsdq).$$

Then

$$M_T = \frac{\psi(t_1,r_1) - \psi(t_2,r_2)}{\rho((t_1,r_1),(t_2,r_2))^\alpha}$$

and $\{M_t\}_{t \in [0,T]}$ is a square integrable martingale with quadratic variation process

$$< M >_t = \int_0^t \int_{\mathcal{O}} \frac{|G(t_1,s,r_1,q) - G(t_2,s,r_2,q)|^2}{\rho((t_1,r_1),(t_2,r_2))^{2\alpha}} f(s,q,\omega)^2 dsdq \le \|f\|_\infty^2 K.$$

It then follows from the same arguments as those leading to (10.2.4) that there exist two constants K_{12}, K_{13} such that

$$E \exp\left(K_{12} M_T^2\right) \le K_{13}.$$

Let

$$\eta = \int_0^t \int_{\mathcal{O}} \int_0^t \int_{\mathcal{O}} \exp\left(K_{12} \frac{|\psi(t_1,r_1) - \psi(t_2,r_2)|^2}{\rho((t_1,r_1),(t_2,r_2))^{2\alpha}}\right) dt_1 dr_1 dt_2 dr_2.$$

Then

$$E\eta \le K_{13} T^2 |\mathcal{O}|^2 < \infty.$$

It follows from Corollary 10.3.1 that, for any $(t_1,r_1),(t_2,r_2) \in [0,T] \times \mathcal{O}$ with $\rho((t_1,r_1),(t_2,r_2)) \le a/2$, we have

$$|\psi(t_1,r_1) - \psi(t_2,r_2)| \le 8 \int_0^{2\rho} \Psi^{-1}\left(\frac{\eta}{K_{11} u^{2d}}\right) p(du).$$

By similar arguments as in (10.2.5), we have

$$[\psi]_\alpha \leq K_{14}\left(\sqrt{|\log \eta|}+1\right),$$

where K_{14} is a constant. The lemma then follows from the same calculations as in (10.2.6). ∎

It is clear that the assumption (A1) holds for $\mathcal{A}_{\mathcal{X}}$. Now we verify the assumptions (A2)-(A5) under the present setup.

Lemma 10.3.4 *The assumption (A2) holds for A.*

Proof: For $x \in \mathcal{X}$, we denote the two terms on the right hand side of (10.3.7) by A_1 and $A_2(x)$ respectively. By (RD4), $A_1 \in \mathcal{X}$. On the other hand, by (RD2) and (RD3), we have

$$\begin{aligned}
&|A(x)(t_1,r_1) - A(x)(t_2,r_2)| \\
\leq\ & \sqrt{\int_0^t \int_{\mathcal{O}} |G(t_1,s,r_1,q) - G(t_2,s,r_2,q)|^2 ds dq} \\
& T|\mathcal{O}|K(R,F,T)(1+\|x\|_{\mathcal{X}}) \\
\leq\ & \sqrt{K(T)}\rho((t_1,r_1),(t_2,r_2))^\alpha(\text{diameter}([0,T]\times\mathcal{O}))^{\alpha_2-\alpha} \\
& T|\mathcal{O}|K(R,F,T)(1+\|x\|_{\mathcal{X}}).
\end{aligned}$$

Then $[A(x)]_\alpha < \infty$ and hence A is a map from $\mathcal{X}$ to $\mathcal{X}$. Similarly, it can be shown that

$$\begin{aligned}
&\|A(x) - A(y)\|_{\mathcal{X}} \\
\leq\ & \sqrt{K(T)}(\text{diameter}([0,T]\times\mathcal{O}))^{\alpha_2-\alpha} T|\mathcal{O}|K(R,F,T)\|x-y\|_{\mathcal{X}}.
\end{aligned}$$

Therefore, (A2) holds for A. ∎

Lemma 10.3.5 *The assumption (A3) holds for B.*

Proof: For $X \in \mathcal{A}_{\mathcal{X}}$, let $f(s,q,\omega) = F(s,q,X(s,q,\omega))$. Then, by (RD5), $\|f\|_\infty \leq K(R,F,T)$. It follows from Lemma 10.3.3 that $[B(X)]_\alpha < \infty$ a.s., i.e. $B(X) \in \mathcal{X}$ a.s. By the definition of the stochastic integral, we see that $B(X)$ is adapted and hence B is a map from $\mathcal{A}_{\mathcal{X}}$ to $\mathcal{A}_{\mathcal{X}}$. Let

$$\hat{B}(x,h)(t,r) = \int_0^t \int_{\mathcal{O}} G(t,s,r,q)F(s,q,x(s,q))\hat{h}(s,q)dsdq \quad \forall x \in \mathcal{X},\ h \in \mathcal{H}.$$

