Singular Stochastic Control in Optimal Investment and Hedging in the Presence of Transaction Costs

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Abstract

In an idealized model without transaction costs, an investor would optimally maintain a proportion of wealth in stock or hold a number of shares of stock to hedge a contingent claim by trading *continuously*. Such continuous strategies are no longer admissible once proportional transaction costs are introduced. The investor must then determine when the stock position is sufficiently "out of line" to make trading worthwhile. Thus, the problems of optimal investment and hedging become, in the presence of transaction costs, singular stochastic control problems, characterized by instantaneous trading at the boundaries of a "no transactions" region whenever the stock position falls on these boundaries. In this paper, we review various formulations of the optimal investment and hedging problems and their solutions, with particular emphasis on the derivation and analysis of Hamilton-Jacobi-Bellman (HJB) equations using the dynamic programming principle. A particular numerical scheme, based on weak convergence of probability measures, is provided for the computation of optimal strategies in the problems we consider.

1 Introduction

The problems of optimal investment and consumption and of option pricing and hedging were initially studied in an idealized setting whereby an investor incurs no transaction costs from trading in a market consisting of a risk-free asset ("bond") with constant rate of return and a risky asset ("stock") whose price is a geometric Brownian motion with constant rate of return and volatility. For example, Merton (1969, 1971) showed that, for an investor acting as a price-taker and seeking to maximize expected utility of consumption, the optimal strategy is to invest a constant proportion (the "Merton proportion") of wealth in the stock and to consume at a rate proportional to wealth. In the related problem of option pricing and hedging, arbitrage considerations of Black and Scholes (1973) demonstrated that, by setting up a portfolio of stock and option that is risk-free, the value of an option must equal the amount of initial capital required for this hedging.

However, both the Merton strategy and the Black-Scholes hedging portfolio require continuous trading and result in an infinite turnover of stock in any finite

time interval. In the presence of transaction costs proportional to the amount of trading, such continuous strategies are prohibitively expensive. Thus, there must be some "no transactions" region inside which the portfolio is insufficiently "out of line" to make trading worthwhile. In such a case, the problems of optimal investment and consumption and of option pricing and hedging involve singular stochastic control. As we shall see, Bellman's *principle of dynamic programming* can often be used to derive (at least formally) the nonlinear partial differential equation (PDE) satisfied by the value function of interest. The derived PDE will then suggest methods (analytic or numerical) to solve for the optimal policies. One such numerical scheme, based on *weak convergence* of probability measures, will be particularly useful to the problems described in this paper. It turns out that some of the resulting free boundary problems can be reduced to optimal stopping problems in ways suggested by Karatzas and Shreve (1984, 1985), thereby simplifying the solutions of the original optimal control problems.

We will focus on the two-asset (one bond and one stock) setting which many authors consider. Besides simplifying the exposition, such a setting can be justified by the so-called "mutual fund theorems" whenever lognormality of prices is assumed; see, for example Merton (1971) in the absence of transaction costs and Magill (1976) in the presence of transaction costs. Specifically, the market consists of two investment instruments: a bond paying a fixed risk-free rate $r > 0$ and a stock whose price is a geometric Brownian motion with mean rate of return $\alpha > 0$ and volatility $\sigma > 0$. Thus, the prices of the bond and stock at time $t \geq 0$ are given respectively by

$$dB_t = rB_t \, dt \qquad \text{and} \qquad dS_t = S_t(\alpha \, dt + \sigma \, dW_t), \qquad (1.1)$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $W_0 = 0$ a.s. The investor's position will be denoted by (X_t, Y_t) (in Section 2) or (X_t, y_t) (in Section 3), where

$$
\begin{aligned}
X_t &= \text{dollar value of investment in bond,} \\
Y_t &= \text{dollar value of investment in stock,} \qquad (1.2)\\
y_t &= \text{number of shares held in stock.}
\end{aligned}
$$

In particular, we note the relation $Y_t = y_t S_t$.

The rest of the paper is organized as follows. In Section 2, we consider optimal investment and consumption, beginning with a treatment of the "Merton problem" (no transaction costs) over a finite horizon, and then proceeding to the transaction costs problem considered by Magill and Constantinides (1976) and, more recently, by ourselves. We also consider the infinite-horizon case, drawing on results from Davis and Norman (1990) and Shreve and Soner (1994), and review the work of Taksar, Klass and Assaf (1988) on the related problem of maximizing the long-run growth rate of the investor's asset value. The problem of option pricing and hedging in the presence of transaction costs is considered in Section 3. Some concluding remarks are given in Section 4.

2 Optimal Consumption and Investment with Transaction Costs

The investment and consumption decisions of an investor comprise three non-negative $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes C, L, and M, such that C is integrable on each finite time interval, and L and M are nondecreasing and right-continuous with left-hand limits. Specifically, the investor consumes at rate C_t from the bond and L_t (resp. M_t) represents the cumulative dollar value of stock bought (resp. sold) within the time interval $[0,t]$, $0 \leq t \leq T$. In the presence of proportional transaction costs, the investor pays fractions $0 \leq \lambda < 1$ and $0 \leq \mu < 1$ of the dollar value transacted on purchase and sale of stock, respectively. Thus, the investor's position (X_t, Y_t) satisfies

$$dX_t = (rX_t - C_t)\,dt - (1+\lambda)\,dL_t + (1-\mu)\,dM_t, \qquad (2.1a)$$

$$dY_t = \alpha Y_t\,dt + \sigma Y_t\,dW_t + dL_t - dM_t. \qquad (2.1b)$$

The factor $1+\lambda$ (resp. $1-\mu$) in (2.1a) reflects the fact that a transaction fee in the amount of $\lambda\,dL$ (resp. $\mu\,dM$) needs to be paid from the bond when purchasing dL (resp. selling dM) dollar value of stock. We define the investor's wealth (or net worth) as

$$Z_t = X_t + (1-\mu)Y_t \quad \text{if } Y_t \geq 0; \qquad Z_t = X_t + (1+\lambda)Y_t \quad \text{if } Y_t < 0.$$

By requiring that the investor remains solvent (i.e., has nonnegative net worth) at all times, the investor's position is constrained to lie in the solvency region $\mathcal{D}$ which is a closed convex set bounded by the line segments

$$\partial_\lambda\mathcal{D} = \{(x,y) : x > 0,\ y < 0 \text{ and } x + (1+\lambda)y = 0\},$$

$$\partial_\mu\mathcal{D} = \{(x,y) : x \leq 0,\ y \geq 0 \text{ and } x + (1-\mu)y = 0\}.$$

