TRIMMED SUMS FROM THE DOMAIN OF GEOMETRIC PARTIAL ATTRACTION OF SEMISTABLE LAWS[1]

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We show that all possible moderately trimmed sums from a domain of geometric partial attraction of a semistable law G are asymptotically normal, along the whole sequence of natural numbers, under the necessary condition that the Lévy functions of G do not have flat stretches. We also show that asymptotic normality prevails still along the whole sequence at least for suitably chosen moderately trimmed sums from such a domain for every G. It then follows that after removing any moderately trimmed sum from the middle the remaining sums of extreme values still produce every semistable limiting distribution G that the original full sums have, along exactly the same geometrically growing subsequences of the natural numbers.

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1 Introduction

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables, and for each natural number $n \in \mathbb{N}$ consider the order statistics $X_{1,n} \leq \cdots \leq X_{n,n}$ pertaining to the sample $X_1, \dots, X_n$. Trimmed sums $\sum_{j=l+1}^{n-m} X_{j,n}$ for $l, m \in \mathbb{N}$, $l+m < n$, are the initial basic objects in statistical theories of robust estimation, so it is not surprising that there has been considerable interest in the investigation of their asymptotic distribution. The large literature on a number of versions of the problem may be traced back from our references; see in particular the collection edited by Hahn, Mason and Weiner (1991). Here we deal only with trimming according to natural order, as in the sums $\sum_{j=l+1}^{n-m} X_{j,n}$, and not with the case when trimming is done with respect to ordering the moduli $|X_1|, \dots, |X_n|$ of the observations.

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Subsequent to specific results on asymptotic normality, Stigler's (1973) theorem completely solves the problem of the asymptotic distribution for the classical trimmed mean

$$\sum_{j=\lfloor n\alpha\rfloor+1}^{n-(n-\lfloor n\beta\rfloor)} X_{j,n}/(\lfloor n\beta\rfloor - \lfloor n\alpha\rfloor) = \sum_{j=\lfloor n\alpha\rfloor+1}^{\lfloor n\beta\rfloor} X_{j,n}/(\lfloor n\beta\rfloor - \lfloor n\alpha\rfloor),$$

where $0 < \alpha < \beta < 1$ and, with $\mathbb{Z}$ standing for the set of integers, $\lfloor x\rfloor = \max\{r \in \mathbb{Z}\colon r \le x\}$ is the integer part of a real number $x \in \mathbb{R}$: In this case of *heavy trimming* enough extreme values are discarded so that, with suitable centering and norming, the remaining mean has an asymptotic distribution as $n \to \infty$ for every underlying distribution function $F(x) = \mathbb{P}\{X_1 \le x\}$, $x \in \mathbb{R}$, where the basic probability space is denoted by $(\Omega, \mathcal{A}, \mathbb{P})$. Introducing the associated quantile function

$$Q(s) = \inf\{x \in \mathbb{R}\colon\ F(x) \ge s\}, \quad 0 < s < 1,$$

this asymptotic distribution is normal if and only if Q is continuous at both α and β. A proof of the general result, different from Stigler's, and one that shows his theorem to be a boundary case of asymptotic distributions for moderately trimmed sums discussed below, is given by Csörgő, Haeusler and Mason (1988b); see also Cheng (1992) for further elaborations.

At the other trimming extreme, it is conceivable that for fixed pairs of positive integers l and m the existence and nature of asymptotic distributions of the *lightly trimmed* sums $S_{n_k}(l,m) = \sum_{j=l+1}^{n_k-m} X_{j,n_k}$, generally along subsequences $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, are closely connected with those of the limiting distributions of the whole untrimmed sums $S_{n_k} = S_{n_k}(0,0) = \sum_{j=1}^{n_k} X_j$. (Asymptotic distributions for any of the sums here and in the sequel are always meant with suitable centering and norming and *all infinite subsequences of* $\mathbb{N}$ *are assumed unbounded throughout.*) Indeed, it was shown in their Corollary 6 by Csörgő, Haeusler and Mason (1988a) that S_{n_k} converges in distribution along some $\{n_k\}$ to a nondegenerate random variable, in other words, F is in the domain of partial attraction of some infinitely divisible distribution, if and only if $S_{n_k}(l,m)$ converges in distribution to nondegenerate random variables for every pair (l,m), along the same $\{n_k\}$. The limiting distributions of the latter are some "trimmed" forms of a special representation of an infinitely divisible random variable, the distribution of which is the limiting distribution of the former; the representation is given in the next section. One may conjecture that it is sufficient to require the distributional convergence of $S_{n_k}(l,m)$ for a single pair $(l,m) \in \mathbb{N}^2$ to achieve the same conclusion for S_{n_k}, and hence also for all $(l,m) \in \mathbb{N}^2$, along the same $\{n_k\}$. For the whole sequence $\{n\} = \mathbb{N}$ this was proved by Kesten (1993), in which case the conclusion is that F is in the domain of attraction of a stable law. The general subsequential version is still open.

Perhaps the most interesting case, the topic of the present note, is that of *moderately trimmed* sums $S_n(l_n, m_n) = \sum_{j=l_n+1}^{n-m_n} X_{j,n}$, where

$$l_n \to \infty, \ \frac{l_n}{n} \to 0 \quad \text{and} \quad m_n \to \infty, \ \frac{m_n}{n} \to 0 \quad \text{as } n \to \infty. \tag{1.1}$$

The first deeper result is due to Csörgő, Horváth and Mason (1986), who proved that if the full sums S_n have a nondegenerate asymptotic distribution along the whole $\{n\} = \mathbb{N}$, i.e. if F is in the domain of attraction of a (normal or nonnormal) stable law, then with $l_n \equiv m_n$ and suitable centering and norming sequences $S_n(m_n, m_n)$ is asymptotically normal as $n \to \infty$. Csörgő, Haeusler and Mason (1988b) then determined the class of all possible asymptotic distributions for $S_n(l_n, m_n)$ along all possible subsequences $\{n_k\}$, together with necessary and sufficient conditions for the convergence in distribution of $S_{n_k}(l_{n_k}, m_{n_k})$ as $k \to \infty$. To formulate at least the condition for asymptotic normality, define for $0 < s < 1 - t < 1$,

$$\begin{aligned} \sigma^2(s\,, 1-t) &= \int_s^{1-t} \int_s^{1-t} [\min(u, v) - uv]\, dQ(u)\, dQ(v) \\ &= sQ^2(s) + tQ^2(1-t) + \int_s^{1-t} Q^2(u)\, du \\ &\quad - \left[sQ(s) + tQ(1-t) + \int_s^{1-t} Q(u)\, du \right]^2, \end{aligned} \tag{1.2}$$

a basic function in Csörgő, Haeusler and Mason (1988a,b). For given sequences $\{l_n\}$ and $\{m_n\}$ set

$$a_n(l_n, m_n) = \sqrt{n}\, \sigma\Big(\frac{l_n}{n}\,, 1 - \frac{m_n}{n}\Big), \tag{1.3}$$

and introduce the two sequences of functions

$$\varphi_{1,n}(x) = \begin{cases} \varphi_{1,n}\Big(-\frac{\sqrt{l_n}}{2}\Big), & -\infty < x < -\frac{\sqrt{l_n}}{2}, \\ \frac{\sqrt{l_n}}{a_n(l_n, m_n)} \Big\{ Q\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big) - Q\Big(\frac{l_n}{n}\Big) \Big\}, & -\frac{\sqrt{l_n}}{2} \le x \le \frac{\sqrt{l_n}}{2}, \\ \varphi_{1,n}\Big(\frac{\sqrt{l_n}}{2}\Big), & \frac{\sqrt{l_n}}{2} < x < \infty, \end{cases}$$

and

$$\varphi_{2,n}(x) = \begin{cases} \varphi_{2,n}\Big(-\frac{\sqrt{m_n}}{2}\Big), & -\infty < x < -\frac{\sqrt{m_n}}{2}, \\ \frac{\sqrt{m_n}}{a_n(l_n, m_n)} \Big\{ Q\Big(1 - \frac{m_n}{n} + x\frac{\sqrt{m_n}}{n}\Big) \\ \qquad - Q\Big(1 - \frac{m_n}{n}\Big) \Big\}, & -\frac{\sqrt{m_n}}{2} \le x \le \frac{\sqrt{m_n}}{2}, \\ \varphi_{2,n}\Big(\frac{\sqrt{m_n}}{2}\Big), & \frac{\sqrt{m_n}}{2} < x < \infty. \end{cases}$$