It is easy to see that for each $x \in \mathcal{X}$, $\hat{B}(x,\cdot) : \mathcal{H} \to \mathcal{X}$ is linear and its lifting $\tilde{B}(x,\cdot) : \Omega \to \mathcal{X}$ is given by

$$\tilde{B}(x,\cdot)(t,r) = \int_0^t \int_{\mathcal{O}} G(t,s,r,q) F(s,q,x(s,q)) W(dsdq) \tag{10.3.9}$$

which is an element of $\mathcal{A}_{\mathcal{X}}$. This verifies i) of (A3).

iii) and iv) of (A3) directly follow from (10.3.9) and the linearity of the stochastic integral. Note that

$$\begin{aligned}
& \left|(\hat{B}(x_1,h) - \hat{B}(x_2,h)(t,r)\right| \\
\le\ & \int_0^t \int_{\mathcal{O}} |G(t,s,r,q)||F(s,q,x_1(s,q)) - F(s,q,x_2(s,q))||\hat{h}(s,q)|dsdq \\
\le\ & K(R,F,T)\sqrt{\int_0^t \int_{\mathcal{O}} |G(t,s,r,q)|^2 |x_1(s,q) - x_2(s,q)|^2 dsdq}\,\|h\|_{\mathcal{H}} \\
\le\ & K(R,F,T)\sqrt{\int_0^t K(T)(t-s)^{-\alpha_1} ds}\,\|h\|_{\mathcal{H}}\|x_1 - x_2\|_{\mathcal{X}} \\
\le\ & K(R,F,T)\sqrt{\frac{K(T)T^{1-\alpha_1}}{1-\alpha_1}}\,\|h\|_{\mathcal{H}}\|x_1 - x_2\|_{\mathcal{X}}
\end{aligned}$$

and

$$\begin{aligned}
& \left|(\hat{B}(x_1,h) - \hat{B}(x_2,h)(t,r_1) - \hat{B}(x_1,h) - \hat{B}(x_2,h)(t,r_2)\right| \\
\le\ & \int_0^t \int_{\mathcal{O}} |G(t,s,r_1,q) - G(t,s,r_2,q)| \\
& |F(s,q,x_1(s,q)) - F(s,q,x_2(s,q))||\hat{h}(s,q)|dsdq \\
\le\ & K(R,F,T)\sqrt{\int_0^t \int_{\mathcal{O}} |G(t,s,r_1,q) - G(t,s,r_2,q)|^2 dqds}\,\|x_1 - x_2\|_{\mathcal{X}}\|h\|_{\mathcal{H}} \\
\le\ & \sqrt{K(T)}K(R,F,T)\|x_1 - x_2\|_{\mathcal{X}}\|h\|_{\mathcal{H}}|r_1 - r_2|^{\alpha_2}.
\end{aligned}$$

Hence, there exists a constant K_{15} such that

$$\|\hat{B}(x_1,h) - \hat{B}(x_2,h)\|_{\mathcal{X}} \le K_{15}\|h\|_{\mathcal{H}}\|x_1 - x_2\|_{\mathcal{X}}.$$

Hence, ii) of (A3) holds.

Finally, we verify the exponential continuity of B. Let $X, Y \in \mathcal{A}_{\mathcal{X}}$ (which may depend on ϵ) and let ψ be given by Lemma 10.3.3 with

$$f(s,q,\omega) = \delta^{-1}(F(s,q,X(s,q,\omega)) - F(s,q,Y(s,q,\omega)))1_{|X(s,q,\omega)-Y(s,q,\omega)|<\delta}.$$

Then

$$\begin{aligned}
|f(s,q,\omega) &\le \delta^{-1}K(R,F,T)|X(s,q,\omega) - Y(s,q,\omega)|1_{|X(s,q,\omega)-Y(s,q,\omega)|<\delta} \\
&\le K(R,F,T).
\end{aligned}$$