We denote by $\mathcal{A}(t,x,y)$ the class of admissible policies, for the position $(X_t, Y_t) = (x,y)$, satisfying $(X_s, Y_s) \in \mathcal{D}$ for $t \leq s \leq T$, or equivalently, $Z_s \geq 0$ for $t \leq s \leq T$. At time t, the investor's objective is to maximize over $\mathcal{A}(t,x,y)$ the expected utility

$$J(t,x,y) = \mathbb{E}\left[\int_t^T e^{-\beta(s-t)}U_1(C_s)\,ds + e^{-\beta(T-t)}U_2(Z_T)\,\bigg|\,X_t = x, Y_t = y\right],$$

where $\beta > 0$ is a discount factor and U_1 and U_2 are concave utility functions of consumption and terminal wealth. We assume that U_1 is differentiable and that the inverse function $(U_1')^{-1}$ exists. Often U_1 and U_2 are chosen from the so-called HARA (hyperbolic absolute risk aversion) class:

$$U(c) = c^\gamma/\gamma \quad \text{if } \gamma < 1,\ \gamma \neq 0; \qquad U(c) = \log c \quad \text{if } \gamma = 0, \qquad (2.2)$$

which has *constant* relative risk aversion $-cU''(c)/U'(c) = 1 - \gamma$. We define the value function by

$$V(t,x,y) = \sup_{(C,L,M)\in\mathcal{A}(t,x,y)} J(t,x,y). \qquad (2.3)$$

2.1   The Merton Problem (No Transaction Costs)

Before presenting the solution to the general transaction costs problem (2.3), we consider the case $\lambda = \mu = 0$ (no transaction costs) analyzed by Merton (1969). In this case, by adding (2.1a) and (2.1b), the total wealth $Z_t = X_t + Y_t$ can be represented as

$$dZ_t = \{rZ_t + (\alpha - r)\theta_t Z_t - C_t\}\, dt + \sigma\theta_t Z_t\, dW_t, \qquad (2.4)$$

where $\theta_t = Y_t/(X_t + Y_t)$ is the proportion of the investment held in stock. Using the reparameterization $z = x + y$, the value function can be expressed as

$$V(t, z) = \sup_{(C,L,M)\in\mathcal{A}(t,z)} \mathbb{E}\left[\int_t^T e^{-\beta(s-t)}U_1(C_s)\, ds + e^{-\beta(T-t)}U_2(Z_T) \,\bigg|\, Z_t = z\right],$$

where $\mathcal{A}(t, z)$ denotes all admissible policies (C, θ) for which $Z_s \geq 0$ for all $t \leq s \leq T$. The Bellman equation for the value function is

$$\max_{C,\theta}\{(\partial/\partial t + \mathcal{L})V(t, z) + U(C) - \beta V(t, z)\} = 0, \qquad (2.5)$$

subject to the terminal condition $V(T, z) = U_2(z)$, where $\mathcal{L}$ is the infinitesimal generator of (2.4):

$$\mathcal{L} = \frac{\sigma^2\theta^2 z^2}{2}\frac{\partial^2}{\partial z^2} + [rz + (\alpha - r)\theta z - C]\frac{\partial}{\partial z}.$$

Formal maximization with respect to C and θ yields $C = (U_1')^{-1}(V_z)$ and $\theta = -(V_z/V_{zz})(\alpha - r)/\sigma^2 z$ (in which subscript denotes partial derivative, e.g., $V_z = \partial V/\partial z$). Substituting for C and θ in (2.5) leads to the PDE

$$\frac{\partial V}{\partial t} - \frac{(\alpha - r)^2}{2\sigma^2}\frac{(\partial V/\partial z)^2}{\partial^2 V/\partial z^2} + (rz - C^*)\frac{\partial V}{\partial z} + U_1(C^*) - \beta V = 0, \qquad (2.6)$$

where $C^* = C^*(t, z) = (U_1')^{-1}(V_z(t, z))$. Let

$$p = \frac{\alpha - r}{(1 - \gamma)\sigma^2}, \qquad c = \frac{1}{1 - \gamma}\left[\beta - \gamma r - \frac{\gamma(\alpha - r)^2}{2(1 - \gamma)\sigma^2}\right],$$

$$C_i(t) = c/\{1 - \phi_i e^{c(t-T)}\} \quad (i = 1, 2), \qquad \phi_1 = 1, \quad \phi_2 = 1 - c. \qquad (2.7)$$

If U_1 takes the form (2.2), then $C^* = (V_z)^{1/(\gamma - 1)}$ and solving the PDE yields the optimal policy: $\theta_t^* \equiv p$ and $C_t^* = C_1(t)Z_t$ when $U_2 \equiv 0$, or $C_t^* = C_2(t)Z_t$ when U_2 takes the form (2.2). Note that $c = \beta$ when $\gamma = 0$. Thus, in the Merton problem, the optimal strategy is to devote a constant proportion (the Merton proportion p) of the investment to the stock and to consume at a rate proportional to wealth. Furthermore, for $i = 1$ or 2 (corresponding to $U_2 \equiv 0$ or to (2.2)), the value function is

$$V(t, z) = \frac{z^\gamma}{\gamma}[C_i(t)]^{\gamma - 1} \quad \text{if } \gamma < 1,\ \gamma \neq 0;$$

$$V(t, z) = a_i(t) + \frac{1}{C_i(t)}\log[C_i(t)z] \quad \text{if } \gamma = 0,$$

where $a_i(t) = \beta^{-2}[r - \beta + (\alpha - r)^2/2\sigma^2]\{1 - e^{\beta(t-T)}[1 + \phi_i\beta(t - T)]\}$. Since $\beta > 0$, $V(t, z) < \infty$ when $\gamma \leq 0$. A necessary and sufficient condition for a finite value function when $0 < \gamma < 1$ is $\beta > \gamma r + \gamma(\alpha - r)^2/\{2(1 - \gamma)\sigma^2\}$. Corresponding results for general utility functions U_1 and U_2 have been given by Cox and Huang (1989), who use a martingale technique instead of the usual dynamic programming principle. By working under the equivalent martingale measure so that differences in mean rates of return among assets are removed, the martingale approach allows candidate optimal policies to be constructed by solving a *linear* (instead of nonlinear) PDE; see also Karatzas, Lehoczky and Shreve (1987).

2.2   Transaction Costs and Singular Stochastic Control

In the presence of transaction costs, analytic solutions are generally unavailable, even for HARA utility functions. One approach to the problem is to apply a discrete time dynamic programming algorithm on a suitable approximating Markov chain for the controlled process. This approach is based on weak convergence of probability measures, which will ensure that the discrete-time value function converges to its continuous-time counterpart as the discretization scheme becomes infinitely fine. Note that the optimal investment and consumption problem involves both singular control (portfolio adjustments) and continuous control (consumption decisions).

We begin with an analysis of the Bellman equation, which will subsequently suggest an appropriate Markov chain approximation for our problem. We can obtain key insights into the nature of the optimal policies by temporarily restricting L and M to be absolutely continuous with derivatives bounded by κ, i.e.,

$$L_t = \int_0^t \ell_s \, ds \quad \text{and} \quad M_t = \int_0^t m_s \, ds, \qquad 0 \leq \ell_s, m_s \leq \kappa < \infty. \tag{2.8}$$

Proceeding as before, the Bellman equation for the value function (2.3) is

$$\max_{C,\ell,m} \{(\partial/\partial t + \mathcal{L})V(t, x, y) + U_1(C) - \beta V(t, x, y)\} = 0, \tag{2.9}$$

subject to $V(T, x, y) = U_2(x + (1 - \mu)y)$ if $y \geq 0$; $V(T, x, y) = U_2(x + (1 + \lambda)y)$ if $y < 0$, where $\mathcal{L}$ is the infinitesimal generator of (2.1a)–(2.1b):

$$\mathcal{L} = \frac{\sigma^2 y^2}{2} \frac{\partial^2}{\partial y^2} + (rx - C)\frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + \left[\frac{\partial}{\partial y} - (1 + \lambda)\frac{\partial}{\partial x}\right]\ell + \left[(1 - \mu)\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]m. \tag{2.10}$$

