34 Proof of Louveau's Theorem

Finally, we arrive at our last section. The following summarizes how I feel now.

> You are walking down the street minding your own business and someone stops you and asks directions. Where's xxx hall? You don't know and you say you don't know. Then they point at the next street and say: Is that xxx street? Well by this time you feel kind of stupid so you say, yea yea that's xxx street, even though you haven't got the slightest idea whether it is or not. After all, who wants to admit they don't know where they are going or where they are.

For $\alpha < \omega_1^{CK}$ define $D \subseteq \omega^\omega$ is Σ^0_α(semihyp) iff there exists S a Π^1_1 set of hyperarithmetic reals such that every element of S is a β-code for some $\beta < \alpha$ and

$$D = \bigcup\{P(T,q) : (T,q) \in S\}.$$

A set is Π^0_α(semihyp) iff it is the complement of a Σ^0_α(semihyp) set. The Π^0_0(semihyp) sets are just the usual clopen basis ($[s]$ for $s \in \omega^{<\omega}$ together with the empty set) and Σ^0_0(semihyp) sets are their complements.

Lemma 34.1 *Σ^0_α(semihyp) sets are Π^1_1 and consequently Π^0_α(semihyp) sets are Σ^1_1.*

proof:

$x \in \bigcup\{P(T,q) : (T,q) \in S\}$ iff there exists $(T,q) \in \Delta^1_1$ such that $(T,q) \in S$ and $x \in P(T,q)$. Quantification over Δ^1_1 preserves Π^1_1 (see Corollary 29.3) and Lemma 33.4 implies that "$x \in P(T,q)$" is Δ^1_1.

■

We will need the following reflection principle in order to prove the Main Lemma 34.3.

A predicate $\Phi \subseteq P(\omega)$ is called Π^1_1 on Π^1_1 iff for any Π^1_1 set $N \subseteq \omega \times \omega$ the set $\{e : \Phi(N_e)\}$ is Π^1_1 (where $N_e = \{n : (e,n) \in N\}$).

Lemma 34.2 *(Harrington [39] Kechris [48]) Π^1_1-Reflection. Suppose $\Phi(X)$ is Π^1_1 on Π^1_1 and Q is a Π^1_1 set.*

If $\Phi(Q)$, then there exists a Δ^1_1 set $D \subseteq Q$ such that $\Phi(D)$.

proof:

By the normal form theorem 17.4 there is a recursive mapping $e \to T_e$ such that $e \in Q$ iff T_e is well-founded. Define for $e \in \omega$

$$N^0_e = \{\hat{e} : T_{\hat{e}} \preceq T_e\}$$

$$N^1_e = \{\hat{e} : \neg(T_e \prec T_{\hat{e}})\}$$

then N^0 is Σ^1_1 and N^1 is Π^1_1. For $e \in Q$ we have $N^0_e = N^1_e = D_e \subseteq Q$ is Δ^1_1; and for $e \notin Q$ we have that $N^1_e = Q$. If we assume for contradiction that $\neg\Phi(N^1_e)$ for all $e \in Q$, then

$$e \notin Q \text{ iff } \phi(N^1_e).$$

But this would mean that Q is Δ^1_1 and this proves the Lemma.

■

Note that a Π^1_1 predicate need not be Π^1_1 on Π^1_1 since the predicate

$$\Phi(X) = \text{“}0 \notin X\text{”}$$

is Δ^0_0 but not Π^1_1 on Π^1_1. Some examples of Π^1_1 on Π^1_1 predicates $\Phi(X)$ are

$$\Phi(X) \text{ iff } \forall x \notin X \;\; \theta(x)$$

or

$$\Phi(X) \text{ iff } \forall x, y \notin X \;\; \theta(x, y)$$

where θ is a Π^1_1 sentence.