As

$$\begin{aligned} & P\left(\sqrt{\epsilon}\|B(X,\omega) - B(Y,\omega)\|_{\mathcal{X}} > \sqrt{\delta}, \|X - Y\|_{\mathcal{X}} < \delta\right) \\ \leq \;& P\left(\|\psi\|_{\mathcal{X}} > \frac{1}{\sqrt{\delta\epsilon}}\right) \end{aligned}$$

and there exists a constant K_{16} such that $\|\psi\|_{\mathcal{X}} \leq K_{16}[\psi]_\alpha$, $\forall \psi \in \mathcal{X}$ s.t. $\psi(0,\cdot) \equiv 0$, the exponential continuity of B follows from Lemma 10.3.3. ∎

Lemma 10.3.6 *The assumption (A4) holds for $\{X^\epsilon\}$.*

Proof: It is clear that the second part of (A4) holds. We only need to prove exponential tightness for $\{X^\epsilon\}$. Let $\beta \in (\alpha, \alpha_2)$. For $M > 0$, let

$$C_M = \{x \in \mathcal{X} : [x]_\beta \leq M,\ x(0,\cdot) = \xi\}.$$

Then C_M is a compact subset of $\mathcal{X}$. Let $f(s,q,\omega) = F(s,q,X^\epsilon(s,q,\omega))$ and let ψ be given by Lemma 10.3.3. Taking $t_2 = 0$, $t_1 = t$ and $r_1 = r_2 = r$ in (4.3.4), we have

$$\int_0^T \int_{\mathcal{O}} |G(t,s,r,q)|^2 ds dq \leq K(T) t^{\alpha_2}.$$

As

$$X_t^\epsilon = (A_1)_t + A_2(X^\epsilon)_t + \sqrt{\epsilon}\psi_t, \tag{10.3.10}$$

by (RD3) and (RD4), we have

$$\begin{aligned} & \|X^\epsilon\|_{t,0} \equiv \sup\{|X^\epsilon(s,r) : s \leq t,\ r \in \mathcal{O}\} \\ \leq\;& \|A_1\|_{t,0} + \sqrt{\epsilon}\|\psi\|_{t,0} + \sup_{r\in\mathcal{O}} \int_0^t \int_{\mathcal{O}} |G(t,s,r,q) R(s,q,X^\epsilon(s,q))| ds dq \\ \leq\;& \|A_1\|_{T,0} + \sqrt{\epsilon}\|\psi\|_{T,0} \\ & + K(R,F,T) T^{\alpha_2} \sqrt{K(T)|\mathcal{O}|} \sqrt{\int_0^t (1 + \|X^\epsilon\|_{s,0})^2 ds}. \end{aligned}$$

It then follows from Gronwall's inequality that

$$1 + \|X^\epsilon\|_{t,0} \leq K_{17}(1 + \|A_1\|_{T,0} + \sqrt{\epsilon}\|\psi\|_{T,0}) \tag{10.3.11}$$

where K_{17} is a constant. Further, by (10.3.10) and (10.3.11), it can be shown that there exist two constants K_{18}, K_{19} s.t.

$$[X^\epsilon]_\beta \leq K_{18} + K_{19}\sqrt{\epsilon}[\psi]_\beta.$$

Letting δ be given by Lemma 10.3.3 and taking $M > 0$ such that

$$(M - K_{18})\sqrt{\delta} = K_{19},$$

it follows from Lemma 10.3.3 that

$$\limsup_{\epsilon \to 0} \epsilon \log P(X^\epsilon \notin C_M) \leq \limsup_{\epsilon \to 0} \epsilon \log P\left([\psi]_\beta > \frac{1}{\sqrt{\epsilon\delta}}\right) \leq -L. \qquad \blacksquare$$

Lemma 10.3.7 *Z^ϵ, given by (10.1.13), tends to 0 in probability as $\epsilon \to 0$.*

Proof: Let ψ be given by the previous lemma. Then

$$\begin{aligned}
& Z^\epsilon(t,r) \\
= & \int_0^t \int_{\mathcal{O}} G(t,s,r,q) \left\{R(s,q,Z^\epsilon(s,q) + x(s,q)) - R(s,q,x(s,q))\right\} dsdq \\
& + \int_0^t \int_{\mathcal{O}} G(t,s,r,q)\{F(s,q,Z^\epsilon(s,q) + x(s,q)) \\
& -F(s,q,x(s,q))\}\hat{h}(s,q)dsdq + \sqrt{\epsilon}\psi(t,r).
\end{aligned}$$