Also, let $\xrightarrow{\mathcal{D}}$ denote convergence in distribution and let Z be a standard normal random variable. According to Theorem 4 of Csörgő, Haeusler and Mason (1988b), for sequences $\{l_n\}$ and $\{m_n\}$ satisfying (1.1), there exist centering and normalizing constants $C_n \in \mathbb{R}$ and $A_n > 0$ such that $A_n^{-1}[S_n(l_n, m_n) - C_n] \xrightarrow{\mathcal{D}} Z$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \varphi_{j,n}(x) = 0 \quad \text{for every } x \in \mathbb{R},\ j = 1, 2, \tag{1.4}$$

in which case $C_n \equiv c_n(l_n, m_n) := n \int_{\frac{l_n+1}{n}}^{1-\frac{m_n+1}{n}} Q(u)\, du$ and $A_n \equiv a_n(l_n, m_n)$ work.

The subsequential version of this result is also true. If at least one of the functions $\varphi_{j,n}(\cdot)$, or one of the renormalized functions $a_n(l_n, m_n)\varphi_{j,n}(\cdot)/A_n$ for some $A_n > 0$ for which $a_n(l_n, m_n)/A_n \to 0$, $j = 1, 2$, converges to a nonzero function either along the whole $\{n\}$ or along a subsequence, then extra terms appear in the limiting random variable so that the asymptotic distribution, typically obtained along a further subsequence, is no longer normal; it does not even have a normal component in the renormalized case. The conditions appearing are optimal; for the precise statements the reader is referred to Csörgő, Haeusler and Mason (1988b, 1991b). Griffin and Pruitt (1989) rederived this theory by a different method, obtaining the conditions and the description of limiting random variables in alternative forms, with numerous additional observations.

While the "asymptotic continuity" condition (1.4) solves the problem of asymptotic normality of moderately trimmed sums completely from a general mathematical point of view, its probabilistic meaning is not so clear until it is tied to better understood conditions that govern the asymptotic distribution of the entire untrimmed sums. Indeed, it was pointed out by Csörgő, Haeusler and Mason (1988b) and then by Griffin and Pruitt (1989) that if F is stochastically compact, meaning that the full sums are stochastically compact in the sense that there exist sequences of constants $b_n \in \mathbb{R}$ and $d_n > 0$ such that every subsequence of $\mathbb{N}$ contains a further subsequence along which $[S_n - b_n]/d_n$ converges in distribution to a nondegenerate random variable, then the sequences of functions $\{\varphi_{j,n}(\cdot)\}_{n=1}^{\infty}$ are uniformly bounded, $j = 1, 2$, and hence the sequence $S_n^*(l_n, m_n) := [S_n(l_n, m_n) - c_n(l_n, m_n)]/a_n(l_n, m_n)$ of centered and normed trimmed sums is also stochastically compact for any pair (l_n, m_n) of sequences satisfying (1.1). However, nonnormal subsequential limiting distributions do arise in this case.

Thus, to date, the only explicitly determined family of underlying distributions for which $S_n^*(m_n, m_n)$ is known to be asymptotically normal along the whole $\mathbb{N}$ for *every* sequence $\{m_n\}$ satisfying (1.1) is the family of those F that are in the domain of attraction of a stable law [Csörgő, Horváth and Mason (1986)], and the only explicit family for which $S_n^*(l_n, m_n)$ is known to be asymptotically normal for *every* sequence $\{(l_n, m_n)\}$ of pairs satisfying (1.1) is the subfamily attracted by not completely asymmetric stable laws [Griffin and Pruitt (1989)]. The question arises whether there is a probabilistically meaningful larger class of distributions, necessarily within the class of stochastically compact distributions, which would respectively contain the families above and for which the same conclusions for the asymptotic normality of trimmed sums would still hold true. A feature of the phenomenon would of course be that the full sums, $[S_n - b_n]/d_n$, would no longer converge in distribution themselves along the whole $\{n\} = \mathbb{N}$. The aim of this paper is to show that a larger class of distributions within the class of stochastically compact distributions does indeed exist with these properties: it is a proper subfamily of the family of distributions in the domain of geometric partial attraction of semistable laws. In the next section we describe this family of distributions, while Section 3 contains the new results and their proofs.

2 Semistable distributions and their domains of geometric partial attraction

Let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $\psi(\cdot)$ defined on the positive half-line $(0,\infty)$ such that $\int_\varepsilon^\infty \psi^2(s)\,ds < \infty$ for all $\varepsilon > 0$. Let $E_1^{(j)}, E_2^{(j)}, \ldots$, $j = 1, 2$, be two independent sequences of independent exponentially distributed random variables with mean 1. With their partial sums $Y_n^{(j)} = E_1^{(j)} + \cdots + E_n^{(j)}$ as jump points, $n \in \mathbb{N}$, consider the standard left-continuous independent Poisson processes $N_j(u) := \sum_{n=1}^{\infty} I(Y_n^{(j)} < u)$, $0 \le u < \infty$, $j = 1, 2$, where $I(\cdot)$ is the indicator function. For a function $\psi \in \Psi$, consider the random variables

$$W_j(\psi) := \int_1^\infty [N_j(s) - s]\,d\psi(s) + \int_0^1 N_j(s)\,d\psi(s) - \psi(1), \quad j = 1, 2,$$

where the first integrals are almost surely well defined, by the condition that $\psi \in \Psi$, as improper Riemann integrals. For $\psi_1 \in \Psi$ and $\psi_2 \in \Psi$, consider the constant

$$\theta(\psi_1, \psi_2) := \int_0^1 \frac{\psi_1(s)}{1 + \psi_1^2(s)}\,ds - \int_1^\infty \frac{\psi_1^3(s)}{1 + \psi_1^2(s)}\,ds - \int_0^1 \frac{\psi_2(s)}{1 + \psi_2^2(s)}\,ds + \int_1^\infty \frac{\psi_2^3(s)}{1 + \psi_2^2(s)}\,ds,$$

let Z be a standard normal random variable such that $N_1(\cdot)$, Z, and $N_2(\cdot)$ are independent, and for a finite constant $\sigma \ge 0$ finally introduce the random variables

$$V(\psi_1, \psi_2, \sigma) := -W_1(\psi_1) + \sigma Z + W_2(\psi_2) \tag{2.1}$$

and $W(\psi_1,\psi_2,\sigma) := V(\psi_1,\psi_2,\sigma) - \theta(\psi_1,\psi_2)$, the latter of which by Theorem 3 in Csörgő, Haeusler and Mason (1988a) has characteristic function

$$\mathbb{E}\Big(e^{itW(\psi_1,\psi_2,\sigma)}\Big) = \exp\bigg\{-\frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}\Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)dL(x) + \int_{0}^{\infty}\Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big)dR(x)\bigg\} \tag{2.2}$$

for all $t \in \mathbb{R}$, where $L(x) = \inf\{s > 0 : \psi_1(s) \geq x\}$ for $x < 0$ and $R(x) = -\inf\{s > 0 : \psi_2(s) \geq -x\}$ for $x > 0$. Here $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $R(\infty) = 0$, and $\int_{-\varepsilon}^{0} x^2 dL(x) + \int_0^{\varepsilon} x^2 dR(x) < \infty$ for every $\varepsilon > 0$ since $\psi_1, \psi_2 \in \Psi$. Thus $V(\psi_1,\psi_2,\sigma)$ is infinitely divisible by Lévy's formula [see e.g. in Gnedenko and Kolmogorov (1954)]. Conversely, given the right side of (2.2) with $L(\cdot)$ and $R(\cdot)$ having the properties just listed, the variable $W(\psi_1,\psi_2,\sigma)$ has this characteristic function again if we choose $\psi_1(s) = \inf\{x < 0 : L(x) > s\}$ and $\psi_2(s) = \inf\{x < 0 : -R(-x) > s\}$, $s > 0$, for then $\psi_1, \psi_2 \in \Psi$.