Lemma 34.3 *Suppose A is Σ^1_1 and $A \subseteq B \in \Sigma^0_\alpha(\text{semihyp})$, then there exists $C \in \Sigma^0_\alpha(\text{hyp})$ with $A \subseteq C \subseteq B$.*

proof:

Let $B = \bigcup\{P(T, q) : (T, q) \in S\}$ where S is a Π^1_1 set of hyperarithmetic $< \alpha$-codes. Let $\hat{S} \subseteq \omega$ be the Π^1_1 set of Δ^1_1-codes for elements of S, i.e.

$$e \in \hat{S} \text{ iff } e \text{ is a } \Delta^1_1\text{-code for } (T_e, q_e) \text{ and } (T_e, q_e) \in S.$$

Now define the predicate $\Phi(X)$ for $X \subseteq \omega$ as follows:

$$\Phi(X) \text{ iff } X \subseteq \hat{S} \text{ and } A \subseteq \textstyle\bigcup_{e \in X} P(T_e, q_e).$$

The predicate $\Phi(X)$ is Π^1_1 on Π^1_1 and $\Phi(\hat{S})$. Therefore by reflection (Lemma 34.2) there exists a Δ^1_1 set $D \subseteq \hat{S}$ such that $\Phi(D)$. Define (T, q) by

$$T = \{e\hat{\ }s : e \in D \text{ and } s \in T_e\} \qquad q(e\hat{\ }s) = q_e(s) \text{ for } e \in D \text{ and } s \in T^0_e.$$

Since D is Δ^1_1 it is easy to check that (T, q) is Δ^1_1 and hence hyperarithmetic. Since $\Phi(D)$ holds it follows that $C = S(T, q)$ the $\Sigma^0_\alpha(\text{hyp})$ set coded by (T, q) has the property that $A \subseteq C$ and since $D \subseteq \hat{S}$ it follows that $C \subseteq B$.

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Define for $\alpha < \omega_1^{CK}$ the α-topology by taking for basic open sets the family

$$\bigcup\{\Pi^0_\beta(\text{semihyp}) : \beta < \alpha\}.$$

As usual, $\text{cl}_\alpha(A)$ denotes the closure of the set A in the α-topology.

The 1-topology is just the standard topology on ω^ω. The α-topology has its basis certain special Σ^1_1 sets so it is intermediate between the standard topology and the Gandy topology corresponding to Gandy forcing.

Lemma 34.4 *If A is Σ^1_1, then* $\mathrm{cl}_\alpha(A)$ *is* $\Pi^0_\alpha(\text{semihyp})$.

proof:
Since the $\Sigma^0_\beta(\text{semihyp})$ sets for $\beta < \alpha$ form a basis for the α-closed sets,

$$\mathrm{cl}_\alpha(A) = \bigcap\{X \supseteq A : \exists\beta < \alpha \;\; X \in \Sigma^0_\beta(\text{semihyp})\}.$$

By Lemma 34.3 this same intersection can be written:

$$\mathrm{cl}_\alpha(A) = \bigcap\{X \supseteq A : \exists\beta < \alpha \;\; X \in \Sigma^0_\beta(\text{hyp})\}.$$

But now define $(T,q) \in Q$ iff $(T,q) \in \Delta^1_1$, (T,q) is a β-code for some $\beta < \alpha$, and $A \subseteq S(T,q)$. Note that Q is a Π^1_1 set and consequently, $\mathrm{cl}_\alpha(A)$ is a $\Pi^0_\alpha(\text{semihyp})$ set, as desired.
■

Note that it follows from the Lemmas that for A a Σ^1_1 set, $\mathrm{cl}_\alpha(A)$ is a Σ^1_1 set which is a basic open set in the β-topology for any $\beta > \alpha$.

Let $\mathbb{P}$ be Gandy forcing, i.e., the partial order of all nonempty Σ^1_1 subsets of ω^ω and let $\mathring{a}$ be a name for the real obtained by forcing with $\mathbb{P}$, so that by Lemma 30.2, for any G which is $\mathbb{P}$-generic, we have that $p \in G$ iff $a^G \in p$.