By (RD2), (RD3) and Hölder's inequality, we have

$$\begin{aligned}
& \|Z_t^\epsilon\|_0 \equiv \sup_{r\in\mathcal{O}} |Z^\epsilon(t,r)| \\
\leq & \sqrt{\epsilon}\|\psi_t\|_0 + K(R,F,T) \sup_{r\in\mathcal{O}} \int_0^t \int_{\mathcal{O}} |G(t,s,r,q)Z^\epsilon(s,q)|dsdq \\
& + K(R,F,T) \sup_{r\in\mathcal{O}} \int_0^t \int_{\mathcal{O}} |G(t,s,r,q)Z^\epsilon(s,q)\hat{h}(s,q)|dsdq \\
\leq & \sqrt{\epsilon}\|\psi_t\|_0 + K(R,F,T)\left(\sqrt{T|\mathcal{O}|} + \|h\|_{\mathcal{H}}\right) \\
& \left(\sup_{r\in\mathcal{O}} \int_0^t \int_{\mathcal{O}} |G(t,s,r,q)Z^\epsilon(s,q)|^2 dsdq\right)^{\frac{1}{2}} \\
\leq & \sqrt{\epsilon}\|\psi\|_{T,0} + K(R,F,T)\left(\sqrt{T|\mathcal{O}|} + \|h\|_{\mathcal{H}}\right) \\
& \left(\int_0^t K(T)(t-s)^{-\alpha_1}\|Z_s^\epsilon\|_0^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, there exists a constant K_{20} such that

$$\|Z_t^\epsilon\|_0^2 \leq 2\epsilon\|\psi\|_{T,0}^2 + K_{20}\int_0^t (t-s)^{-\alpha_1}\|Z_s^\epsilon\|_0^2 ds. \qquad (10.3.12)$$

Applying (10.3.12) to $\|Z_s^\epsilon\|_0^2$ on the right hand side of (10.3.12), we have

$$\begin{aligned}
\|Z_t^\epsilon\|_0 &\leq 2\epsilon\|\psi\|_{T,0}^2 + K_{20}\int_0^t (t-s)^{-\alpha_1} 2\epsilon\|\psi\|_{T,0}^2 ds \\
&\quad + K_{20}\int_0^t (t-s)^{-\alpha_1} K_{20}\int_0^s (s-s_1)^{-\alpha_1}\|Z_{s_1}^\epsilon\|_0^2 ds_1 ds \\
&= 2\epsilon\|\psi\|_{T,0}^2\left(1 + K_{20}\frac{T^{1-\alpha_1}}{1-\alpha_1}\right) \\
&\quad + K_{20}^2\int_0^t\left(\int_{s_1}^t (t-s)^{-\alpha_1}(s-s_1)^{-\alpha_1} ds\right)\|Z_{s_1}^\epsilon\|_0^2 ds_1 \\
&= 2\epsilon\|\psi\|_{T,0}^2\left(1 + K_{20}\frac{T^{1-\alpha_1}}{1-\alpha_1}\right) \\
&\quad + K_{20}^2\int_0^t 2\left(\int_{s_1}^{\frac{s_1+t}{2}} (t-s)^{-\alpha_1}(s-s_1)^{-\alpha_1} ds\right)\|Z_{s_1}^\epsilon\|_0^2 ds_1 \\
&\leq 2\epsilon\|\psi\|_{T,0}^2\left(1 + K_{20}\frac{T^{1-\alpha_1}}{1-\alpha_1}\right) \\
&\quad + \frac{K_{20}^2 2^{2\alpha_1}}{1-\alpha_1}\int_0^t (t-s)^{1-2\alpha_1}\|Z_s^\epsilon\|_0^2 ds.
\end{aligned}$$

If $1-2\alpha_1 \geq 0$, we stop here; Otherwise, as $1-2\alpha_1 > -\alpha_1$, continuing the above estimate we will find two constants K_{21} and K_{22} such that

$$\|Z_t^\epsilon\|_0 \leq \epsilon K_{21}\|\psi\|_{T,0}^2 + K_{22}\int_0^t \|Z_s^\epsilon\|_0^2 ds.$$

It follows from Gronwall's inequality that

$$\|Z^\epsilon\|_{T,0}^2 \leq \epsilon K_{23}\|\psi\|_{T,0}^2, \tag{10.3.13}$$

where K_{23} is a constant. By (RD2), (RD3) and Hölder's inequality again, we have

$$[Z^\epsilon - \sqrt{\epsilon}\psi]_\alpha \leq \sqrt{K(T)}\left(\sqrt{T|\mathcal{O}|} + \|h\|_{\mathcal{H}}\right) K(R,F,T)\|Z^\epsilon\|_{T,0}. \tag{10.3.14}$$

