## Index

Affine dependence:
invariance of, 405
measure of, 404, 418, 419
between random vectors, 403
Affinely equivalent, 404
Almost invariant function, 287
Ancillary, 285
Ancillary statistic, 465
Angles between subspaces:
definition, 61
geometric interpretation, 61
Action group, see Group

## Beta distribution:

definition, 320
noncentral, 320
relation to $F, 320$
Beta random variables, products of, 236,321, 323
Bilinear, 33
Bivariate correlation coefficient:
density has monotone likelihood ratio, 459
distribution of, 429, 432
Borel $\sigma$-algebra, 70
Borel measurable, 71
Canonical correlation coefficients:
as angles between subspaces, 408, 409
definition, 408
density of, 442
interpretation of sample, 421
as maximal correlations, 413
model interpretation, 456
and prediction, 418
population, 408
sample, as maximal invariant, 425
Canonical variates:
definition, 415
properties of, 415
Cauchy-Schwarz Inequality, 26
Characteristic function, 76
Characteristic polynomial, 44
Chi-square distribution:
definition, 109, 110
density, 110, 111
Compact group, invariant integral on, 207
Completeness:
bounded, 466
independence, sufficiency and, 466
Complex covariance structure:
discussion of, 381
example of, 370
Complex normal distribution:
definition, 374, 375
discussion of, 373
independence in, 378
relation to real normal distribution, 377
Complex random variables:
covariance of, 372
covariance matrix of, 375
mean of, 372
variance of, 372
Complex vector space, 39
Complex Wishart distribution, 378
Conditional distribution:
for normal variables, 116, 117
in normal random matrix, 118

Conjugate transpose, 39
Correlation coefficient, density of in normal sample, 329
Covariance:
characterization of, 75
of complex random variables, 372
definition, 74
of outer products, 96,97
partitioned, 86
properties of, 74
of random sample, 89
between two random variables, 28
Covariance matrix, 73
Cyclic covariance:
characterization of, 365
definition, 362
diagonalization of, 366
multivariate, 368

Density, of maximal invariant, 272, 273-277
Density function, 72
Determinant:
definition, 41
properties of, 41
Determinant function:
alternating, 40
characterization of, 40
definition, 39
as $n$-linear function, 40
Direct product, 212
Distribution, induced, 71
Eigenvalue:
and angles between subspaces, 61
definition, 44
of real linear transformations, 47
Eigenvector, 45
Equivariant, 218
Equivariant estimator, in simple linear model, 157
Equivariant function:
description of, 249
example of, 250
Error subspace, see Linear model
Estimation:
Gauss-Markov Theorem, 134
linear, 133
of variance in linear model, 139
Expectation, 71

Factorization, see Matrix
$F$ Distribution:
definition, 320
noncentral, 320
relation to beta, 320
$F$ test, in simple linear model, 155
Gauss-Markov estimator:
definition, 135
definition in general linear model, 146
discussion of, 140-143
equal to least squares, 145
existence of, 147
in $k$-sample problem, 148
for linear functions, 136
in MANOVA, 151
in regression model, 135
in weakly spherical linear model, 134
Generalized inverse, 87
Generalized variance:
definition, 315
example of, 298
Gram-Schmidt Orthogonalization, 15
Group:
action, 186, 187
affine, 187, 188
commutative, 185
definition, 185
direct product, 212
general linear, 186
isotropy subgroup, 191
lower triangular, 185, 186
normal subgroup, 189, 190
orthogonal, 185
permutation matrices, 188, 189
sign changes, 188,189
subgroup, 186
topological, 195
transitive, 191
unimodular, 200
upper triangular, 186
Hermitian matrix, 371
Homomorphism:
definition, 218
on lower triangular matrices, 230, 231
matrices, on non-singular, 230
Hoteling's $T^{2}$ :
complex case of, 381
as likelihood ratio statistic, 402