Thus the class $\mathcal{I}$ of all nondegenerate infinitely divisible distributions can be identified with the class $\{(\psi_1,\psi_2,\sigma) \neq (0,0,0) : \psi_1,\psi_2 \in \Psi, \sigma \geq 0\}$ of triplets. Then F being in the domain of partial attraction of a $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{I}$, written $F \in \mathbb{D}_{\mathrm{p}}(G)$, means that there exists a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ and centering and norming constants $C_{k_n} \in \mathbb{R}$ and $A_{k_n} > 0$ such that

$$\frac{1}{A_{k_n}}\left\{\sum_{j=1}^{k_n} X_j - C_{k_n}\right\} \xrightarrow{\mathcal{D}} V(\psi_1,\psi_2,\sigma), \tag{2.3}$$

where a convergence relation is meant to hold as $n \to \infty$ unless otherwise specified and $G_{\psi_1,\psi_2,\sigma}$ is the distribution function of the random variable $V(\psi_1,\psi_2,\sigma)$ from (2.1); the characteristic function of $V(\psi_1,\psi_2,\sigma) - \theta(\psi_1,\psi_2)$ is in (2.2). By classical theory [Gnedenko and Kolmogorov (1954) or Corollary 5* in Csörgő (1990)] this happens for $\{k_n\} = \{n\} = \mathbb{N}$ if and only if either $(\psi_1,\psi_2,\sigma) = (0,0,\sigma)$ for some $\sigma > 0$, in which case F is in the domain of attraction of the normal distribution, written $F \in \mathbb{D}(2)$, or $(\psi_1,\psi_2,\sigma) = (m_1\psi^{\alpha}, m_2\psi^{\alpha}, 0)$ for some constants $\alpha \in (0,2)$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$, where $\psi^{\alpha}(s) = -s^{-1/\alpha}$, $s > 0$, in which case F is in the domain of attraction of a stable distribution of exponent α, written $F \in \mathbb{D}(\alpha)$. (The superscript α in ψ^{α}, and in ψ_1^{α} and ψ_2^{α} beginning with (2.4) below, is meant as a label, not as a power exponent.) The normal being the stable law of exponent 2, let $\mathcal{S}$ denote the class of all stable laws.

Lévy (1937) introduced the class $\mathcal{S}_* \subset \mathcal{I}$ of semistable laws by extending a defining property of stable characteristic functions and, as translated into the framework of the present description of infinitely divisible laws, showed that $G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$ if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, giving the normal distribution as a semistable distribution of exponent 2, or $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where

$$\psi_j^\alpha(s) = M_j(s)\psi^\alpha(s) = -M_j(s)s^{-1/\alpha}, \quad s > 0, \quad j = 1, 2, \tag{2.4}$$

for some $\alpha \in (0,2)$, defining a semistable distribution of exponent $\alpha \in (0,2)$, where M_1 and M_2 are nonnegative, right-continuous functions on $(0,\infty)$, either identically zero or bounded away from both zero and infinity, such that $M_1 + M_2$ is not identically zero, the functions $M_j(\cdot)\psi^\alpha(\cdot)$ are nondecreasing, $j = 1, 2$, and $M_j(cs) = M_j(s)$ for all $s > 0$, $j = 1, 2$, for some constant $c > 1$; the latter property will be referred to as multiplicative periodicity with period c. For $\alpha \in (0,2)$, Lévy's original description of the property in (2.4) in terms of L and R in (2.2) is that there exist nonnegative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0,\infty)$, one of which has a strictly positive infimum and the other one either has a strictly positive infimum or is identically zero, such that $L(x) = M_L(x)/|x|^\alpha$, $x < 0$, is left-continuous and nondecreasing on $(-\infty, 0)$ and $R(x) = -M_R(x)/x^\alpha$, $x > 0$, is right-continuous and nondecreasing on $(0,\infty)$, while $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x < 0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x > 0$, for the same period $c > 1$. Because of the inversions given above, the two descriptions are equivalent.

The realization of a tangible significance of $\mathcal{S}_* \supset \mathcal{S}$ starts with a remark of Doeblin (1940), without any elaboration or, for that matter, even a precise statement, to the effect that semistable laws arise in the limit in (2.3) if the normalizing constants A_{k_n} satisfy a geometric growth condition. Thirty years later, Shimizu (1970) and Pillai (1971) came close while Kruglov (1972) and Mejzler (1973) fully achieved that realization, all four of them acting independently of one another. It turned out that the following *Characterization Theorem* is true: If (2.3) holds along a subsequence $\{k_n\} \subset \mathbb{N}$ for which

$$\liminf_{n\to\infty} \frac{k_{n+1}}{k_n} = c \quad \text{for some } c \in (1,\infty), \tag{2.5a}$$

then the distribution $G_{\psi_1,\psi_2,\sigma}$ of $V(\psi_1, \psi_2, \sigma)$ is in $\mathcal{S}_*$ such that, in the case when the exponent of $G_{\psi_1,\psi_2,\sigma} = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ is $\alpha < 2$, the multiplicative period of the functions M_1 and M_2 in (2.4) is the c from (2.5a). Conversely, for every $G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$ there exists an F such that if $X_1, X_2, \ldots$ are independent random variables with the common distribution function F, then there exists a subsequence $\{k_n\} \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \quad \text{for some } c \in [1,\infty) \tag{2.5b}$$

and (2.3) holds along $\{k_n\}$. An equivalent version of this theorem, in terms of the Lévy type description of $\mathcal{S}_*$ was proved by Kruglov (1972) and Mejzler (1973), while the present version was obtained by Megyesi (2000) with an independent proof within the framework of the 'probabilistic' or 'quantile-transform' approach of Csörgő, Haeusler and Mason (1988a,b; 1991a,b) and Csörgő (1990) to domains of attraction and partial attraction.

For $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$, we say that F is in the *domain of geometric partial attraction of G with rank $c \geq 1$*, in short $F \in \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$, if (2.3) holds along a subsequence $\{k_n\} \subset \mathbb{N}$ satisfying (2.5b). Of course, the geometric subsequence $k_n \equiv \lfloor c^n \rfloor$, the integer part of c^n, is unbounded and satisfies the (quasi)geometric growth condition (2.5b) if $c > 1$. Recalling that $(\psi_1, \psi_2, \sigma) \neq (0,0,0)$ for $G = G_{\psi_1,\psi_2,\sigma} \in \mathcal{S}_*$, define $\mathbf{c} = \mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma > 0$ and $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \inf\{c > 1\colon\ M_j(cs) = M_j(s),\ s > 0,\ j = 1,2\}$, the minimal common period c of the factor functions M_1 and M_2 in ψ_1^α and ψ_2^α in (2.4) for $\alpha \in (0,2)$. Thus $\mathbf{c} = \mathbf{c}(G)$ is defined for all $G \in \mathcal{S}_*$. It turns out for the whole domain $\mathbb{D}_{\mathrm{gp}}(G) := \bigcup_{c \geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$ of geometric partial attraction of $G \in \mathcal{S}_*$ that $\mathbb{D}_{\mathrm{gp}}(G) = \bigcap_{m \in \mathbb{N}} \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^m)}(G) = \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G)$. Also, if $\mathbf{c}(G) = 1$ for $G \in \mathcal{S}_*$, then $G \in \mathcal{S}$ and $\mathbb{D}_{\mathrm{gp}}(G) = \mathbb{D}(G)$, the domain of attraction of the stable G. In other words, if $\mathbb{D}(\mathcal{S}) := \bigcup_{G \in \mathcal{S}} \mathbb{D}(G) = \bigcup_{0<\alpha\leq 2} \mathbb{D}(\alpha)$ is the classical domain of attraction and $\mathbb{D}_{\mathrm{gp}}(\mathcal{G}) := \bigcup_{G \in \mathcal{G}} \mathbb{D}_{\mathrm{gp}}(G)$ is the domain of geometric partial attraction of a class $\mathcal{G} \subset \mathcal{S}_*$, then $\mathbb{D}_{\mathrm{gp}}(\mathcal{S}) = \mathbb{D}(\mathcal{S})$. Some of these results were first proved by Mejzler (1973), all of them and related other observations are obtained by Megyesi (2000).