The maximum in (2.9) is attained by $C = (U_1')^{-1}(V_x)$, $\ell = \kappa \mathbb{I}_{\{V_y \geq (1+\lambda)V_x\}}$, and $m = \kappa \mathbb{I}_{\{V_y \leq (1-\mu)V_x\}}$. Thus, it can be conjectured that buying or selling either takes place at maximum rate or not at all, and the solvency region $\mathcal{D}$ can be partitioned into three regions corresponding to "buy stock" ($\mathcal{B}$), "sell stock" ($\mathcal{S}$), and "no transactions" ($\mathcal{N}$). Instantaneous transition from $\mathcal{B}$ to the buy boundary $\partial\mathcal{B}$ or from $\mathcal{S}$ to the sell boundary $\partial\mathcal{S}$ takes place by letting $\kappa \to \infty$ and moving the portfolio parallel to $\partial_\lambda\mathcal{D}$ or $\partial_\mu\mathcal{D}$ (i.e., in the direction of

$(-1, (1+\lambda)^{-1})^T$ or $(1, -(1-\mu)^{-1})^T$, where T denotes transpose). This suggests that $V(t, x, y) = V(t, x + (1 - \mu)\delta y, y - \delta y)$ for $(t, x, y) \in \mathcal{S}$ and $V(t, x, y) = V(t, x - (1 + \lambda)\delta y, y + \delta y)$ for $(t, x, y) \in \mathcal{B}$. In the limit as $\delta y \to 0$, we have

$$V_y(t, x, y) = (1 - \mu)V_x(t, x, y), \qquad (t, x, y) \in \mathcal{S}, \qquad (2.11a)$$

$$V_y(t, x, y) = (1 + \lambda)V_x(t, x, y), \qquad (t, x, y) \in \mathcal{B}. \qquad (2.11b)$$

In $\mathcal{N}$ the value function satisfies (2.9) with $\ell = m = 0$, leading to the PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 y^2}{2} \frac{\partial^2 V}{\partial y^2} + (rx - C^*)\frac{\partial V}{\partial x} + \alpha y \frac{\partial V}{\partial y} + U_1(C^*) - \beta V = 0, \qquad (t, x, y) \in \mathcal{N},$$
$$(2.11c)$$

where $C^* = C^*(t, x, y) = (U_1')^{-1}(V_x(t, x, y))$ as in (2.6).

To solve (2.11a)–(2.11c), the first step is to find an approximating Markov chain which is *locally consistent* with the controlled diffusion (2.1a)–(2.1b). Following Kushner and Dupuis (1992), we will use the "finite difference" method to obtain the transition probabilities of the approximating Markov chain. Specifically, for a candidate consumption decision (i.e., continuous control) C, we make the following (standard) approximations to the derivatives in equation (2.11c):

$$V_t(t, x, y) \to [V(t + \delta, x, y) - V(t, x, y)]/\delta,$$

$$V_x(t, x, y) \to \begin{cases} [V(t + \delta, x + \epsilon, y) - V(t + \delta, x, y)]/\epsilon & \text{if } rx - C \geq 0, \\ [V(t + \delta, x, y) - V(t + \delta, x - \epsilon, y)]/\epsilon & \text{if } rx - C < 0, \end{cases}$$

$$V_y(t, x, y) \to \begin{cases} [V(t + \delta, x, y + \epsilon) - V(t + \delta, x, y)]/\epsilon & \text{if } y \geq 0, \\ [V(t + \delta, x, y) - V(t + \delta, x, y - \epsilon)]/\epsilon & \text{if } y < 0, \end{cases}$$

$$V_{yy}(t, x, y) \to [V(t + \delta, x, y + \epsilon) + V(t + \delta, x, y - \epsilon) - 2V(t + \delta, x, y)]/\epsilon^2.$$
$$(2.12)$$

Collecting terms and noting that C^* in (2.11c) is the *optimal* control, we obtain the following backward induction equation for the "consumption step":

$$V^0(t, x, y) = e^{-\beta \delta} \max_C \left\{ \sum_{\tilde{x}, \tilde{y}} p(\tilde{x}, \tilde{y} \,|\, x, y) V(t + \delta, \tilde{x}, \tilde{y}) + \delta U_1(C) \right\}, \qquad (2.13)$$

where only the following five transition probabilities are nonzero:

$$p(x \pm \epsilon, y \,|\, x, y) = (rx - C)^{\pm} \delta/\epsilon,$$

$$p(x, y \pm \epsilon \,|\, x, y) = \alpha y^{\pm} \delta/\epsilon + (\sigma^2 y^2/2)\delta/\epsilon^2,$$

$$p(x, y \,|\, x, y) = 1 - (|rx - C| + \alpha|y|)\delta/\epsilon - (\sigma^2 y^2)\delta/\epsilon^2.$$

Equation (2.13) is to be evaluated for $t \in \mathbb{T} = \{0, \delta, 2\delta, \dots, N\delta\}$ with $\delta = T/N$ and (x, y) belonging to some grid $\mathbb{X} \times \mathbb{Y}$ made up of multiples of $\pm\epsilon$. Given δ, the choice of ϵ must ensure that $p(x, y \,|\, x, y) \geq 0$. Let $A_1 = \max_{x \in \mathbb{X}, C} |rx - C|$ and $A_2 = \max_{y \in \mathbb{Y}} |y|$. Then one could set

$$\epsilon = (\delta/2)(A_1 + \alpha A_2) \left[1 + \sqrt{1 + 4\sigma^2/(A_1/A_2 + \alpha)^2 \delta} \right].$$

A similar treatment of equations (2.11a)–(2.11b) yields respective relations for the "sell step" and the "buy step" (singular controls):

$$V^s(t, x, y) = \mu V(t, x, y - \epsilon) + (1 - \mu)V(t, x + \epsilon, y - \epsilon),$$
$$V^b(t, x, y) = (1 + \lambda)^{-1}[\lambda V(t, x - \epsilon, y) + V(t, x - \epsilon, y + \epsilon)].$$

Since only one of buy, sell or no transactions can happen at each step, the dynamic programming equation for the (discrete-time) finite horizon value function is therefore

$$V(t, x, y) = \max\{V^0(t, x, y), V^s(t, x, y), V^b(t, x, y)\},$$

with terminal condition $V(T, x, y) = U_2(x + (1 - \mu)y)$ if $y \geq 0$; $V(T, x, y) = U_2(x + (1 + \lambda)y)$ if $y < 0$. For a sufficiently fine grid $\mathbb{T} \times \mathbb{X} \times \mathbb{Y}$, this gives good approximations to the value function (2.3) and the transaction regions: $(t, x, y) \in \mathcal{S}$ if $V(t, x, y) = V^s(t, x, y)$ and $(t, x, y) \in \mathcal{B}$ if $V(t, x, y) = V^b(t, x, y)$.

When U_1 and U_2 take the form (2.2), we find that V is concave and homothetic in (x, y): for $\eta > 0$,

$$V(t, \eta x, \eta y) = \eta^\gamma V(t, x, y) \quad \text{if } \gamma < 1, \gamma \neq 0;$$
$$V(t, \eta x, \eta y) = \{\beta^{-1}[1 - e^{\beta(t-T)}] + e^{\beta(t-T)}\} \log \eta + V(t, x, y) \quad \text{if } \gamma = 0.$$

Homotheticity of V suggests that if equations (2.11a) and (2.11b) are satisfied for some $(t, x, y) \in \partial\mathcal{S}$ and $\partial\mathcal{B}$, respectively, then the same is true for any $(t, \eta x, \eta y)$ with $\eta > 0$. Thus, it can further be conjectured that the boundaries between the transaction and no transactions regions are straight lines (rays) through the origin for each $t \in [0, T]$. Moreover, since $C^* = (V_x)^{1/(\gamma-1)}$, equation (2.11c) becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 y^2}{2}\frac{\partial^2 V}{\partial y^2} + rx\frac{\partial V}{\partial x} + \alpha y\frac{\partial V}{\partial y} + \frac{1 - \gamma}{\gamma}\left(\frac{\partial V}{\partial x}\right)^{\gamma/(\gamma-1)} - \beta V = 0, \quad (t, x, y) \in \mathcal{N}, \tag{2.14}$$

with the fifth term on the l.h.s. of (2.14) replaced by $-(1 + \log V_x)$ when $\gamma = 0$.