Therefore, there exists a constant K_{24} such that

$$\|Z^\epsilon\|_{T,\alpha} \leq \sqrt{\epsilon}K_{24}[\psi]_\alpha$$

and our result then follows from Lemma 10.3.3. ∎

In Lemmas 10.3.4-10.3.7, we verified the assumptions (A1)-(A5) under the SPDE setup. Therefore, $\{X^\epsilon\}$ satisfies LDP. Before stating our main theorem in this section, we study the map γ defined in Definition 10.1.2.

Lemma 10.3.8 *γ is a single valued map from $\mathcal{H}$ to $\mathcal{X}$.*

Proof: First we consider $h \in \mathcal{H}$ such that $\|\hat{h}\|_{T,0} < \infty$. For $(t, r, x) \in [0, T] \times \mathcal{O} \times \mathbf{R}$, let

$$\tilde{R} = R(t, r, x) + F(t, r, x)\hat{h}(t, r) \quad \text{and} \quad \tilde{F} = 0.$$

Then

$$\begin{aligned} & |\tilde{R}(t, r, x) - \tilde{R}(t, r, y)| \\ \leq\ & |R(t, r, x) - R(t, r, y)| + |F(t, r, x) - F(t, r, y)||\hat{h}(t, r)| \\ \leq\ & K(R, F, T)\left(1 + \|\hat{h}\|_{T,0}\right)|x - y| \end{aligned}$$

and

$$\begin{aligned} |\tilde{R}(t, r, x)| & \leq |R(t, r, x)| + |F(t, r, x)||\hat{h}(t, r)| \\ & \leq K(R, F, T)\left(1 + \|\hat{h}\|_{T,0}\right)(1 + |x|). \end{aligned}$$

It follows from Theorem 4.3.2 that the following equation

$$x(t, r) = \int_{\mathcal{O}} G(t, 0, r, q)\xi(q)dq + \int_0^t \int_{\mathcal{O}} G(t, s, r, q)\tilde{R}(t, r, x(s, q))dsdq \tag{10.3.15}$$

has a unique solution, denoting it by $x = \gamma(h)$, in $\mathcal{X}$.

For general h, let $h^n \in \mathcal{H}$ be such that $\|h^n\|_{T,0} < \infty$, $\forall n \geq 1$, and $\|h^n - h\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Let $x^n = \gamma(h^n)$. Then

$$x^n(t, r) = \int_{\mathcal{O}} G(t, 0, r, q)\xi(q)dq + \int_0^t \int_{\mathcal{O}} G(t, s, r, q)\tilde{R}(t, r, x^n(s, q))dsdq. \tag{10.3.16}$$

Similar to the proof of the previous lemma, it can be shown that there exist two constants K_{25} and K_{26} s.t.

$$\|x^n\|_{\mathcal{X}} \leq K_{25}(1 + \|h^n\|_{\mathcal{H}}) \qquad \forall n \geq 1$$

and

$$\|x^n - x^m\|_{\mathcal{X}} \leq K_{26}\|h^n - h^m\|_{\mathcal{H}}) \qquad \forall n, m \geq 1. \tag{10.3.17}$$

Hence $\{x^n\}$ converges to an element x in $\mathcal{X}$. By (10.3.16), x is a solution of (10.3.15). The uniqueness for the solution of (10.3.16) directly follows from (10.3.17). ∎

Finally we summarize our results to the following main theorem of this section.

Theorem 10.3.1 *Suppose that $\mathcal{O}$ satisfies the cone condition. Then, under assumptions (RD1)-(RD5), $\{X^\epsilon\}$ satisfies the large deviation principle on $C([0,T],\mathbf{B}_\alpha)$ ($\alpha < \alpha_2$) with rate function I given by*

$$I(x) = \inf\left\{\frac{1}{2}\int_0^T\int_{\mathcal{O}} |\hat{h}(s,r)|^2 dsdr : h \in \mathcal{H} \text{ s.t. } x = \gamma(h)\right\} \tag{10.3.18}$$

where $x = \gamma(h) \in C([0,T],\mathbf{B}_\alpha)$ is given by

$$\begin{aligned} x(t,r) &= \int_{\mathcal{O}} G(t,0,r,q)\xi(q)dq \qquad (10.3.19)\\ &+ \int_0^t\int_{\mathcal{O}} G(t,s,r,q)(R(s,q,x(s,q)) + F(s,q,x(s,q))\hat{h}(s,q))dqds. \end{aligned}$$