The first characterization of an $F \in \mathbb{D}_{\mathrm{gp}}(\mathcal{S}_*)$ was obtained by Grinevich and Khokhlov (1995). However, besides the fact that it contained an error, this characterization is in terms of the norming factors A_{k_n} in (2.3) and the tails of F, and so it is not useful when trying to apply the criterion (1.4) to trimmed sums. The following alternative characterization is due to Megyesi (2000).

Consider a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfying (2.5b). If $c = 1$ in (2.5b), then put $\gamma(s) = 1$ for every $s \in (0,1)$. If $c > 1$, then the sequence $\{k_n\}$ is eventually strictly increasing to ∞. Hence, for all $s \in (0,1)$ small enough there exists a uniquely determined $k_{n^*(s)}$ such that $k_{n^*(s)}^{-1} \leq s < k_{n^*(s)-1}^{-1}$. For any such s we define $\gamma(s) = s k_{n^*(s)}$, so that for any fixed $\varepsilon > 0$ and all $s \in (0,1)$ small enough we have $1 \leq \gamma(s) < c + \varepsilon$ for the limiting $c \geq 1$ from (2.5b). In particular, for any sequence $s_m > 0$ for which $\lim_{m\to\infty} s_m = 0$, the limit points of the sequence $\{\gamma(s_m)\}_{m=1}^\infty$ are in the interval $[1, c]$. Let $Q_+(\cdot)$ denote the right-continuous version of the quantile function $Q(\cdot)$ of the underlying distribution function $F(\cdot)$. Since $\mathbb{D}_{\mathrm{gp}}(G) = \mathbb{D}(G)$ for a normal $G \in \mathcal{S}_*$, we only have to describe the domain of geometric partial attraction of nonnormal semistable laws, for which the *Domain Theorem* is this: If $G_{\psi_1^\alpha,\psi_2^\alpha,0} \in \mathcal{S}_*$ is semistable with exponent $\alpha \in (0,2)$, so that ψ_1^α and ψ_2^α

satisfy (2.4), and $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ such that (2.3) holds for $V(\psi_1^\alpha,\psi_2^\alpha,0)$ and a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfying (2.5b), then for all $s \in (0,1)$,

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\ell(s)[M_1(\gamma(s)) + h_1(s)] \qquad \text{and} \\ Q(1-s) &= s^{-1/\alpha}\ell(s)[M_2(\gamma(s)) + h_2(s)] \end{aligned} \tag{2.6}$$

for some $\alpha \in (0,2)$, where $\ell(\cdot)$ is a right-continuous function, slowly varying at zero, and the errors h_1 and h_2 are right-continuous functions such that if M_j is continuous, then $\lim_{s\downarrow 0} h_j(s) = 0$ for the corresponding h_j, while if M_j has discontinuities, then the corresponding $h_j(s)$ may not go to zero as $s \downarrow 0$ but $\lim_{n\to\infty} h_j(t/k_n) = 0$ for every continuity point $t > 0$ of M_j, $j = 1,2$. Conversely, if for the quantile function pertaining to F the equations in (2.6) hold with the properties of ℓ and of h_1 and h_2 just described, for some $\alpha \in (0,2)$ and functions M_1 and M_2 satisfying the properties described at (2.4), and for $\gamma(\cdot)$ determined by a given subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfying (2.5b), then $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for the ψ_1^α and ψ_2^α given by (2.4), and, in particular, the relation (2.3) can be specified as

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{k_n} X_j - k_n\int_{\frac{1}{k_n}}^{1-\frac{1}{k_n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha,\psi_2^\alpha,0). \tag{2.7}$$

Finally we note that if $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for some $\alpha \in (0,2)$, so that (2.6) holds with all the properties of the ingredients described above, then it is easy to see that

$$\begin{aligned} C_1 s^{-1/\alpha}\ell(s) &< |Q_+(s)| < D_1 s^{-1/\alpha}\ell(s) \qquad \text{and} \\ C_2 s^{-1/\alpha}\ell(s) &< |Q(1-s)| < D_2 s^{-1/\alpha}\ell(s) \end{aligned} \tag{2.8}$$

for all $s > 0$ sufficiently small, where $0 \le C_1 < D_1 < \infty$ and $0 \le C_2 < D_2 < \infty$ are constants such that $C_1 + C_2 > 0$ and $C_j = 0$ if and only if $D_j > 0$ can be chosen as small as we wish, which happens if and only if $M_j(\cdot) \equiv 0$, $j = 1,2$.

3 Asymptotic normality of moderately trimmed sums from $\mathbb{D}_{\mathrm{gp}}(\mathcal{S}_*)$

Our main result is

Theorem 3.1 *Suppose that $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for some nondegenerate semistable law $G = G_{\psi_1,\psi_2,\sigma}$ such that both ψ_1 and ψ_2 are continuous on $(0,\infty)$.*

(i) If neither of ψ_1 and ψ_2 is identically zero, then for any two sequences $\{l_n\}_{n=1}^\infty$ and $\{m_n\}_{n=1}^\infty$ of positive integers satisfying (1.1),

$$\frac{1}{a_n(l_n,m_n)}\left\{\sum_{j=l_n+1}^{n-m_n} X_{j,n} - n\int_{\frac{l_n}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} Z, \tag{3.1}$$

where $a_n(l_n, m_n)$ is as in (1.3) and Z is a standard normal random variable.

(ii) If at least one of ψ_1 and ψ_2 is identically zero, then (3.1) holds true for any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying (1.1) such that

$$0 < \liminf_{n\to\infty} \frac{l_n}{m_n} \le \limsup_{n\to\infty} \frac{l_n}{m_n} < \infty. \tag{3.2}$$

In this theorem, the distribution G is either normal, i.e. $G = G_{0,0,\sigma}$ for some $\sigma > 0$, or $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$, a semistable law of exponent $\alpha \in (0,2)$ with continuous ψ_1^α and ψ_2^α satisfying (2.4). In the first case, the continuity condition is trivially satisfied and part (ii) for this case is just a restatement of part of Theorem 1 in Csörgő, Horváth and Mason (1986) when $l_n \equiv m_n$. In the second case, the two parts (i) and (ii) here extend results of Csörgő, Haeusler and Mason (1988) and Griffin and Pruitt (1989) mentioned in the introduction. By (2.4), the theorem's continuity condition is nothing but the requirement of continuity of the corresponding functions M_1 and M_2. This condition cannot be dropped in general as the example of the St. Petersburg game shows, where the underlying distribution is in the domain of geometric partial attraction of a semistable law with exponent 1 and Theorem 3.2 of Csörgő and Dodunekova (1991) shows that nonnormal limits do arise for moderately trimmed sums along subsequences of $\mathbb{N}$. The generalized St. Petersburg games considered by Csörgő and Simons (1996) in a different context and their symmetrized versions may serve to show the same for all exponents $\alpha \in (0,2)$. In terms of the Lévy functions L and R in (2.2), we see that a nonzero ψ_1^α (or ψ_2^α) is continuous, or equivalently the corresponding M_1 (or M_2) is continuous if and only if L (or R) does not have flat stretches in the sense that it is not constant on intervals with positive length.

We emphasize that even though (2.7) holds for the full sums only along a subsequence satisfying (2.5b), the convergence in (3.1) takes place along the whole $\mathbb{N}$. If the continuity condition is violated, we still have an existence result along the whole $\mathbb{N}$.