We can further exploit homotheticity of V to reduce the nonlinear PDE (2.14) to an equation in one *state* variable. Indeed, let $\psi(x) = V(t, x, 1)$ so that $V(t, x, y) = y^\gamma \psi(t, x/y)$. Then, for some functions $A_*(t)$, $A^*(t)$, and $-(1 - \mu) < x_*(t) < x^*(t) < \infty$, equations (2.11a)–(2.11b) and (2.14) are equivalent to the following when $\gamma < 1$ and $\gamma \neq 0$:

$$\psi(t, x) = \gamma^{-1}A_*(t)(x + 1 - \mu)^\gamma, \qquad x \leq x_*(t), \tag{2.15a}$$
$$\psi(t, x) = \gamma^{-1}A^*(t)(x + 1 + \lambda)^\gamma, \qquad x \geq x^*(t), \tag{2.15b}$$
$$\frac{\partial\psi}{\partial t} + b_3 x^2\frac{\partial^2\psi}{\partial x^2} + b_2 x\frac{\partial\psi}{\partial x} + \frac{1 - \gamma}{\gamma}\left(\frac{\partial\psi}{\partial x}\right)^{\gamma/(\gamma-1)} + b_1\psi = 0, \quad x \in [x_*(t), x^*(t)], \tag{2.15c}$$

where

$$b_1 = \sigma^2\gamma(\gamma - 1)/2 + \alpha\gamma - \beta, \qquad b_2 = \sigma^2(1 - \gamma) + r - \alpha, \qquad b_3 = \sigma^2/2. \tag{2.16}$$

A similar set of equations can also be obtained for $\gamma = 0$. A simplified version of the numerical scheme described earlier in this section can be implemented to solve for $\psi(t,x)$ as well as the boundaries $x_*(t)$ and $x^*(t)$. For details and numerical examples, see Lai and Lim (2002a).

Hence, for HARA utility functions, the optimal policy for the transaction costs problem (2.3) is given by the triple (C^*, L^*, M^*), where

$$C_t^* = Y_t[\psi_x(t, X_t/Y_t)]^{1/(\gamma-1)}$$

and

$$L_t^* = \int_0^t \mathbb{I}_{\{X_s/Y_s=x^*(s)\}}\, dL_s^*, \qquad M_t^* = \int_0^t \mathbb{I}_{\{X_s/Y_s=x_*(s)\}}\, dM_s^*, \qquad t \in [0, T].$$

The introduction of transaction costs into Merton's problem in Section 2.1 has the following consequence. The investor should optimally maintain the proportion of investment in stock between $\theta_*(t) := [1 + x^*(t)]^{-1} > 0$ and $\theta^*(t) := [1 + x_*(t)]^{-1} < \mu^{-1}$, i.e., $\theta_*(t) \leq \theta_t^* \leq \theta^*(t)$ in our earlier notation. Thus, the no transactions region $\mathcal{N}$ is a "wedge" in the solvency region $\mathcal{D}$. Such an observation can be traced back to Magill and Constantinides (1976), who found that "the investor trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transaction costs."

The foregoing analysis and solution of problem (2.3) can be extended to the case of more than one stock. While a straightforward application of the principle of dynamic programming would suffice to derive the Bellman equation, computational aspects of the problem become much more involved. As pointed out by Magill and Constantinides (1976), m stocks imply 3^m possible partitions of the solvency region so even for moderately large m (e.g., $3^5 \approx 250$, $3^{10} \approx 60000$) it is unclear how to systematically solve for the transaction regions. When the stock prices are geometric Brownian motions, Magill (1976) established a mutual fund theorem on the reduction of the optimal investment and consumption problem to the case consisting of a bond and only one stock.

2.3   Stationary Policies for Infinite-Horizon Problems

We can view the infinite-horizon optimal investment and consumption problem as the limiting case of the finite-horizon problem in Section 2.2. By setting $t = 0$ and letting $T \to \infty$, the finite-horizon value function (2.3) approaches the following infinite-horizon value function (dropping the subscript on U_1):

$$V(x, y) = \sup_{(C,L,M) \in \mathcal{A}(x,y)} \mathbb{E} \int_0^\infty e^{-\beta t} U(C_t)\, dt, \qquad (x, y) \in \mathcal{D}, \qquad (2.17)$$

where $\mathcal{A}(x, y)$ denotes the set of all *admissible* policies (C, L, M) for an *initial* position $(x, y) \in \mathcal{D}$ such that $(X_t, Y_t) \in \mathcal{D}$ for all $t \geq 0$ a.s. Because the problem no longer depends on time t, the regions $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{N}$ are stationary over time. The Bellman equation is given by (2.9) without $\partial/\partial t$. The analysis of Section

2.2 carries over, leading to analogs of equations (2.11a)–(2.11c) (i.e., without t and $\partial V/\partial t$).

For a general utility function U, the numerical procedure described in Section 2.2 can be modified to give a solution of the infinite-horizon investment and consumption problem. With the finite difference approximations given by (2.12) but without t or $t + \delta$, we obtain, after normalization, the following analog of (2.13):

$$V^0(x,y) = \max_C e^{-(\beta+\sigma^2 y^2/\epsilon^2)\delta} \left\{ \sum_{\tilde{x},\tilde{y}} p(\tilde{x},\tilde{y} \,|\, x,y) V(\tilde{x},\tilde{y}) + \delta U_1(C) \right\}, \quad (2.18)$$

where $\delta = \epsilon/\Sigma$, $\Sigma = |rx - C| + \alpha|y|$, and

$$p(x \pm \epsilon, y \,|\, x,y) = (rx - C)^{\pm}\delta/\epsilon, \qquad p(x, y \pm \epsilon \,|\, x,y) = \alpha y^{\pm}\delta/\epsilon.$$

Thus, proceeding as in Section 2.2, the dynamic programming equation is

$$V(x,y) = \max\{V^0(x,y), V^s(x,y), V^b(x,y)\}, \qquad (2.19)$$

where $V^s(x,y) = \mu V(x, y - \epsilon) + (1 - \mu)V(x + \epsilon, y - \epsilon)$ and $V^b(x,y) = (1 + \lambda)^{-1}[\lambda V(x - \epsilon, y) + (1 - \lambda)V(x - \epsilon, y + \epsilon)]$. According to which value on the r.h.s. of (2.19) $V(x,y)$ takes, the position (x,y) is classified as belong to $\mathcal{N}$, $\mathcal{S}$, or $\mathcal{B}$.

