We shall now say that P is <u>reducible</u> to Q if

$$P(\vec{\alpha},\vec{x}) \longleftrightarrow Q(\lambda y G_1(y,\vec{\alpha},\vec{x}),\ldots,\lambda y G_m(y,\vec{\alpha},\vec{x}),F_1(\vec{\alpha},\vec{x}),\ldots,F_k(\vec{\alpha},\vec{x}))$$

where $G_1,\ldots,G_m,F_1,\ldots,F_k$ are total and recursive.

19.3. PROPOSITION. If P is Π^1_n and Q is reducible to P, then P is Π^1_n; and similarly with Σ^1_n or Δ^1_n in place of Π^1_n. □

The analogue of the table in §12 is the following table.

P,Q	$\neg P$	$P \vee Q$	$P \,\&\, Q$	$\forall \alpha P$	$\exists \alpha P$	QxP
Π^1_n	Σ^1_n	Π^1_n	Π^1_n	Π^1_n	Σ^1_{n+1}	Π^1_n
Σ^1_n	Π^1_n	Σ^1_n	Σ^1_n	Π^1_{n+1}	Σ^1_n	Σ^1_n
Δ^1_n	Δ^1_n	Δ^1_n	Δ^1_n	Π^1_n	Σ^1_n	Δ^1_n

It is proved and used in the same way as the earlier table.

The classification of analytical relations into the Π^1_n and Σ^1_n relations is called the <u>analytical hierarchy</u>.

19.4. ANALYTICAL ENUMERATION THEOREM. For every n, m, and k, there is a Π^1_n $(m,k+1)$–ary function which enumerates the class of Π^1_n (m,k)–ary relations; and similarly with Σ^1_n for Π^1_n.

Proof. Suppose, for example,we want to enumerate the Π^1_2 $(1,1)$–ary relations. Every such relation R is of the form $\forall\alpha\exists\beta P$ where P is Π^0_1 by the remarks after 19.1. Thus if Q is Π^0_1 and enumerates the Π^0_1 $(3,1)$–ary relations, then $\forall\alpha\exists\beta Q(\alpha,\beta,\gamma,x,e)$ is the desired enumerating function. □

19.5. ANALYTICAL HIERARCHY THEOREM. For each n, there is a Π^1_n set which is not Σ^1_n, hence not Π^1_k or Σ^1_k for any $k < n$. The same holds with Π^1_n and Σ^1_n interchanged.

Proof. As in the arithmetical case. □

20. The Projective Hierarchy

The results of the last section can be relativized to a class Φ of total functions of number variables. A particularly interesting case is that in which Φ

is the class of all such functions. Replacing the functions by their contractions, we see we are relativizing to the class $\mathbb{R}$ of reals. Note that by 18.1, a function is recursive in $\mathbb{R}$ iff it is obtained from a recursive function by replacing some of the unary function variables by names of particular reals. The same then holds with *recursive* replaced by Π^1_k or Σ^1_k.

A relation is <u>projective</u> if it is analytical in $\mathbb{R}$. The analytical hierarchy relativized to $\mathbb{R}$ is called the <u>projective</u> <u>hierarchy</u>. (It is customary to write a boldface Π^1_n for Π^1_n in $\mathbb{R}$ and similarly for Σ and Δ. We avoid this notation, since boldface is sometimes hard to distinguish from lightface.) The theory of the projective hierarchy antedates that of the analylytical hierearchy; it was begun by Lusin, Suslin, and Sierpinski.

The Enumeration Theorem does not hold in its usual form for Π^1_n in $\mathbb{R}$; but we shall prove a modified form. We say that a $(m+1,k)$–ary relation Q <u>$\mathbb{R}$–enumerates</u> a class Φ of (m,k)–ary relations if for every R in Φ, there is a β such that $R(\vec{\alpha},\vec{x}) \leftrightarrow Q(\vec{\alpha},\vec{x},\beta)$ for all $\vec{\alpha}$ and $\vec{x}$.

20.1. PROJECTIVE ENUMERATION THEOREM. For every n, m, and k, there is a $(m+1,k)$–ary Π^1_n relation which $\mathbb{R}$–enumerates the class of (m,k)–ary Π^1_n in $\mathbb{R}$ relations; and similarly with Σ^1_n for Π^1_n.