Theorem 3.2 *If $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for a nondegenerate semistable law G, then there exist two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of integers satisfying (1.1) such that (3.1) holds.*

With some extra work the proof can be modified to allow the choice $l_n \equiv m_n$. Also, if $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ for some exponent $\alpha \in (0,2)$, neither of ψ_1^α and ψ_2^α is identically zero and ψ_1^α is continuous, then there is an $\{m_n\}_{n=1}^{\infty}$ satisfying (1.1) such that (3.1) holds for every $\{l_n\}_{n=1}^{\infty}$ satisfying (1.1); an analogous statement is true when ψ_2^α is continuous.

Our last result extends Theorem 3 of Csörgő, Horváth and Mason (1986) and demonstrates that asymptotic semistability in (2.7) is determined only by arbitrarily small moderate portions of upper and lower order statistics in the sample.

Theorem 3.3 *If $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for a nonnormal semistable law $G = G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0,2)$, so that (2.7) holds along a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (2.5b), then, for the slowly varying function $\ell(\cdot)$ from (2.6) and (2.7),*

$$\text{(3.3)} \qquad \frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{l_{k_n}} X_{j,k_n} - k_n \int_{\frac{1}{k_n}}^{\frac{l_{k_n}}{n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} -W_1(\psi_1^\alpha),$$

$$\text{(3.4)} \qquad \frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=l_n+1}^{n-m_n} X_{j,n} - n \int_{\frac{l_n}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0,$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, and

$$\text{(3.5)} \quad \frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=k_n-m_{k_n}+1}^{k_n} X_{j,k_n} - k_n \int_{1-\frac{m_{k_n}}{k_n}}^{1-\frac{k_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} W_2(\psi_2^\alpha),$$

where the independent random variables $W_1(\psi_1^\alpha)$ and $W_2(\psi_1^\alpha)$ are given at (2.1), and so

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{l_{k_n}} X_{j,k_n} + \sum_{j=k_n-m_{k_n}+1}^{k_n} X_{j,k_n} - k_n\left[\int_{\frac{1}{k_n}}^{\frac{l_{k_n}}{n}} Q(u)\,du + \int_{1-\frac{m_{k_n}}{k_n}}^{1-\frac{k_n}{n}} Q(u)\,du\right]\right\} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0) = -W_1(\psi_1^\alpha) + W_2(\psi_2^\alpha)$$

for any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying (1.1).

The general theory in Csörgő, Haeusler and Mason (1988a, 1991b) and Csörgő (1990) ensures the *existence* of sequences $\{l_n\}$ and $\{m_n\}$ satisfying (1.1) for which these statements hold, the point of Theorem 3.3 is that they hold for *all* such sequences. If $M_j = \psi_j^\alpha \equiv 0$, which is allowed in (2.7) and in Theorem 3.3 above for one of the j, then of course $W_j(0) = 0$. A more general version of Theorem 3.3, in which a fixed number of the smallest and the largest extremes may be discarded from the sums in (3.3) and (3.5) is also true; the way in which the centering sequences and the limiting random variables should be changed in (3.3) and (3.5) for this version is clear from the general scheme in Csörgő, Haeusler and Mason (1988a), Csörgő (1990), or Megyesi (2000). The formulation of Theorem 3.3 above suits well the genuinely two-sided case. In the completely asymmetric case when one of ψ_1^α and ψ_2^α is identically zero, a somewhat stronger statement can be made, even in the more general version with possible light trimming: see the end of the proof of Theorem 3.3 for this in the present case of full extreme sums.

Turning now to the proofs and recalling the notation in (1.2) and the statement in (2.8), Theorem 3.1 requires the following

Lemma 3.4 *If $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, then*

(i) there exist some constants $K_1, K_2 \in (0,\infty)$ such that

$$K_1 \le \liminf_{s\downarrow 0} \frac{\sigma^2(s\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\sigma^2(s\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2\,;$$

(ii) if $C_1 > 0$ in (2.8), then there exist some constants $K_1^{(1)}, K_2^{(1)} \in (0,\infty)$ such that

$$K_1^{(1)} \le \liminf_{s\downarrow 0} \frac{\sigma^2(s\,,1/2)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\sigma^2(s\,,1/2)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2^{(1)},$$

and if $C_2 > 0$ in (2.8), then there exist some constants $K_1^{(2)}, K_2^{(2)} \in (0,\infty)$ such that

$$K_1^{(2)} \le \liminf_{s\downarrow 0} \frac{\sigma^2(1/2\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\sigma^2(1/2\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2^{(2)}.$$

Proof We follow the proof of Lemma 1 in Csörgő, Horváth and Mason (1986). Obviously the inequalities in (2.8) directly imply that

$$C_1^2 + C_2^2 \le \liminf_{s\downarrow 0} \frac{sQ^2(s) + sQ^2(1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{sQ^2(s) + sQ^2(1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le D_1^2 + D_2^2$$

and

$$\lim_{s\downarrow 0} \frac{s|Q(s)| + s|Q(1-s)|}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0.$$

Also from (2.8), similarly as at (3.11) and (3.12) in Csörgő, Horváth and Mason (1986),

$$K_1^* \le \liminf_{s\downarrow 0} \frac{\int_s^{1-s} Q^2(u)\,du}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\int_s^{1-s} Q^2(u)\,du}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2^* \tag{3.6}$$

for some constants $K_1^*, K_2^* \in (0,\infty)$ and

$$\text{(3.7)} \qquad \lim_{s\downarrow 0} \frac{\int_s^{1-s} |Q(u)|\,du}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0.$$

Using these four relations in the second formula in (1.2), the inequalities in (i) follow.

The symmetric two statements in (ii) are obtained in a similar fashion. Considering the first, for example, the first pair of inequalities in (2.8) imply that

$$C_1^2 \le \liminf_{s\downarrow 0} \frac{sQ^2(s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{sQ^2(s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le D_1^2 \text{ and } \lim_{s\downarrow 0} \frac{s|Q(s)|}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0,$$

and (3.6) and (3.7) remain true by the same argument if $1-s$ in the upper limits of the integrals is replaced by $1/2$. ∎

Proof of Theorem 3.1 To prove part (i), consider any two sequences $\{l_n\}_{n=1}^\infty$ and $\{m_n\}_{n=1}^\infty$ satisfying (1.1) and introduce the "renormalized" half-sided functions

$$\varphi_{n,l_n}^{(1)}(x) = \frac{a_n(l_n,m_n)}{a_{1,n}(l_n)}\,\varphi_{1,n}(x) \quad \text{and} \quad \varphi_{n,m_n}^{(2)}(x) = \frac{a_n(l_n,m_n)}{a_{2,n}(m_n)}\,\varphi_{2,n}(x),$$

($x \in \mathbb{R}$), the original functions $\varphi_{1,n}(\cdot)$ and $\varphi_{2,n}(\cdot)$ being given between (1.3) and (1.4), where

$$a_{1,n}(l_n) = \sqrt{n}\,\sigma\Big(\frac{l_n}{n}, \frac{1}{2}\Big) \quad \text{and} \quad a_{2,n}(m_n) = \sqrt{n}\,\sigma\Big(\frac{1}{2}, 1-\frac{m_n}{n}\Big).$$

Since none of $\psi_1 = \psi_1^\alpha$ and $\psi_1 = \psi_2^\alpha$ is zero anywhere, $0<\alpha<2$, it follows from (2.6) and (1.2) that $a_{1,n}(l_n), a_{2,n}(m_n) > 0$, and so the definitions of the renormalized functions are meaningful for all n large enough and, of course, $a_{1,n}(l_n), a_{2,n}(m_n) \le a_n(l_n,m_n)$. Hence, to prove (1.4), it suffices to show that

$$\text{(3.8)} \qquad \lim_{n\to\infty} \varphi_{n,l_n}^{(1)}(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi_{n,m_n}^{(2)}(x) = 0 \quad \text{for every } x \in \mathbb{R}.$$