Hypothesis testing, invariance in, 263

## Independence:

in blocks, testing for, 446
characterization of, 78
completeness, sufficiency and, 466
decomposition of test for, 449
distribution of likelihood ratio test for, 447
likelihood ratio test for, 444,446
MANOVA model and, 453, 454
of normal variables, 106, 107
of quadratic forms, 114
of random vectors, 77
regression model and, 451
sample mean and sample covariance, 126, 127
testing for, 443, 444
Inner product:
definition, 14
for linear transformations, 32
norm defined by, 14
standard, 15
Inner product space, 15
Integral:
definition, 194
left invariant, 195
right invariant, 195
Intraclass covariance:
characterization of, 355
definition, 131,356
multivariate version of, 360
Invariance:
in hypothesis testing, 263
and independence, 289
of likelihood ratio, 263
in linear model, 296, 256, 257
in MANOVA model with block covariance, 353
in MANOVA testing problem, 341
of maximum likelihood estimators, 258
Invariance and independence, example of, 290-291, 292-295
Invariant densities:
definition, 254
example of, 255
Invariant distribution:
example of, 282-283
on nxp matrices, 235
representation of, 280
Invariant function:
definition, 242
maximal invariant, 242
Invariant integral:
on affine group, 202, 203
on compact group, 207
existence of, 196
on homogeneous space, 208, 210
on lower triangular matrices, 200
on matrices (nxp of rank p), 213-218
on $m$-frames, 210, 211
and multiplier, 197
on non-singular matrices, 199
on positive definite matrices, 209, 210
relatively left, 197
relatively right, 197, 198
uniqueness of, 196
see also Integral
Invariant measure, on a vector space, 121-122
Invariant probability model, 251
Invariant subspace, 49
Inverse Wishart distribution:
definition, 330
properties of, 330
Isotropy subgroup, 191
Jacobian:
definition, 166
example of, 168, 169, 171, 172, 177
Kronecker product:
definition, 34
determinant of, 67
properties of, 36,68
trace of, 67
Lawley-Hotelling trace test, 348
Least squares estimator:
definition, 135
equal to Gauss-Markov estimator, 145
in $k$-sample problem, 148
in MANOVA, 151
in regression model, 135
Lebesque measure, on a vector space, 121-125
Left homogeneous space:
definition, 207
relatively invariant integral on, 208-210
Left translate, 195
Likelihood ratio test:
decomposition of in MANOVA, 349
definition, 263

Likelihood ratio test (Continued)
in MANOVA model with block covariance, 351
in MANOVA problem 340
in mean testing problem, 384, 390
Linear isometry:
definition, 36
properties of, 37
Linear model:
error subspace, 133
error vector, 133
invariance in, 156, 157, 256, 257
with normal errors, 137
regression model, 132
regression subspace, 133
weakly spherical, 133
Linear transformation:
adjoint of, 17, 29
definition, 7
eigenvalues, see Eigenvalues
invertible, 9
matrix of, 10
non-negative definite, 18
null space of, 9
orthogonal, 18
positive definite, 18
range of, 9
rank, 9
rank one, 19
self-adjoint, 18
skew symmetric, 18
transpose of, 17
vector space of, 7
Locally compact, 194
MANOVA:
definition, 150
maximum likelihood estimator in, 151
with normal errors, 151
MANOVA model:
with block diagonal covariance, 350
canonical form of, 339
with cyclic covariance, 366
description of, 336
example of, 308
and independence, 453, 454
with intraclass covariance, 356
maximum likelihood estimators in, 337
under non-normality, 398
with non-normal density, 462
testing problem in, 337

MANOVA testing problem:
canonical form of, 339
complex case of, 379
description of, 337
with intraclass covariance structure, 359
invariance in, 341
likelihood ratio test in, 340, 347
maximal invariant in, 342,346
maximal invariant parameter in, 344
uniformly most powerful test in, 345
Matric $t$ distribution:
definition, 330
properties of, 330
Matrix:
definition, 10
eigenvalue of, 44
factorization, 160, 162, 163, 164
lower triangular, 44, 159
orthogonal, 25
partitioned positive definite, 161, 162
positive definite, 25
product, 10
skew symmetric, 25
symmetric, 25
upper triangular, 159
Maximal invariant:
density of, 278-279
example of, 242, 243, 246
parameter, 268
and product spaces, 246
representing density of, 272
Maximum likelihood estimator:
of covariance matrix, 261
invariance of, 258
in MANOVA model, 151
in simple linear model, 138
Mean value, of random variable, 72
Mean vector:
of coordinate random vector, 72
definition, 72
for outer products, 93
properties of, 72
of random sample, 89
$M$-frame, 38
Modulus, right hand, 196
Monotone likelihood ratio:
for non-central chi-squared, 468, 469
for non-central $F, 469$
for non-central Student's :t,470
and totally positive of order 2,467
Multiple correlation coefficient:
definition, 434
distribution of, 434
Multiplier:
on affine group, 204
definition, 197
and invariant integral, 197
on lower triangular matrices, 201
on non-singular matrices, 199
Multivariate beta distribution:
definition, 331
properties of, 331, 332
Multivariate $F$ distribution:
definition, 331
properties of, 331
Maximal invariant, see Invariant function
Multivariate General Linear Model, see MANOVA