We next specialize U to take the form (2.2) to simplify the dynamic programming equation. For future reference, we begin with some results for the case of no transaction costs ($\lambda = \mu = 0$). An analysis of the infinite-horizon analog of (2.5) (i.e., without $\partial/\partial t$) yields $\theta_t^* \equiv p$ and $C_t^* = cZ_t$ for all $t \geq 0$, where p and c are given by (2.7). The value function is

$$V(z) = \frac{z^\gamma}{\gamma} c^{\gamma-1} \quad \text{if } \gamma < 1, \gamma \neq 0;$$

$$V(z) = \frac{1}{\beta^2}\left[r - \beta + \frac{(\alpha - r)^2}{2\sigma^2}\right] + \frac{1}{\beta}\log(\beta z) \quad \text{if } \gamma = 0.$$

These results can also be derived from those in Section 2.1 on the Merton problem by letting $T \to \infty$, since then $C_i(0) \to c$ ($i = 1, 2$). In the presence of transaction costs, the control problem has been independently considered by Davis and Norman (1990) using the principle of smooth fit and by Shreve and Soner (1994) using the concept of viscosity solutions to second-order PDEs. Earlier Constantinides (1986) obtained an approximate solution of the problem under the restriction that the investor consumes at a rate proportion to his holding in bond. A general numerical procedure when there are $m > 1$ stocks has been developed by Akian, Menaldi and Sulem (1996).

Because V is concave and homothetic, it is possible to reduce the problem to solving ordinary differential equations (ODEs). Indeed, the control problem can be solved by finding a C^2 function ψ and constants $\infty > x^* > x_* > -(1 - \mu)$ and A_*, A^* satisfying equations (2.15a)–(2.15c) *without time dependence*. It can be shown that $\theta_* \leq p \leq \theta^*$, with $\theta_* = (1 + x^*)^{-1}$, $\theta^* = (1 + x_*)^{-1}$. Two sufficient conditions for finiteness of the value function

V are $\beta > \gamma r + \gamma(\alpha - r)^2/\{2(1-\gamma)\sigma^2\}$ and $(\beta - \alpha\gamma)(1+\lambda) > (\beta - r\gamma)(1-\mu)$; see Shreve, Soner and Xu (1991). Interestingly, if lump-sum transaction costs proportional to portfolio value (e.g., portfolio management fees) are imposed in addition to proportional transaction costs, then portfolio selection and withdrawal for consumption are made optimally at *regular* intervals (as opposed to trading at randomly spaced instants of time), with the investor consuming deterministically between transactions, as shown by Duffie and Sun (1990).

To find the constants x_*, x^*, A_*, A^*, and the function ψ, the principle of smooth fit can be first applied to ψ'' at x_* and x^* to solve for A_* and A^* (which depend on x_* and x^* respectively). Next, the second order ODE (2.15c) (without t and $\partial\psi/\partial t$) can be written as a pair of first-order equations after a change of variables. Specifically, for $\gamma \neq 0$ (so $U(c) = c^\gamma/\gamma$), let $Q(f) = -b_1/\gamma - b_2 f + (1-\gamma)b_3 f^2$ and $R(f) = -b_1/\gamma + (b_3 - b_2)f - \gamma b_3 f^2$, where b_1, b_2, and b_3 are defined in (2.16). Then there exist functions $f(x)$ and $h(x)$ satisfying the system of differential equations

$$f' = \frac{1}{b_3 x}[R(f) - h], \qquad f(x_*) = f_* := \frac{x_*}{x_* + 1 - \mu}, \quad f(x^*) = f^* := \frac{x^*}{x^* + 1 + \lambda},$$
$$\text{(2.20a)}$$

$$h' = \frac{\gamma}{1-\gamma}\frac{h}{b_3 x f}[h - Q(f)], \qquad h(x_*) = Q(f_*), \quad h(x^*) = Q(f^*), \qquad \text{(2.20b)}$$

such that

$$\psi(x) = \frac{1}{\gamma}\left[\frac{\gamma h(x)}{1-\gamma}\right]^{\gamma-1}\left[\frac{x}{f(x)}\right]^\gamma$$

satisfies (2.15c) (without t and $\partial\psi/\partial t$). In this case, the optimal consumption policy is $C_t^* = C^*(X_t, Y_t)$, where $C^*(x,y) = \gamma(1-\gamma)^{-1}xh(x/y)/f(x/y)$. The case $\gamma = 0$ can be treated similarly.

Davis and Norman (1990) suggested the following algorithm for the numerical solution of (2.20a)–(2.20b) (in which f, h, x_*, x^* need to be determined). The iterative procedure starts with an arbitrary value $\hat{x}^*$ of $x^* > 1 - p$, and the corresponding values $\hat{f}^* = \hat{x}^*/(\hat{x}^* + 1 + \lambda)$ and $\hat{h}^* = Q(\hat{f}^*)$. It uses numerical integration to evaluate

$$\hat{f}(x) = \hat{f}^* - \int_x^{\hat{x}^*} \frac{R(\hat{f}(u)) - \hat{h}(u)}{b_3 u}\,du,$$

$$\hat{h}(x) = \hat{h}^* - \frac{\gamma}{1-\gamma}\int_x^{\hat{x}^*} \frac{\hat{h}(u)[\hat{h}(u) - Q(\hat{f}(u))]}{b_3 u \hat{f}(u)}\,du$$

for a sequence of decreasing x values until the *first* value $\hat{x}_*$ of x for which $\hat{h}(\hat{x}_*) \doteq Q(\hat{f}(\hat{x}_*))$. At this point, we have a solution of (2.20a)–(2.20b) with μ replaced by $\hat{x}_* + 1 - \hat{x}_*/f(\hat{x}_*)$. The iterative procedure continues by adjusting the initial guess $\hat{x}^*$ and computing the resulting $\hat{x}_*$, terminating when $\hat{x}_* + 1 - \hat{x}_*/f(\hat{x}_*)$ differs from μ by no more than some prescribed error bound.

2.4 Maximization of Long-Run Growth Rate

An alternative optimality criterion was considered by Taksar, Klass and Assaf (1988). Instead of maximizing expected utility of consumption as in (2.17), suppose the objective is to maximize, in the model (2.1a)–(2.1b) without consumption (i.e., $C_t \equiv 0$), the expected rate of growth of investor assets (equivalently the long-run growth rate). This optimality criterion can be reformulated in terms of $R_t = Y_t / X_t$ alone so that the problem is to minimize the following limiting expected "cost" per unit time:

$$\limsup_{t \to \infty} t^{-1} \mathbb{E} \left\{ \int_0^t h(R_t)\, dt + \int_0^t g(R_t)\, d\tilde{L}_t + \int_0^t f(R_t)\, d\tilde{M}_t \right\}, \qquad (2.21)$$

where

$$f(x) = \frac{\mu x}{x+1}, \qquad g(x) = \frac{\lambda}{x+1}, \qquad h(x) = \frac{\sigma^2 x^2}{2(x+1)^2} - \left(\alpha - r + \frac{\sigma^2}{2} \right) \frac{x}{x+1}. \tag{2.22}$$

In (2.21), $\tilde{L}_t$ (resp. $\tilde{M}_t$) can be interpreted as the cumulative percentage of stock bought (resp. sold) within the time interval $[0, t]$, and is related to L_t (resp. M_t) via $d\tilde{L}_t = (1/X_t)\, dL_t$ (resp. $d\tilde{M}_t = Y_t^{-1}\, dM_t$). If $\lambda = \mu = 0$ (no transaction costs), the second and third terms in (2.21) vanish and the optimal policy is to keep R_t equal to the optimal proportion obtained as the minimizer of $h(x)$. This is tantamount to setting $\theta_t \ (= Y_t/(X_t + Y_t))$ equal to $p^* := (\alpha - r)/\sigma^2 + 1/2$, which resembles the Merton proportion p in (2.7).