To deal with $\varphi_{n,l_n}^{(1)}(x)$ at any fixed $x \in \mathbb{R}$, note that by the domain theorem at (2.6),

$$Q\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big) = -\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big)^{-\frac{1}{\alpha}} \ell\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big) \times \Big[M_1\Big(\gamma\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\Big)\Big]$$

and

$$Q\Big(\frac{l_n}{n}\Big) = -\Big(\frac{l_n}{n}\Big)^{-\frac{1}{\alpha}}\ell\Big(\frac{l_n}{n}\Big)\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]$$

for all n large enough. We substitute these into the formula for $\varphi^{(1)}_{n,l_n}(x)$ through the formula given for $\varphi_{1,n}(x)$. Using then the fact that

$$\begin{aligned} K := \sqrt{K_1^{(1)}} &\le \liminf_{n\to\infty} \frac{\sigma(l_n/n\,,1/2)}{(l_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(l_n/n)} \\ &\le \limsup_{n\to\infty} \frac{\sigma(l_n/n\,,1/2)}{(l_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(l_n/n)} \le \sqrt{K_2^{(1)}} \end{aligned}$$

by the first statement of Lemma 3.4(ii), for all n large enough we obtain

$$\begin{aligned} \big|\varphi^{(1)}_{n,l_n}(x)\big| &\le \frac{2}{K}\frac{\big(\frac{l_n}{n}\big)^{\frac{1}{\alpha}}}{\ell\big(\frac{l_n}{n}\big)}\left|\frac{\ell\big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\big)}{\big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\big)^{\frac{1}{\alpha}}}\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big)\right. \\ &\qquad \left. + h_1\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big] - \frac{\ell\big(\frac{l_n}{n}\big)}{\big(\frac{l_n}{n}\big)^{\frac{1}{\alpha}}}\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]\right| \\ &= \frac{2}{K}\left|u_n(x)\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big] - \Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]\right|, \end{aligned}$$

where

$$u_n(x) = \Big(1+\frac{x}{\sqrt{l_n}}\Big)^{-\frac{1}{\alpha}}\frac{\ell\big(\frac{l_n}{n}\big[1+\frac{x}{\sqrt{l_n}}\big]\big)}{\ell\big(\frac{l_n}{n}\big)} \to 1$$

by the slow variation of $\ell(\cdot)$ at zero. Since $M_1(\cdot)$ is bounded, we see, therefore, that the first convergence in (3.8) will follow if we show that

$$\text{(3.9)} \qquad \big|[M_1(\gamma(t_n(x))) + h_1(t_n(x))] - [M_1(\gamma(s_n)) + h_1(s_n)]\big| \to 0,$$

where $s_n = l_n n^{-1} \to 0$ and $t_n(x) = l_n n^{-1} + x\sqrt{l_n}\,n^{-1} = s_n[1 + x l_n^{-1/2}] \to 0$. Since also, as a result of our continuity assumption, $\lim_{s\downarrow 0} h_1(s) = 0$ by the domain theorem at (2.6), and $t_n(0) = s_n$ of course, for (3.9) it suffices to show that

$$\text{(3.10)} \qquad v_n(x) := \big|M_1(\gamma(t_n(x))) - M_1(\gamma(s_n))\big| \to 0 \quad \text{for each } x \ne 0.$$

Let $c \ge 1$ be the limit in (2.5b) for the sequence $\{k_n\}_{n=1}^{\infty}$ which defines $\gamma(\cdot)$ preceding (2.6). We may and do assume that $c > 1$ since in the case of $c = 1$, when $F \in \mathbb{D}(\alpha)$ for the given $\alpha \in (0,2)$ at hand and $M_1(\cdot)$ is a constant function, (3.10) is trivial. Then for all n large enough, $\gamma(s_n), \gamma(t_n(x)) \in [1, c^2]$, say, the definitions

$$\gamma_n(x) := \begin{cases} \gamma(t_n(x)), & \text{if } \gamma(t_n(x)) > \gamma(s_n), \\ c\,\gamma(t_n(x)), & \text{if } \gamma(t_n(x)) \leq \gamma(s_n), \end{cases} \qquad \text{for } x > 0,$$

and

$$\gamma_n(x) := \begin{cases} \gamma(t_n(x)), & \text{if } \gamma(t_n(x)) < \gamma(s_n), \\ \frac{1}{c}\,\gamma(t_n(x)), & \text{if } \gamma(t_n(x)) \geq \gamma(s_n), \end{cases} \qquad \text{for } x < 0,$$

are meaningful and $c^{-1} \leq \gamma_n(x) \leq c^2$ for $x < 0$ and $1 \leq \gamma_n(x) \leq c^3$ for $x > 0$. Since $M_1(\gamma(t_n(x))) = M_1(\gamma_n(x))$ by the multiplicative periodicity of $M_1(\cdot)$, we have $v_n(x) = |M_1(\gamma_n(x)) - M_1(\gamma(s_n))|$ and, using the continuity condition for the second time, the function $M_1(\cdot)$ is uniformly continuous on the closed interval $[c^{-1}, c^3]$. Now, based on the definition of $\gamma(\cdot)$ above (2.6), the asymptotic equality

$$\frac{\gamma_n(x)}{\gamma(s_n)} \sim \frac{t_n(x)}{s_n}, \quad \text{where} \quad \frac{t_n(x)}{s_n} \to 1,$$

can be shown by elementary arguments, which since the sequence $\{\gamma(s_n)\}$ is bounded, implies that $|\gamma_n(x) - \gamma(s_n)| \to 0$. The uniform continuity of $M_1(\cdot)$ then implies (3.10), proving the first statement in (3.8). Using the second statement of Lemma 3.4(ii), the proof of the second statement in (3.8) is completely analogous, and hence we have part (i) of the theorem.

Condition (3.2) for part (ii) of the theorem implies the existence of some finite positive constants $A_1 < 1 < A_2$ such that $A_1 m_n \leq l_n \leq A_2 m_n$ and $A_2^{-1} l_n \leq m_n \leq A_1^{-1} l_n$ for all n large enough. When proving (1.4), we renormalize $\varphi_{1,n}(\cdot)$ and $\varphi_{2,n}(\cdot)$ replacing $a_n(l_n, m_n)$ in the denominator by the sequences $a_n(A_1^{-1} l_n, A_1^{-1} l_n) \leq a_n(l_n, m_n)$ and $a_n(A_2 m_n, A_2 m_n) \leq a_n(l_n, m_n)$ to obtain the present versions of $\varphi^{(1)}_{n,l_n}(\cdot)$ and $\varphi^{(2)}_{n,m_n}(\cdot)$ of the proof above, respectively. For unified notation, we write $r_{1,n} = l_n$ and $r_{2,n} = m_n$.

Part (ii) itself has two cases. When $G = G_{0,0,\sigma}$ is normal for some $\sigma > 0$, we see by the criterion (1.26a) in Corollary 1 of Csörgő, Haeusler and Mason (1988a) for the domain of attraction of a normal distribution that both terms in $\varphi^{(j)}_{n,r_{j,n}}(x)$ go to zero separately at every $x \in \mathbb{R}$, $j = 1, 2$, and hence (3.8) holds and implies (1.4) again.

Finally, the other case of part (ii) is when one of $M_1(\cdot)$ and $M_2(\cdot)$ in (2.4) and (2.6) is identically zero while the other is nowhere zero. Replacing K by $\sqrt{K_1}$ of part (i) of Lemma 3.4, the proof of (3.8) for that one of the present two sequences $\{\varphi^{(j)}_{n,r_{j,n}}(\cdot)\}$ for which $M_j(\cdot) > 0$, $j \in \{1, 2\}$, is practically the same as the one above for case (i), while it is simpler for the other $j \in \{1, 2\}$

for which $M_j(\cdot) = 0$ because (3.9) for that $M_j(\cdot)$ is trivial. Thus condition (1.4) for asymptotic normality holds true once more. ■

We also separate two lemmas for the proofs of Theorems 3.2 and 3.3, respectively.