Remark 10.3.1 *LDP considered in this section has also been studied by Peszat [44] and Chow [4]. The present section differs from their results in the following aspects:*
(i) Their methods are similar to the finite dimensional case, i.e. they resort to a sequence of approximate solutions for which the LDP is satisfied, and then show that the LDP is preserved in the limit. Our method is to approximate the probability that the solution lies in a small neighborhood by the probability that a Gaussian process, obtained by freezing the right hand side of the SPDE, lies in the same neighborhood.
(ii) The stochastic integral on the right hand side of (10.3.1) is different from the one in their papers. They consider it to be the integral of a Hilbert-Schmidt valued process with respect to a Wiener process. We regard it as the integral of a real valued random field (both the time and space variables as parameters) with respect to a Brownian sheet in space-time. The advantage of this point of view is that the Hilbert-Schmidt property is not required and hence, some of the conditions in their papers can be relaxed.

10.4 Reaction-diffusion SPDEs

Now we apply our results to a class of reaction-diffusion SPDEs. In this case, $\{L(t)\}$ is a family of second order differential operators. Let $d = 1$ and $\mathcal{O} = (0,\ell)$. Let $\{X^\epsilon\}$ be the solution of the following equation

$$\begin{aligned} dX^\epsilon(t,r) &= (L(t)X^\epsilon(t,r) + R(t,r,X^\epsilon(t,r)))drdt\\ &\quad +\sqrt{\epsilon}F(t,r,X^\epsilon(t,r))W(drdt) \qquad (10.4.1)\\ X^\epsilon(0,r) &= \xi(r). \end{aligned}$$

Theorem 10.4.1 *Suppose that $\{L(t)\}$ generates a two-parameter evolution semigroup $\{U(t,s) : 0 \le s \le t\}$ on $C([0,\ell])$ which has kernel function*

$G(t,s,r,q)$, $0 \le s < t$, $0 < r,q < \ell$, satisfying the following conditions:
(i) There exists a constant K such that $\forall 0 \le t_1, t_2 \le T$ and $0 < r_1, r_2 < \ell$, we have

$$\int_0^T \int_{\mathcal{O}} |G(t_1,s,r_1,q) - G(t_2,s,r_2,q)|^2 dqds \le K \left(|t_1 - t_2|^2 + |r_1 - r_2|^2\right)^{\frac{1}{4}} \tag{10.4.2}$$

where $G(t,s,r,q) \equiv 0$ for $s > t$.
(ii) $\forall 0 \le s < t \le T$ and $r \in (0,\ell)$, we have

$$\int_0^T \int_{\mathcal{O}} |G(t,s,r,q)|^2 dq \le K(t-s)^{-\frac{1}{2}}. \tag{10.4.3}$$

(iii) (RD4) holds.
We also assume that R and F satisfy the condition (RD3).

Then for and $\alpha < \frac{1}{4}$, $\{X^\epsilon\}$ satisfies LDP with rate function

$$I(x) = \inf \left\{ \begin{array}{ll} \frac{1}{2}\int_0^T \int_0^\ell |\hat{h}(t,r)|^2 dtdr : & \hat{h} \in L^2([0,T] \times (0,\ell)) \text{ s.t.} \\ & (10.3.19) \text{ holds with } \mathcal{O} = (0,\ell) \end{array} \right\}.$$

Remark 10.4.1 *The conditions (i)-(iii) hold for most parabolic operators $\frac{\partial}{\partial t} - L(t)$ (cf. Friedman [10]).*

Example 10.4.1 *Nonlinear stochastic cable equation*

Consider the following nonlinear stochastic cable equation with small noise:

$$\begin{aligned} \frac{\partial}{\partial t} v^\epsilon(t,x) &= \left(\frac{\partial^2}{\partial x^2} - 1\right) v^\epsilon(t,x) + f(x, v^\epsilon(t,x)) \\ &\quad + \sqrt{\epsilon}\sigma(x, v^\epsilon(t,x)) \frac{\partial^2 W}{\partial t \partial x} \\ v^\epsilon(0,x) &= \xi(x) \\ \frac{\partial v^\epsilon(t,0)}{\partial x} &= \frac{\partial v^\epsilon(t,\pi)}{\partial x} = 0. \end{aligned} \tag{10.4.4}$$