Proof. As in the proof of the analytical case, it is enough to do this for Σ^0_1, i.e., RE. If R is RE in $\mathbb{R}$, it is RE in a finite sequence $\vec{\alpha}$ of reals. If e is a $\vec{\alpha}$–index of R, then by (3) of §18,

$$\begin{aligned} R(\vec{\beta},\vec{x}) &\leftrightarrow \{e\}^{\vec{\alpha}}(\vec{\beta},\vec{x}) \text{ is defined} \\ &\leftrightarrow \{e\}(\vec{\beta},\vec{\alpha},\vec{x}) \text{ is defined} \\ &\leftrightarrow W_e(\vec{\beta},\vec{\alpha},\vec{x}). \end{aligned}$$

Choose γ so that $(\gamma)_0(0) = e$ and $(\gamma)_i = \alpha_i$ for $1 \leq i \leq n$. Then the right side becomes $W_{(\gamma)_0(0)}(\vec{\beta},(\gamma)_1,\ldots,(\gamma)_m,\vec{x})$. This is an RE relation P of $\vec{\beta},\vec{x},\gamma$; and P is the desired enumerating relation. □

We leave it to the reader to derive a Projective Hierarchy Theorem from this; the examples will now be $(1,0)$–ary. (Every $(0,k)$–ary relation is recursive in $\mathbb{R}$.)

20.2. PROPOSITION. Let P be defined by $P(\vec{\alpha},\vec{x},y) \leftrightarrow P_y(\vec{\alpha},\vec{x})$. Then P is Π^1_n in $\mathbb{R}$ iff each P_y is Π^1_n in $\mathbb{R}$; and similarly with Σ^1_n or Δ^1_n in place of Π^1_n.

Proof. If P is Π^1_n in $\mathbb{R}$, each P_y is clearly Π^1_n in $\mathbb{R}$. Now suppose that each P_y is Π^1_n in $\mathbb{R}$. By the Projective Enumeration Theorem, there is a Π^1_n relation Q and a β_y for each y such that $P_y(\vec{\alpha},\vec{x}) \leftrightarrow Q(\vec{\alpha},\vec{x},\beta_y)$. Choose β so that $(\beta)_y = \beta_y$ for all y. Then $P(\vec{\alpha},\vec{x},y) \leftrightarrow Q(\vec{\alpha},\vec{x},(\beta)_y)$. Thus P is Π^1_n in β and hence in $\mathbb{R}$. □

The further study of the analytical and projective hierarchies is known as Descriptive Set Theory, and is a hybrid of Recursion Theory and Set Theory. We shall prove only one result. We shall prove it for the projective hierarchy; the analogue for the analytical hierarchy is more difficult both to state and to prove.

We recall a definition from measure theory. Let X be a space and let Λ be a class of subsets of X. We say that Λ is a <u>σ–ring</u> if: (a) the complement of every set in Λ is in Λ; (b) every countable union of sets in Λ is in Λ; (c) $X \in \Lambda$. From (a) and (b) it follows that: (d) every countable intersection of sets in Λ is in Λ. If Γ is any collection of subsets of X, there is a smallest σ–ring including Γ; it is the intersection of all of the Σ–rings which include Γ.

20.3 PROPOSITION. The class of Δ^1_n in $\mathbb{R}$ (m,k)–ary relations is a σ–ring in $\mathbb{R}^{m,k}$.

Proof. In view of the table, it is enough to show that the union Q of a sequence $\{P_j\}$ of such relations is Δ^1_n in $\mathbb{R}$. Defining $P(\vec{\alpha},\vec{x},j) \leftrightarrow P_j(\vec{\alpha},\vec{x})$, P is Δ^1_n by 20.2. Since $Q(\vec{\alpha},\vec{x}) \leftrightarrow \exists j P(\vec{\alpha},\vec{x},j)$, Q is Δ^1_n in $\mathbb{R}$ by the table. □

An (m,k)–ary relation is <u>Borel</u> if it belongs to the smallest σ–ring in $\mathbb{R}^{m,k}$ which contains all the recursive (m,k)–ary relations. By 20.3, every Borel relation is Δ^1_1 in $\mathbb{R}$. We shall prove that the converse also holds.