Lemma 3.5 *Suppose that $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ with a quantile function given by (2.6). Then for any $a,b \in [1,c)$, $a < b$, that are continuity points of both M_j, $j = 1,2$, and for any $\delta > 0$ and $\varepsilon \in (0,1)$ there exists a threshold number $N(a,b,\delta,\varepsilon)$ such that the inequality*

$$\begin{aligned}
&|M_{n,j}^*(y_1) - M_{n,j}^*(y_2)| \\
&\quad = \left| \left[M_j\left(\gamma\left(\frac{y_1}{k_n}\right)\right) + h_j\left(\frac{y_1}{k_n}\right)\right] - \left[M_j\left(\gamma\left(\frac{y_2}{k_n}\right)\right) + h_j\left(\frac{y_2}{k_n}\right)\right]\right| \\
&\quad \le 2\left[|M_j(a) - M_j(b)| + C(a,b,\alpha,\varepsilon) + \delta\right]
\end{aligned} \tag{3.11}$$

holds true for all $n \ge N(a,b,\delta,\varepsilon)$ and $y_1, y_2 \in [a,b]$, $j = 1,2$, where

$$C(a,b,\alpha,\varepsilon) = \max\{D_1, D_2\}\left[1 - (1-\varepsilon)\left(\frac{a}{b}\right)^{1/\alpha}\right]$$

with the constants D_1 and D_2 from (2.8) and $M_{n,j}^*(\cdot) = M_j(\gamma(\cdot/k_n)) + h_j(\cdot/k_n)$, $j = 1,2$.

Proof Notice first that (3.11) is trivial if $M_j \equiv 0$. Thus, since the half-sided version of the proof below will be an obvious special case when exactly one $M_j \equiv 0$, it suffices to deal with the situation when $M_j \not\equiv 0$, $j = 1,2$. In this situation M_1 and M_2 both have positive infima on $(0,\infty)$ and we see by applying (2.6) for $s = t/k_n$, where $t > 0$ is a continuity point of M_1 and M_2, and by the monotone nondecreasing nature of Q that $\lim_{s\downarrow 0} Q(s) = -\infty$ and $\lim_{s\uparrow 1} Q(s) = \infty$. We choose $N^* = N(a,b,\delta,\varepsilon)$ so that, for $j = 1,2$,

$$Q_+\left(\frac{b}{k_n}\right) < 0 \quad \text{and} \quad Q\left(1 - \frac{b}{k_n}\right) > 0, \quad \text{and so} \quad M_{n,j}^*(y) > 0, \;\; a \le y \le b,$$

$$\gamma(a/k_n) = a \quad \text{and} \quad \gamma(b/k_n) = b\,,$$

$$\left|1 - \frac{\ell(y_1/k_n)}{\ell(y_2/k_n)}\right| < \varepsilon, \;\; y_1, y_2 \in [a,b], \quad \text{and} \quad \left|h_j\left(\frac{a}{k_n}\right)\right| + \left|h_j\left(\frac{b}{k_n}\right)\right| < \delta,$$

hold simultaneously whenever $n \ge N^*$ and show (3.11) with this choice of the threshold.

Assuming without loss of generality that $a \le y_1 \le y_2 \le b$, notice that

$$\frac{Q_+\left(\frac{y_1}{k_n}\right)}{Q_+\left(\frac{y_2}{k_n}\right)} \ge 1 \;\; \text{and} \;\; \frac{Q\left(1 - \frac{y_1}{k_n}\right)}{Q\left(1 - \frac{y_2}{k_n}\right)} \ge 1, \quad \text{that is,} \quad \frac{\left(\frac{y_1}{k_n}\right)^{-\frac{1}{\alpha}} \ell\left(\frac{y_1}{k_n}\right) M_{n,j}^*(y_1)}{\left(\frac{y_2}{k_n}\right)^{-\frac{1}{\alpha}} \ell\left(\frac{y_2}{k_n}\right) M_{n,j}^*(y_2)} \ge 1,$$

($j = 1, 2$), and so

$$\frac{M^*_{n,j}(y_1)}{M^*_{n,j}(y_2)} \geq \left(\frac{y_1}{y_2}\right)^{1/\alpha} \frac{\ell(y_1/k_n)}{\ell(y_2/k_n)} \geq \left(\frac{a}{b}\right)^{1/\alpha} (1-\varepsilon), \;\; j = 1, 2,$$

if $n \geq N^*$. Recalling (2.8) we see that for $n \geq N^*$ the inequality

$$|M^*_{n,j}(y_1) - M^*_{n,j}(y_2)| \leq C(a, b, \alpha, \varepsilon) \tag{3.12}$$

holds true for any choice of $y_1, y_2 \in [a, b]$, $y_1 \leq y_2$, provided $M^*_{n,j}(y_1) \leq M^*_{n,j}(y_2)$, $j = 1, 2$. If this is the case indeed, then (3.12) in itself proves (3.11), but the following considerations apply in general. Indeed, observe that the choice of N^* ensures that

$$|M^*_{n,j}(a) - M^*_{n,j}(b)| < |M_j(a) - M_j(b)| + \delta, \quad j = 1, 2, \tag{3.13}$$

for all $n \geq N^*$, and note also that

$$|M^*_{n,j}(y_1) - M^*_{n,j}(y_2)| \leq |M^*_{n,j}(a) - M^*_{n,j}(y_1)| + |M^*_{n,j}(a) - M^*_{n,j}(y_2)|.$$

Here $|M^*_{n,j}(a) - M^*_{n,j}(y_l)| \leq C(a, b, \alpha, \varepsilon)$ by (3.12) if $M^*_{n,j}(a) \leq M^*_{n,j}(y_l)$, $l, j \in \{1, 2\}$, but if this fails for some $l, j \in \{1, 2\}$ then we still have

$$|M^*_{n,j}(a) - M^*_{n,j}(y_l)| \leq |M^*_{n,j}(a) - M^*_{n,j}(b)| + |M^*_{n,j}(b) - M^*_{n,j}(y_l)|,$$

where the first term can be estimated using (3.13) and $C(a, b, \alpha, \varepsilon)$ is an upper bound on the second one, provided $M^*_{n,j}(y_l) \leq M^*_{n,j}(b)$. However, if the latter inequality is not the case either, then $|M^*_{n,j}(a) - M^*_{n,j}(y_l)| \leq |M^*_{n,j}(a) - M^*_{n,j}(b)|$, since $M^*_{n,j}(a) > M^*_{n,j}(y_l) > M^*_{n,j}(b)$. All this together imply (3.11). ∎

Proof of Theorem 3.2 We only have to deal with the case when $G = G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ for some $\alpha \in (0, 2)$, where at least one of ψ_1^α and ψ_2^α is not identically 0. The other case being analogous, suppose that $\psi_1^\alpha \not\equiv 0$. Retaining the notation in the proof of Theorem 3.1, we show that a sequence $\{l_n\}$ of positive integers can be chosen to satisfy both (1.1) and the first convergence relation in (3.8). The latter follows, through the same considerations as there, if $\{l_n\}$ is chosen to make sure that (3.9) holds.

By the monotonicity of ψ_1^α we can pick a sequence of pairs (a_j, b_j), $1 \leq a_j < b_j < c$, and constants $\varepsilon_j \in (0, 1)$ such that both a_j and b_j are continuity points of M_1 and the inequalities $|M_1(a_j) - M_1(b_j)| < \frac{1}{6j}$ and $C(a_j, b_j, \alpha, \varepsilon_j) < \frac{1}{6j}$ hold for all $j \in \mathbb{N}$. Next, put $N_0^\diamond := 0$ and, by means of the threshold numbers of Lemma 3.5, define an increasing sequence $\{N_j^\diamond\}$ of positive integers by setting $N_j^\diamond := \max\{N(a_j, b_j, \frac{1}{6j}, \varepsilon_j), N_{j-1}^\diamond + 1\}$, $j \in \mathbb{N}$. Elementary consideration shows now that for each $j \in \mathbb{N}$ there exists a threshold number $N_j^* \in \mathbb{N}$ such that for every $n \geq N_j^*$ one can choose an $l_{n,j}^* \in \mathbb{N}$ with the properties that

(3.14)
$$\left[\frac{l_{n,j}^*}{n} - j\frac{\sqrt{l_{n,j}^*}}{n}, \frac{l_{n,j}^*}{n} + j\frac{\sqrt{l_{n,j}^*}}{n}\right] \subset \left[\frac{a_j}{k_{N_j^\diamond}}, \frac{b_j}{k_{N_j^\diamond}}\right] \quad \text{and} \quad l_{n+1,j}^* \geq l_{n,j}^*,$$

and it can clearly be stipulated that $1 < N_1^* < N_2^* < \cdots$. By Lemma 3.5 we see that

$$\left|\left[M_1\left(\gamma\left(\frac{l_{n,j}^*}{n}+x\frac{\sqrt{l_{n,j}^*}}{n}\right)\right)+h_1\left(\frac{l_{n,j}^*}{n}+x\frac{\sqrt{l_{n,j}^*}}{n}\right)\right]-\left[M_1\left(\gamma\left(\frac{l_{n,j}^*}{n}\right)\right)+h_1\left(\frac{l_{n,j}^*}{n}\right)\right]\right| \leq \frac{1}{j}$$

for all $x \in [-j, j]$ and $n \geq N_j^*$.