Non-central chi-squared distribution:
definition, 110, 111
for quadratic forms, 112
Noncentral Wishart distribution:
covariance of, 317
definition, 316
density of, 317
mean of, 317
properties of, 316
as quadratic form in normals, 318
Normal distribution:
characteristic function of, 105, 106
complex, see Complex normal distribution
conditional distribution in, 116, 117
covariance of, 105, 106
definition, 104
density of, 120-126
density of normal matrix, 125
existence of, 105, 106
independence in 106, 107
mean of, 105,106
and non-central chi-square, 111
and quadratic forms, 109
relation to Wishart, 307
representation of, 108
for random matrix, 118
scale mixture of, 129, 130
sufficient statistic in, 126, 127, 131
of symmetric matrices, 130
Normal equations, 155, 156
Orbit, 241
Order statistic, 276-277

Orthogonal:
complement, 16
decomposition, 17
definition, 15
Orthogonal group, definition, 23
Orthogonally invariant distribution, 81
Orthogonally invariant function, 82
Orthogonal projection:
characterization of, 21
definition, 17
in Gauss-Markov Theorem, 134
random, 439, 440
Orthogonal transformation, characterization of, 22
Orthonormal:
basis, 15
set, 15
Outer product:
definition, 19
properties of, 19, 30
Parameter, maximal invariant, 268
Parameter set:
definition, 146
in linear models, 146
Parameter space, 252
Partitioning, a Wishart matrix, 310
Pillai trace test, 348
Pitman estimator, 264-267
Prediction:
affine, 94
and affine dependence, 416
Principal components:
and closest flat, 457, 458
definition, 457
low rank approximation, 457
Probability model, invariant, 251
Projection:
characterization of, 13
definition, 12
Quadratic forms:
independence of, 114,115
in normal variables, 109
Radon measure:
definition, 194
factorization of, 224
Random vector, 71
Regression:
multivariate, 451

Regression(Continued)
and testing for independence, 451
Right translate, 195
Roy maximum root test, 348,349
Regression model, see Linear model
Regression subspace, see Linear model
Relatively invariant integral, see Invariant integral

Sample correlation coefficient, as a maximal invariant, 268-271
Scale mixture of normals, 129, 130
Self adjoint transformations, functions of, 52
Singular Value Decomposition Theorem, 58
Spectral Theorem:
and positive definiteness, 51
statement of, 50, 53
Spherical distributions, 84
Sufficiency:
completeness, independence and, 466
definition, 465
Symmetry model:
definition, 361
examples of, 361

Topological group, 195
Trace:
of linear transformation, 47
of matrix, 33
sub-k, 56
Transitive group action, 191
Two way layout, 155

Uniform distribution:
on $M$-frames, 234
on unit sphere, 101
Uncorrelated:
characterization of, 98
definition, 87
random vectors, 88

Uniformly most powerful invariant test, in MANOVA problem, 345

Vector space:
basis, 3
complementary subspaces, 5
definition, 2
dimension, 4
direct sum, 6
dual space, 8
finite dimensional, 3
linearly dependent, 3
linearly independent, 3
linear manifold, 4
subspace, 4
Weakly spherical:
characterization of, 83
definition, 83
linear model, 133
Wishart constant, 175
Wishart density, 239, 240
Wishart distribution:
characteristic function of, 305
convolution of two, 306
covariance of, 305
definition, 303
density of, 239, 240
for nonintegral degrees of freedom, 329
in MANOVA model, 308
mean of, 305
noncentral, see Noncentral Wishart distribution
nonsingular, 304
partitioned matrix and, 310
of quadratic form, 307
representation of, 303
triangular decomposition of, 313, 314
Wishart matrix:
distribution of partitioned, 311
ratio of determinants of, 319