Lemma 34.5 *For any $\alpha < \omega_1^{CK}$, $p \in \mathbb{P}$, and $C \in \underset{\sim}{\Pi}^0_\alpha$ (coded in V) if*

$$p \Vdash \mathring{a} \in C,$$

then

$$cl_\alpha(p) \Vdash \mathring{a} \in C.$$

proof:
This is proved by induction on α.

For $\alpha = 1$ recall that the α-topology is the standard topology and C is a standard closed set. If $p \Vdash \mathring{a} \in C$, then it better be that $p \subseteq C$, else there exists $s \in \omega^{<\omega}$ with $q = p \cap [s]$ nonempty and $[s] \cap C = \emptyset$. But then $q \leq p$ and $q \Vdash \mathring{a} \notin C$. Hence $p \subseteq C$ and since C is closed, $\mathrm{cl}(p) \subseteq C$. Since $\mathrm{cl}(p) \Vdash a \in \mathrm{cl}(p)$, it follows that $\mathrm{cl}(p) \Vdash a \in C$.

For $\alpha > 1$ let

$$C = \bigcap_{n<\omega} \sim C_n$$

where each C_n is $\underset{\sim}{\Pi}^0_\beta$ for some $\beta < \alpha$. Suppose for contradiction that

$$cl_\alpha(p) \not\Vdash \mathring{a} \in C \cdot$$

Then for some $n < \omega$ and $r \leq cl_\alpha(p)$ it must be that

$$r \Vdash \mathring{a} \in C_n.$$

Suppose that C_n is $\underset{\sim}{\Pi}^0_\beta$ for some $\beta < \alpha$. Then by induction

$$\mathrm{cl}_\beta(r) \Vdash \mathring{a} \in C_n.$$

But $\mathrm{cl}_\beta(r)$ is a Π^0_β(semihyp) set by Lemma 34.4 and hence a basic open set in the α-topology. Note that since they force contradictory information ($\mathrm{cl}_\beta(r) \Vdash \mathring{a} \notin C$ and $p \Vdash \mathring{a} \in C$) it must be that $\mathrm{cl}_\beta(r) \cap p = \emptyset$, (otherwise the two conditions would be compatible in $\mathbb{P}$). But since $\mathrm{cl}_\beta(r)$ is α-open this means that

$$\mathrm{cl}_\beta(r) \cap \mathrm{cl}_\alpha(p) = \emptyset$$

which contradicts the fact that $r \leq \mathrm{cl}_\alpha(p)$.

■

Now we are ready to prove Louveau's Theorem 33.1. Suppose A and B are Σ^1_1 sets and C is a $\underset{\sim}{\Pi}^0_\alpha$ set with $A \subseteq C$ and $C \cap B = \emptyset$. Since $A \subseteq C$ it follows that

$$A \Vdash \mathring{a} \in C.$$

By Lemma 34.5 it follows that

$$\mathrm{cl}_\alpha(A) \Vdash \mathring{a} \in C.$$

Now it must be that $\mathrm{cl}_\alpha(A) \cap B = \emptyset$, otherwise letting $p = \mathrm{cl}_\alpha(A) \cap B$ would be a condition of $\mathbb{P}$ such that

$$p \Vdash \mathring{a} \in C$$

and

$$p \Vdash \mathring{a} \in B$$

which would imply that $B \cap C \neq \emptyset$ in the generic extension. But by absoluteness B and C must remain disjoint. So $\mathrm{cl}_\alpha(A)$ is a Π_α(semihyp)-set (Lemma 34.4) which is disjoint from the set B and thus by applying Lemma 34.3 to its complement there exists a Π^0_α(hyp)-set C with $\mathrm{cl}_\alpha(A) \subseteq C$ and $C \cap B = \emptyset$.

■

The argument presented here is partially from Harrington [34], but contains even more simplification brought about by using forcing and absoluteness. Louveau's Theorem is also proved in Sacks [93] and Mansfield and Weitkamp [71].