Now we are ready to choose the desired sequence $\{l_n\}$. We set $l_n := 1$ for $n < N_1^*$ and define $\{l_n\}_{n=N_1^*}^{\infty}$ by the following algorithm, in which $T \in \mathbb{N}$ is a new auxiliary variable:

Step 1. Let the initial values of j and n be $j := 1$ and $n := N_1^*$, and put $T := N_1^*$.

Step 2. If $N_j^* \leq n < N_{j+1}^*$ then let $l_n := l_{n,j}^*$.

Step 3. If $n \geq N_{j+1}^*$ then put $l_n := l_{n,j}^*$ or $l_n := l_{n,j+1}^*$ according as $l_{n,j+1}^* \leq l_T$ or $l_{n,j+1}^* > l_T$, and if $l_{n,j+1}^* > l_T$ then set also $j := j + 1$ and $T := n$.

Step 4. Set $n := n + 1$ and go to Step 2.

Then $l_n \to \infty$ by the choices of T and, since $N_j^\diamond \to \infty$ as $j \to \infty$, we also have $l_n/n \to 0$ by (3.14). Thus (1.1) holds for the chosen sequence $\{l_n\}$ and the displayed inequality following (3.14) above shows that (3.9) is also satisfied for any fixed $x \in \mathbb{R}$.

If $\psi_2^\alpha \not\equiv 0$, then the sequence $\{m_n\}$ can be chosen in a similar fashion. If $\psi_2^\alpha \equiv 0$, then simply put $m_n := l_n$ for every $n \in \mathbb{N}$, and the desired asymptotic normality follows as in the proof of part (ii) of Theorem 3.1. ∎

Lemma 3.6 *If a function $\ell(\cdot)$ on $(0,1)$ is slowly varying at zero and $\{r_n\}$ is a sequence of positive numbers such that $r_n \to \infty$ and $r_n/n \to 0$, then*

$$\frac{(r_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(r_n/n)}{(1/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(1/n)} \to 0.$$

If, in addition, $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, then

$$\frac{\sigma\left(\frac{r_n}{n}, 1-\frac{r_n}{n}\right)}{(1/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(1/n)} \to 0.$$

Proof The first statement is just a special case of Lemma 2 in Csörgő, Horváth and Mason (1986), while the second follows from the first and part (i) of Lemma 3.4. ∎

Proof of Theorem 3.3 We see by (2.8) and the first statement of Lemma 3.6 that for all $x \in \mathbb{R}$,

$$\frac{\sqrt{l_{k_n}}\, Q_+\Big(\frac{l_{k_n}}{k_n} + x\frac{\sqrt{l_{k_n}}}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0 \quad \text{and} \quad \frac{\sqrt{m_{k_n}}\, Q\Big(1 - \frac{m_{k_n}}{k_n} + x\frac{\sqrt{m_{k_n}}}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0.$$

In fact, these convergences take place along the whole sequence $\{n\} = \mathbb{N}$. Next, according to Megyesi (2000), since the main source of the domain theorem at (2.6) is that $M_j(\gamma(y/k_n)) \to M_j(y)$ for every continuity point $y > 0$ of M_j, we have

$$\text{(3.15)} \qquad \frac{Q_+\Big(\frac{y}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to \psi_1^\alpha(y) \quad \text{and} \quad \frac{-Q\Big(1 - \frac{y}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to \psi_2^\alpha(y)$$

at all the respective continuity points $y > 0$ of the limiting functions. Furthermore, Lemma 3.4(i) implies that

$$\limsup_{n\to\infty} \frac{\sqrt{k_n}\, \sigma\Big(\frac{1}{k_n}, \frac{l_{k_n}}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \le \sqrt{K_2},$$

$$\limsup_{n\to\infty} \frac{\sqrt{k_n}\, \sigma\Big(1 - \frac{m_{k_n}}{k_n}, 1 - \frac{1}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \le \sqrt{K_2}.$$

Finally, Lemma 3.6 implies

$$\frac{\sqrt{k_n}\, \sigma\Big(\frac{r_{k_n}}{k_n}, \frac{l_{k_n}}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0 \quad \text{and} \quad \frac{\sqrt{k_n}\, \sigma\Big(1 - \frac{m_{k_n}}{k_n}, 1 - \frac{r_{k_n}}{k_n}\Big)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0$$

for any sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$, $r_n/l_n \to 0$ and $r_n/m_n \to 0$; in fact, these are true along the whole $\mathbb{N}$ again.

These four pairs of facts allow a subsequential application of that variant of a two-sided version of Theorem 1 in Csörgő, Haeusler and Mason (1991a), the version alluded to on p. 789 there, in which the basic functions $Q_+(s)$ and $Q(1-s)$, $0 < s < 1$, are taken right-continuous and the Poisson processes $N_1(\cdot)$ and $N_2(\cdot)$ are taken left-continuous as in the present paper. Using the eight facts above, this variant implies that every subsequence of $\mathbb{N}$ contains a further subsequence such that (3.3) and (3.5) hold jointly along that subsequence. This implies that (3.3) and (3.5) hold jointly as stated.

By the convergence of types theorem, (3.3) and (3.5) already imply (3.4) for the subsequence $\{k_n\}$. However, if neither of M_1 and M_2, or equivalently, neither of ψ_1^α and ψ_2^α is identically zero, then the left side of (3.9) is bounded, by $2(D_1 + D_2)$ from (2.8), for both M_1 and M_2 even if they are not continuous, implying that the two sequences of functions in (3.8) are pointwise bounded. Hence the same is true for the sequences $\{\varphi_{j,n}(\cdot)\}$, $j \in \{1,2\}$. Also, setting $r_n \equiv \min(l_n, m_n)$, we have $a_n(l_n, m_n) \le a_n(r_n, r_n)$ for all $n \in \mathbb{N}$ and $a_n(r_n, r_n)/[n^{1/\alpha}\ell(1/n)] \to 0$ by Lemma 3.6. Therefore, the discussion at (1.13) in Csörgő, Haeusler and Mason (1988b) yields (3.4) as stated.

If, on the other hand, $M_1 \equiv 0$ and $M_2 \not\equiv 0$, then by the same argument

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=m_n+1}^{n-m_n} X_{j,n} - n\int_{\frac{m_n}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

and, since in this case the first convergence in (3.15) takes place along the whole $\{n\} = \mathbb{N}$ with an identically zero limiting function, we also get

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=1}^{r_n} X_{j,n} - n\int_{\frac{1}{n}}^{\frac{r_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0,$$

for both $r_n \equiv l_n$ and $r_n \equiv m_n$, which together prove (3.4).

We see that if $M_1(\cdot) \equiv 0$ and $M_2(\cdot) > 0$, then in fact we have

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=1}^{n-m_n} X_{j,n} - n\int_{\frac{1}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

along with (3.5). Similarly, if $M_2(\cdot) \equiv 0$ and $M_1(\cdot) > 0$, then again we have (3.4) and, in fact,

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=l_n+1}^{n} X_{j,n} - n\int_{\frac{l_n}{n}}^{1-\frac{1}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

along with (3.3). ∎

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