We study the general problem of minimizing (2.21) under the condition $|\alpha - r| < \sigma^2/2$. (If this condition is violated, the optimal policy is to transfer all the investment to bond or stock at time 0 and to do no more transfer thereafter.) Since

$$dR_t = (\alpha - r + \sigma^2/2) R_t\, dt + \sigma R_t\, dW_t + (1 + (1 + \lambda) R_t)\, d\tilde{L}_t - R_t(1 + (1 - \mu) R_t)\, d\tilde{M}_t,$$

an analysis of the value function V using the Bellman equation shows (in a manner similar to the previous section) that there exist constants x_*, x^*, A (optimal value) such that

$$(\sigma^2/2) x^2 V''(x) + (\alpha - r + \sigma^2/2) x V'(x) + h(x) - A = 0, \qquad x \in [x_*, x^*], \tag{2.23a}$$

$$V'(x) = F(x), \quad x \le x_*, \qquad V'(x) = G(x), \quad x \ge x^*, \tag{2.23b}$$

where $F(x) = -\lambda(1+x)^{-1}(1+(1+\lambda)x)^{-1}$ and $G(x) = \mu(1+x)^{-1}(1+(1-\mu)x)^{-1}$. Using the principle of smooth fit at x_* and x^*, we find that $A = h((1+\lambda)x_*) = h((1-\mu)x^*)$, from which it follows that

$$\text{either} \quad x^* = \frac{1+\lambda}{1-\mu} x_*, \qquad \text{or} \quad x^* = \left(\frac{1}{1-\mu} \right) \frac{(p^* - 1/2)(1+\lambda)x_* + p^*}{(1-p^*)(1+\lambda)x_* - p^*}. \tag{2.24}$$

Hence, even though an alternative criterion (of maximizing long-run growth rate) is used to assess the optimality of investment policies, the above analysis shows that like Section 2.3 the investor should again optimally maintain the

proportion of investment in stock between $\theta_* := x_*/(1 + x_*)$ and $\theta^* := x^*/(1 + x^*)$. The constants x_* and x^* can be computed by solving the second-order nonhomogeneous ODE

$$V'(x) = \frac{2}{\sigma^2 x^{2p^*}} \int_{x_*}^{x} [h(x_*) - h(y)] y^{2(p^*-1)} \, dy + \frac{F_* + x_* F_*'}{1 - 2p^*} \left(\frac{x_*}{x}\right)^{2p^*}$$
$$- \frac{2p^* F_* + x_* F_*'}{1 - 2p^*} \left(\frac{x_*}{x}\right), \tag{2.25}$$

with initial conditions $V'(x_*) = F_* := F(x_*)$ and $V''(x_*) = F_*' := F'(x_*)$ at x_*, which is obtained by differentiating (2.23a)–(2.23b). A search procedure can then be employed to find that value of x_* for which x^* given by (2.24) satisfies $V'(x^*) = G(x^*)$ in view of (2.23b).

3 Option Pricing and Hedging

This section considers the problem of constructing hedging strategies which best replicate the outcomes from options (and other contingent claims) in the presence of transaction costs, which can be formulated as the minimization of some loss function defined on the replication error. In our recent work, we directly minimize the (expected) cumulative variance of the replicating portfolio in the presence of additional rebalancing costs due to transaction costs. As shown in Section 3.3, this leads to substantial simplification as the optimal hedging strategy can be obtained by solving an optimal stopping (instead of control) problem. In Sections 3.1 and 3.2 we review an alternative approach, developed by Hodges and Neuberger (1989), Davis, Panas and Zariphopoulou (1993) and Clewlow and Hodges (1997), which is based on the maximization of the expected utility of terminal wealth and which generally results in a free boundary problem in four-dimensional space. Instead of solving the free boundary problem, Constantinides and Zariphopoulou (1999) derived analytic bounds on option prices.

3.1 Formulation via Utility Maximization

The utility-based approach adopts a paradigm similar to Section 2. Suppose the investor trades only in the underlying stock on which the option is written and proportional transaction costs are imposed on purchase and sale of stock. Following the notation in (1.2), his holding of bond (dollar value) and stock (number of shares) is given by

$$dX_t = rX_t \, dt - (1 + \lambda)S_t \, dL_t + (1 - \mu)S_t \, dM_t, \tag{3.1a}$$
$$dy_t = dL_t - dM_t, \tag{3.1b}$$

where L_t (resp. M_t) represents the cumulative number of shares bought (resp. sold) within the time interval $[0, t]$. Define the cash value of y shares of stock when the stock price is S by

$$Y(y, S) = (1 + \lambda)yS \quad \text{if } y < 0; \qquad Y(y, S) = (1 - \mu)yS \quad \text{if } y \geq 0.$$

For technical reasons, the investor's position is constrained to lie in the region

$$\mathcal{D} = \{(x, y, S) \in \mathbb{R}^2 \times \mathbb{R}_+ : x + Y(y, S) > -a\} \qquad (3.2)$$

for some prescribed positive constant a. We denote by $\mathcal{A}(t, x, y, S)$ the class of *admissible* trading strategies (L, M) for the position $(x, y, S) \in \mathcal{D}$ at time t such that $(X_s, y_s, S_s) \in \mathcal{D}$ for all $s \in [t, T]$. The objective is to maximize the expected utility of terminal wealth, giving rise to the value functions

$$V^i(t, x, y, S) = \sup_{(L,M) \in \mathcal{A}(t,x,y,S)} \mathbb{E}\left[U(Z_T^i)\right], \qquad i = 0, \text{s}, \text{b}, \qquad (3.3)$$

where $U : \mathbb{R} \to \mathbb{R}$ is a concave increasing function (so it is a risk-averse utility function). The terminal wealth of the investor (with or without an option position) is given by

$$Z_T^0 = X_T + Y(y_T, S_T) \qquad \text{(no call)},$$
$$Z_T^s = X_T + Y(y_T, S_T)\,\mathbb{I}_{\{S_T \leq K\}} + [Y(y_T - 1, S_T) + K]\,\mathbb{I}_{\{S_T > K\}} \qquad \text{(sell a call)},$$
$$Z_T^b = X_T + Y(y_T, S_T)\,\mathbb{I}_{\{S_T \leq K\}} + [Y(y_T + 1, S_T) - K]\,\mathbb{I}_{\{S_T > K\}} \qquad \text{(buy a call)},$$

in which we have assumed that the option is *asset settled* so that the writer delivers one share of stock in return for a payment of K when the holder chooses to exercise the option at maturity T. In the case of *cash settled* options, the writer delivers $(S_T - K)^+$ in cash, so $Z_T^s = X_T + Y(y_T, S_T) - (S_T - K)^+$ and $Z_T^b = X_T + Y(y_T, S_T) + (S_T - K)^+$.