Let A and B be subsets of a space X. We say that a subset C of X separates A and B if $A \subseteq C$ and $B \subseteq C^c$. This clearly implies that A and B are disjoint.

20.4. SEPARATION THEOREM. Any two disjoint Σ^1_1 in $\mathbb{R}$ (m,k)–ary relations can be separated by a Borel relation.

Proof. To make the notation simpler, let $m = 1$ and $k = 0$. Say that A is inseparable from B if no Borel relation separates A and B. We shall first prove the following lemma: If $\cup_{i \in \omega} A_i$ is inseparable from $\cup_{j \in \omega} B_j$, then there are i and j such that A_i is inseparable from B_j. Suppose, on the contrary, that for every i and j, there is a Borel relation $C_{i,j}$ which separates A_i and B_j. If $C = \cap_{j \in \omega} \cup_{i \in \omega} C_{i,j}$, then C is Borel and separates $\cup_{i \in \omega} A_i$ and $\cup_{j \in \omega} B_j$.

Now assume that P and Q are inseparable Σ^1_1 in $\mathbb{R}$ relations; we shall show that P and Q are not disjoint. Using the remarks after 19.1, we can write

$$P(\alpha) \leftrightarrow \exists \beta \forall n R(\bar{\alpha}(n), \bar{\beta}(n)),$$

$$Q(\alpha) \leftrightarrow \exists \gamma \forall n R'(\bar{\alpha}(n), \bar{\gamma}(n)),$$

where R and R' are recursive in $\mathbb{R}$. For $z,w \in Seq$, let

$$P_{z,w}(\alpha) \leftrightarrow z = \bar{\alpha}(lh(z)) \ \& \ \exists \beta (w = \bar{\beta}(lh(w)) \ \& \ \forall n R(\bar{\alpha}(n), \bar{\beta}(n))),$$

and define $Q_{z,w}$ similarly but with R replaced by R'. It is clear that

$$P_{z,w} = \cup_{m \in \omega} \cup_{p \in \omega} P_{z * <m>, w * <p>}$$

and similarly for $Q_{z,w}$.

We shall define $\alpha(n)$, $\beta(n)$, and $\gamma(n)$ by induction on n so that $P_{\bar{\alpha}(n), \bar{\beta}(n)}$ and $Q_{\bar{\alpha}(n), \bar{\gamma}(n)}$ are inseparable. Since $P = P_{< >, < >}$ and $Q = Q_{< >, < >}$, this holds for $n = 0$. Suppose it holds for some n. By our lemma, there are i, j, k, and l so that $P_{\bar{\alpha}(n) * <i>, \bar{\beta}(n) * <j>}$ is inseparable from $Q_{\bar{\alpha}(n) * <k>, \bar{\gamma}(n) * <l>}$. Then $i = k$; for otherwise, $\{\delta\colon \bar{\delta}(n+1) = \bar{\alpha}(n) * <i>\}$ is a recursive (and hence Borel) set which separates $P_{\bar{\alpha}(n) * <i>, \bar{\beta}(n) * <j>}$ from $Q_{\bar{\alpha}(n) * <k>, \bar{\gamma}(n) * <l>}$.

Thus we may take $\alpha(n) = i$, $\beta(n) = j$, and $\gamma(n) = l$.

For each n, $P_{\overline{\alpha}(n),\overline{\beta}(n)}$ is inseparable from $Q_{\overline{\alpha}(n),\overline{\gamma}(n)}$; so, they are both non–empty. This implies that $R(\overline{\alpha}(n),\overline{\beta}(n))$ and $R'(\overline{\alpha}(n),\overline{\gamma}(n))$ for all n. Hence $P(\alpha)$ and $Q(\alpha)$; so P and Q are not disjoint. □

20.5. SUSLIN'S THEOREM. A relation is Borel iff it is Δ_1^1 in $\mathbb{R}$.

Proof. We have already seen that every Borel relation is Δ_1^1 in $\mathbb{R}$. Now let P be Δ_1^1 in $\mathbb{R}$. Then P and $\neg P$ are Σ_1^1 in $\mathbb{R}$; so by the Separation Theorem, there is a Δ_1^1 in $\mathbb{R}$ relation which separates P and $\neg P$. But the only relation which separates P and $\neg P$ is P. □