Suppose that f and σ satisfy the condition (RD3) and $\sigma(x,v) \neq 0$ for all $(x,v) \in (0,\pi) \times \mathbf{R}$. For any $\alpha < \frac{1}{4}$ and $\phi \in C([0,T], \mathbf{B}_\alpha)$, let

$$S_\xi(\phi) = \frac{1}{2}\int_0^T \int_0^{2\pi} \left| \frac{\frac{\partial}{\partial t}\phi(t,x) - (\frac{\partial^2}{\partial x^2} - 1)\phi(t,x) - f(x,\phi(t,x))}{\sigma(x,\phi(t,x))} \right|^2 dtdx \tag{10.4.5}$$

if $\phi \in W_2^{1,2}$ and $\phi(0,\cdot) = \xi$; Otherwise, $S_\xi(\phi) = \infty$. It follows from the proof of Theorem 4.4.1 that the conditions (i)-(iii) of Theorem 10.4.1 are satisfied for $L = \frac{\partial^2}{\partial x^2} - 1$.

Theorem 10.4.2 *$\{v^\epsilon\}$ satisfies the LDP on $C([0,T],\mathbf{B}_\alpha)$ ($\alpha<\frac{1}{4}$) with rate function S_ξ.*

Proof: It follows from Theorem 10.4.1 that $\{v^\epsilon\}$ satisfies the LDP on $C([0,T],\mathbf{B}_\alpha)$ with rate function I. We only need to show that $I=S_\xi$. If $S_\xi(\phi)<\infty$, then $\phi\in W_2^{1,2}$, $\phi(0,\cdot)=\xi$ and

$$\hat{h}(t,x)\equiv\frac{\frac{\partial}{\partial t}\phi(t,x)-(\frac{\partial^2}{\partial x^2}-1)\phi(t,x)-f(x,\phi(t,x))}{\sigma(x,\phi(t,x))} \tag{10.4.6}$$

is in $L^2([0,T]\times\mathcal{O})$. Note that (10.4.6) implies that

$$\begin{aligned}\phi(t,r) &= \int_{\mathcal{O}}G(t,0,r,q)\xi(q)dq \qquad (10.4.7)\\ &+\int_0^t\int_{\mathcal{O}}G(t,s,r,q)\{f(q,\phi(s,q))+\sigma(q,\phi(s,q))\hat{h}(s,q)\}dqds.\end{aligned}$$

Hence $I(\phi)<\infty$.

On the other hand, if $I(\phi)<\infty$, then, for any $\delta>0$, there exists $\hat{h}\in L^2([0,T]\times\mathcal{O})$ such that (10.4.7) holds and

$$\frac{1}{2}\int_0^T\int_{\mathcal{O}}|\hat{h}(s,r)|^2dsdr\in[I(\phi),I(\phi)+\delta].$$

It is easy to see that $\hat{h}$ is uniquely determined and coincides with the right hand side of (10.4.6) (see Walsh [57] for details) and hence,

$$\frac{1}{2}\int_0^T\int_{\mathcal{O}}|\hat{h}(s,r)|^2dsdr=I(\phi).$$

Therefore $S_\xi(\phi)<\infty$ and $S_\xi(\phi)=I(\phi)$. ∎

Remark 10.4.2 *Recently, Sowers [50] has derived the LDP for SPDE (10.4.4) with periodic boundary condition. The following conditions were imposed: there exist constants F, f^-, m, M and σ^- such that, for any $x\in[0,2\pi]$ and $y,z\in\mathbf{R}$, we have*

$$|f(x,y)|\le F(1+|y|),\ |f(x,y)-f(x,z)|\le f^-|y-z|, \tag{10.4.8}$$

and

$$0<m\le\sigma(x,y)\le M,\ |\sigma(x,y)-\sigma(x,z)|\le\sigma^-|y-z|, \tag{10.4.9}$$

It is clear that the conditions (10.4.8) and (10.4.9) are stronger than the condition (RD3). Further, by similar arguments as in the proof of Theorem 4.4.1 that the conditions (i)-(iii) of Theorem 10.4.1 are satisfied for the periodic boundary condition. Therefore, Sowers' case can be derived as a special case of Theorem 10.3.1.