From the definition of the value functions (3.3), it is evident that an application of the principle of dynamic programming will yield the same PDE for each value function ($i = 0, \text{s}, \text{b}$), with the terminal condition governed by utility of the respective terminal wealth. By temporarily restricting L and M as in (2.8) (and then letting $\kappa \to \infty$), the Bellman equation for V^i is $\max_{\ell, m}(\partial/\partial t + \mathcal{L})V^i(t, x, y, S) = 0$, where $\mathcal{L}$ is the infinitesimal generator of (3.1a)–(3.1b) and $dS_t = S_t(\alpha\,dt + \sigma\,dW_t)$:

$$\mathcal{L} = rx\frac{\partial}{\partial x} + \alpha S\frac{\partial}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + \left[\frac{\partial}{\partial y} - (1 + \lambda)\frac{\partial}{\partial x}\right]\ell + \left[(1 - \mu)\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]m.$$

Thus, once again, the state space can be partitioned into regions in which it is optimal to buy stock at the maximum rate, or to sell stock at the maximum rate, or not to do any transaction. Arguments similar to those in Section 2 show that there exist functions $y_*(t, x, S)$ (buy boundary) and $y^*(t, x, S)$ (sell boundary) for each $i = 0, \text{s}, \text{b}$ such that

$$V_y^i(t, x, y, S) = (1 + \lambda)SV_x^i(t, x, y, S), \qquad y \leq y_*(t, x, S), \qquad (3.4a)$$
$$V_y^i(t, x, y, S) = (1 - \mu)SV_x^i(t, x, y, S), \qquad y \geq y^*(t, x, S), \qquad (3.4b)$$
$$V_t^i + rxV_x^i + \alpha SV_S^i + (\sigma^2 S^2/2)V_{SS}^i = 0, \qquad y \in [y_*(t, x, S), y^*(t, x, S)], \quad (3.4c)$$

The optimal hedging strategy associated with (3.3) is given by the pair (L^*, M^*), where for each $i = 0, \text{s}, \text{b}$,

$$L_t^* = \int_0^t \mathbb{I}_{\{y_s = y_*(s, X_s, S_s)\}}\,dL_s^*, \qquad M_t^* = \int_0^t \mathbb{I}_{\{y_s = y^*(s, X_s, S_s)\}}\,dM_s^*, \qquad t \in [0, T].$$

Two different definitions of option prices have been proposed. In Hodges and Neuberger (1989) and subsequently in Clewlow and Hodges (1997), the reservation selling (resp. buying) price is defined as the amount of cash P^{s} (resp. P^{b}) required *initially* to provide the same expected utility as not selling (resp. buying) the option. Thus, P^{s} and P^{b} satisfy the following equations:

$$V^{\mathrm{s}}(0, P^{\mathrm{s}}, 0, S) = V^0(0, 0, 0, S) = V^{\mathrm{b}}(0, -P^{\mathrm{b}}, 0, S). \qquad (3.5)$$

An alternative definition is used by Davis, Panas and Zariphopoulou (1993). Assuming that $U(0) = 0$, define

$$x^i = \inf\{x : V^i(0, x, 0, S) \geq 0\}, \qquad i = 0, \mathrm{s}, \mathrm{b},$$

so in particular, $x^0 \leq 0$ because $V^0(0, 0, 0, S) \geq 0$ (investing in neither bond nor stock is admissible). Thus, an investor pays an "entry fee" $-x^0$ to trade in the market strictly on his own account. The selling price P^{s} and buying price P^{b} of the option are then constructed such that the investor is indifferent between going into the market with and without an option position: $P^{\mathrm{s}} = x^{\mathrm{s}} - x^0$ and $P^{\mathrm{b}} = -(x^{\mathrm{b}} - x^0)$. Although they advocate this definition for the option writer's price, Davis, Panas and Zariphopoulou (1993, pp. 492–493) express reservations of using it to define the buyer's price.

3.2   Solution for Exponential Utility Functions

A reduction in dimensionality (from four to three) can be achieved by specializing to the negative exponential utility function $U(z) = 1 - e^{-\gamma z}$ (with *constant* index of risk aversion $-U''(z)/U'(z) = \gamma$). Using this utility function, the bond position can be managed through time independently of the stock holding and

$$V^i(t, x, y, S) = 1 - \exp\left\{-\gamma x e^{r(T-t)}\right\} H^i(t, y, S), \qquad i = 0, \mathrm{s}, \mathrm{b},$$

where $H^i(t, y, S) := 1 - V^i(t, 0, y, S)$. As a consequence, the free boundary problem (3.4a)–(3.4c) for each $i = 0, \mathrm{s}, \mathrm{b}$ is transformed into the following problem:

$$H^i_y(t, y, S) = -\gamma e^{r(T-t)}(1 + \lambda) S H^i(t, y, S), \qquad y \leq y_*(t, S), \qquad (3.6\mathrm{a})$$

$$H^i_y(t, y, S) = -\gamma e^{r(T-t)}(1 - \mu) S H^i(t, y, S), \qquad y \geq y^*(t, S), \qquad (3.6\mathrm{b})$$

$$H^i_t + \alpha S H^i_S + (\sigma^2 S^2/2) H^i_{SS} = 0, \qquad y \in [y_*(t, S), y^*(t, S)]. \qquad (3.6\mathrm{c})$$

It is also straightforward to observe that the price definitions are equivalent to

$$P^{\mathrm{s}} = \gamma^{-1} e^{-rT} \log\left[\frac{H^{\mathrm{s}}(0, 0, S)}{H^0(0, 0, S)}\right], \qquad P^{\mathrm{b}} = -\gamma^{-1} e^{-rT} \log\left[\frac{H^{\mathrm{b}}(0, 0, S)}{H^0(0, 0, S)}\right].$$
$$(3.7)$$

The solution of the free boundary problem (3.6a)–(3.6c) can be obtained by approximating $dy_t = dL_t - dM_t$ and $dS_t = S_t(\alpha\, dt + \sigma\, dW_t)$ with Markov chains and applying a that discrete-time dynamic programming algorithm as in

Section 2.2. To this end, it is useful to note from (3.6a)–(3.6b) that

$$H^i(t, y_1, S) = H^i(t, y_2, S) \exp\left\{-\gamma e^{r(T-t)}(1+\lambda)S(y_1 - y_2)\right\}, \quad y_1 \leq y_2 \leq y_*(t, S),$$

$$H^i(t, y_1, S) = H^i(t, y_2, S) \exp\left\{-\gamma e^{r(T-t)}(1-\mu)S(y_1 - y_2)\right\}, \quad y_1 \geq y_2 \geq y^*(t, S).$$

We discretize time t so that it takes values in $\mathbb{T} = \{0, \delta, 2\delta, \ldots, N\delta\}$, where $\delta = T/N$. The number of shares is also discretized so that y is a multiple of ϵ. Then we can approximate the stock price process using the following random walk:

$$S_{t+\delta} = \begin{cases} e^u S_t & \text{with probability } p, \\ e^{-u} S_t & \text{with probability } 1-p, \end{cases}$$

where $u = \sqrt{\sigma^2\delta + (\alpha - \sigma^2/2)^2\delta^2}$ and $p = [1 + (\alpha - \sigma^2/2)\delta/u]/2$. Let $\mathbb{Y} = \{k\epsilon : k \text{ is an integer}\}$ and $\mathbb{S} = \{e^{ku}S_0 : k \text{ is an integer}\}$ This discretization scheme leads to the following algorithm for $(t, y, S) \in \mathbb{T} \times \mathbb{Y} \times \mathbb{S}$:

$$H^i(t, y, S) = \min\left\{ H^i(t, y+\epsilon, S) \exp\left[\gamma e^{r(T-t)}(1+\lambda)S\epsilon\right], \right.$$
$$H^i(t, y-\epsilon, S) \exp\left[-\gamma e^{r(T-t)}(1-\mu)S\epsilon\right],$$
$$\left. pH^i(t+\delta, y, e^u S) + (1-p)H^i(t+\delta, y, e^{-u}S) \right\}; \quad (3.8)$$

see Davis, Panas and Zariphopoulou (1993) and Clewlow and Hodges (1997) for details. Depending on which term on the r.h.s. of (3.8) is the smallest, the point (t, y, S) is classified as belonging to $\mathcal{B}$, $\mathcal{S}$, or $\mathcal{N}$, respectively. We set $y_*(t, S)$ (resp. $y^*(t, S)$) to be the largest (resp. smallest) value of y for which $(t, y, S) \in \mathcal{B}$ (resp. $\mathcal{S}$).

3.3 A New Approach

The previous analysis shows that, in the presence of transaction costs, perfect hedging of an option is not possible and trading in options involves an element of risk. Indeed, if the region $\mathcal{D}$ defined in (3.2) is replaced by the solvency region of Section 2, Soner, Shreve and Cvitanić (1995) showed that "the least costly way of hedging the call option in a market with proportional transaction costs is the trivial one—to buy a share of the stock and hold it." By relaxing the requirement of perfect hedging, Leland (1985) and Boyle and Vorst (1992) demonstrated that discrete-time hedging strategies, for which trading takes place at *regular* intervals, can nearly replicate the option payoff at maturity. The option price is essentially the Black-Scholes value with an adjusted volatility. While hedging error can be reduced to zero as the time between trades approaches zero, the adjusted volatility approaches infinity and the option value approaches the value of one share of stock.

A new approach has been recently proposed in Lai and Lim (2002b). The formulation is motivated by the original analysis of Black and Scholes (1973) in the following way: form a hedging portfolio that minimizes hedging error and price the option by the (expected) initial capital require to set up the hedge.

For the hedging portfolio, the objective is to minimize the expected cumulative instantaneous variance and additional rebalancing costs due to transaction fees, given by

$$J(t, S, y) = \mathbb{E}\left[\int_t^T F(s, S_s, y_s)\, ds + \lambda \int_t^T (S_s/K)\, dL_s \right.$$

$$\left. + \mu \int_t^T (S_s/K)\, dM_s \;\middle|\; S_t = S, y_t = y\right].$$

where $F(t, S, y) = \sigma^2 (S/K)^2 [y - \Delta(t, S)]^2$ for the option writer and $F(t, S, y) = \sigma^2 (S/K)^2 [y + \Delta(t, S)]^2$ for the option buyer. Here, $\Delta(t, S) = N(d_1(t, S))$ is the Black-Scholes delta (i.e., the number of shares in the option's perfectly replicating portfolio) with

$$d_1(t, S) = \{\log(S/K) + r(T - t)\}/\sigma\sqrt{T - t} + \sigma\sqrt{T - t}/2.$$

Taking $\alpha = r$, analysis of the Bellman equation for the value function $V(t, S, y) = \min_{L,M} J(t, S, y)$ leads to the following free boundary problem:

$$V_y(t, S, y) = -\lambda S/K \qquad \text{in } \mathcal{N}^c \cap \{y < \Delta(t, S)\},$$
$$V_y(t, S, y) = \mu S/K \qquad \text{in } \mathcal{N}^c \cap \{y > \Delta(t, S)\},$$
$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + F = 0 \qquad \text{in } \mathcal{N}.$$

By working with V_y instead of directly with V, we deduce from the previous set of equations that $V_y(t, S, y)$ satisfies another free boundary problem associated with an *optimal stopping* problem. It is this reduction to optimal stopping that greatly simplifies the hedging problem.

Applying the transformations $s = \sigma^2(t - T)$ and $z = \log(S/K) - (\rho - 1/2)s$, where $\rho = r/\sigma^2$, it suffices to work with $v(s, z, y) = V_y(t(s), S(s, z), y)$. For each y, we obtain the following discrete-time dynamic programming equation for the option writer, utilizing a symmetric Bernoulli walk approximation to Brownian motion:

$$v(s, z, y) = \min\{\mu e^{z+\beta s}, \tilde{v}(s, z, y)\}\mathbb{I}_{\{y > D(s,z)\}}$$
$$+ \max\{-\lambda e^{z+\beta s}, \tilde{v}(s, z, y)\}\mathbb{I}_{\{y < D(s,z)\}}, \qquad (3.9)$$

with $v(0, z, y) = [\mu\mathbb{I}_{\{y > D(0,z)\}} - \lambda\mathbb{I}_{\{y < D(0,z)\}}]e^z$, where $\tilde{v}(s, z, y) = \delta g(s, z, y) + \frac{1}{2}[v(s + \delta, z + \sqrt{\delta}, y) + v(s + \delta, z - \sqrt{\delta}, y)]$, $g(s, z, y) = 2e^{2(z+(\rho-1/2)s)}[y - D(s, z)]$, $D(s, z) = e^{\alpha\rho s}\Phi(z/\sqrt{-s} + \sqrt{-s})$, and $s = -\delta, -2\delta, \ldots$. Each point $(s, z, y) \in (-\infty, 0] \times \mathbb{R} \times [0, 1]$ can be classified as belonging to the sell region, buy region, or no transactions region, according to whether $v(s, z, y) = \mu e^{z+\beta s}$, $v(s, z, y) = -\lambda e^{z+\beta s}$, or $-\lambda e^{z+\beta s} < v(s, z, y) < \mu e^{z+\beta s}$, respectively. Since $v(s, z, y)$ is nondecreasing in y, there exist sell and buy boundaries, denoted respectively by $y^s(s, z)$ and $y^b(s, z)$, such that if $y > y^s(s, z)$ (resp. $y < y^b(s, z)$), the option writer must immediately sell $y - y^s(s, z)$ (resp. buy $y^b(s, z) - y$) shares of stock to form an optimal hedge. The optimal hedging portfolio for the option buyer can also be obtained from (3.9) by symmetry: the optimal sell and buy boundaries for the option buyer with sell rate μ and buy rate λ are $-y^b(s, z)$ and $-y^s(s, z)$

respectively, where $y^s(s, z)$ and $y^b(s, z)$ are the optimal sell and buy boundaries for the option writer with sell rate λ and buy rate μ. Simulation studies have shown the approach to be efficient in the sense that it results in the smallest standard error of hedging error for any specified mean hedging error, where hedging error is defined to the difference between the Black-Scholes value and the initial capital needed to replicate the option payoff at maturity. For details and refinements, see Lai and Lim (2002b).

4   Conclusion

Optimal investment portfolios and hedging strategies derived in the absence of transaction costs involve continuous trading to maintain the optimal positions. Such continuous policies are at best approximations to what can be achieved in the real world, and a frequent practice is to execute the policies discretely so that transactions take place at regular (or predetermined) intervals. With appropriate adjustments, these policies can also be implemented in the presence of transaction costs since they do not lead to an infinite turnover of asset. However, in the absence of a clearly defined objective, it is difficult to argue that a discrete policy is optimal in any sense.

This difficulty can be overcome in investment and consumption problems through utility maximization, and in option pricing and hedging problems through the minimization of hedging error. Many formulations of these problems lead naturally to singular stochastic control problems, in which transactions either occur at maximum rate ("bang-bang") or not at all. In the analysis of these singular control problems, the principle of dynamic programming is used to derive the Bellman equations, which are nonlinear PDEs whose solutions in the classical sense have posed formidable existence and uniqueness problems. The development of *viscosity solutions* to these PDEs in the 1980s is a major breakthrough that circumvents these difficulties; see Crandall, Ishii and Lions (1992). In contrast to discrete policies, singular control policies require trading to take place at *random* instants of time, when asset holdings fall too "out of line" from a "target." Besides being naturally intuitive, singular control policies lend further insight into optimal investor behavior when faced with investment decisions (with or without consumption). Efficient numerical procedures can be developed to solve for the singular control policies based on Markov chain approximations of the controlled diffusion process. In some instances, a reduction to optimal stopping reduces the computational effort considerably.

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