32 Σ^1_1 equivalence relations

Theorem 32.1 *(Burgess [14]) Suppose E is a $\boldsymbol{\Sigma}^1_1$ equivalence relation. Then either E has $\leq \omega_1$ equivalence classes or there exists a perfect set of pairwise E-inequivalent reals.*

proof:

We will need to prove the boundedness theorem for this result. Define

$$WF = \{T \subseteq \omega^{<\omega} : T \text{is a well-founded tree}\}.$$

For $\alpha < \omega_1$ define $WF_{<\alpha}$ to the subset of WF of all well-founded trees of rank $< \alpha$. WF is a complete Π^1_1 set, i.e., for every $B \subseteq \omega^\omega$ which is $\boldsymbol{\Pi}^1_1$ there exists a continuous map f such that $f^{-1}(WF) = B$ (see Theorem 17.4). Consequently, WF is not Borel. On the other hand each of the $WF_{<\alpha}$ are Borel.

Lemma 32.2 *For each $\alpha < \omega_1$ the set $WF_{<\alpha}$ is Borel.*

proof:

Define for $s \in \omega^{<\omega}$ and $\alpha < \omega_1$

$$WF^s_{<\alpha} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree, } s \in T, \ \ r_T(s) < \alpha\}.$$

The fact that $WF^s_{<\alpha}$ is Borel is proved by induction on α. The set of trees is Π^0_1. For λ a limit

$$WF^s_{<\lambda} = \bigcup_{\alpha<\lambda} WF^s_{<\alpha}.$$

For a successor $\alpha + 1$

$$T \in WF^s_{<\alpha+1} \text{ iff } s \in T \text{ and } \forall n \ \ (s\hat{\ }n \in T \rightarrow T \in WF^{s\hat{\ }n}_{<\alpha}).$$

■

Another way to prove this is take a tree T of rank α and note that $WF_{<\alpha} = \{\hat{T} : \hat{T} \prec T\}$ and this set is $\boldsymbol{\Delta}^1_1$ and hence Borel by Theorem 26.1.

Lemma 32.3 *(Boundedness) If $A \subseteq WF$ is $\boldsymbol{\Sigma}^1_1$, then there exists $\alpha < \omega_1$ such that $A \subseteq WF_\alpha$.*

proof:

Suppose no such α exists. Then

$$T \in WF \text{ iff there exists } \hat{T} \in A \text{ such that } T \preceq \hat{T}.$$

But this would give a $\boldsymbol{\Sigma}^1_1$ definition of WF, contradiction.

■

There is also a lightface version of the boundedness theorem, i.e., if A is a Σ^1_1 subset of WF, then there exists a recursive ordinal $\alpha < \omega_1^{CK}$ such that $A \subseteq WF_{<\alpha}$. Otherwise,

$$\{e \in \omega : e \text{ is the code of a recursive well-founded tree }\}$$

would be Σ_1^1.

Now suppose that E is a Σ_1^1 equivalence relation. By the Normal Form Theorem 17.4 we know there exists a continuous mapping $(x, y) \mapsto T_{xy}$ such that T_{xy} is always a tree and

$$xEy \text{ iff } T_{xy} \notin WF.$$

Define

$$xE_\alpha y \text{ iff } T_{xy} \notin WF_{<\alpha}.$$

By Lemma 32.2 we know that the binary relation E_α is Borel. Note that E_α refines E_β for $\alpha > \beta$. Clearly,

$$E = \bigcap_{\alpha<\omega_1} E_\alpha$$

and for any limit ordinal λ

$$E_\lambda = \bigcap_{\alpha<\lambda} E_\alpha.$$

While there is no reason to expect that any of the E_α are equivalence relations, we use the boundedness theorem to show that many are.

Lemma 32.4 *For unboundedly many $\alpha < \omega_1$ the binary relation E_α is an equivalence relation.*

proof:

Note that every E_α must be reflexive, since E is reflexive and $E = \bigcap_{\alpha<\omega_1} E_\alpha$. The following claim will allow us to handle symmetry.

Claim: For every $\alpha < \omega_1$ there exists $\beta < \omega_1$ such that for every x, y

$$\text{if } xE_\alpha y \text{ and } y \not E_\alpha x, \text{ then } x \not E_\beta y.$$

proof:

Let

$$A = \{T_{xy} : xE_\alpha y \text{ and } y \not E_\alpha x\}.$$

Then A is a Borel set. Since $y \not E_\alpha x$ implies $y \not E x$ and hence $x \not E y$, it follows that $A \subseteq WF$. By the Boundedness Theorem 32.3 there exists $\beta < \omega_1$ such that $A \subseteq WF_{<\beta}$.

■

The next claim is to take care of transitivity.

Claim: For every $\alpha < \omega_1$ there exists $\beta < \omega_1$ such that for every x, y, z

$$\text{if } xE_\alpha y \text{ and } yE_\alpha z, \text{ and } x \not E_\alpha z, \text{ then either } x \not E_\beta y \text{ or } y \not E_\beta z.$$

proof:

Let

$$B = \{T_{xy} \oplus T_{yz} : xE_\alpha y,\ \ yE_\alpha z, \text{ and } x \not\!E_\alpha z\}.$$

The operation $\oplus$ on a pair of trees T_0 and T_1 is defined by

$$T_0 \oplus T_1 = \{(s,t) : s \in T_0,\ \ t \in T_1, \text{ and } |s| = |t|\}.$$

Note that the rank of $T_0 \oplus T_1$ is the minimum of the rank of T_0 and the rank of T_1. (Define the rank function on $T_0 \oplus T_1$ by taking the minimum of the rank functions on the two trees.)

The set B is Borel because the relation E_α is. Note also that since $x \not\!E_\alpha z$ implies $x \not\!E z$ and E is an equivalence relation, then either $x \not\!E y$ or $y \not\!E z$. It follows that either $T_{xy} \in WF$ or $T_{yz} \in WF$ and so in either case $T_{xy} \oplus T_{yz} \in WF$ and so $B \subseteq WF$. Again, by the Boundedness Theorem there is a $\beta < \omega_1$ such that $B \subseteq WF_{<\beta}$ and this proves the Claim.

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Now we use the Claims to prove the Lemma. Using the usual Lowenheim-Skolem argument we can find arbitrarily large countable ordinals λ such that for every $\alpha < \lambda$ there is a $\beta < \lambda$ which satisfies both Claims for α. But this means that E_λ is an equivalence relation. For suppose $xE_\lambda y$ and $y \not\!E_\lambda x$. Then since $E_\lambda = \bigcap_{\alpha<\lambda} E_\alpha$ there must be $\alpha < \lambda$ such that $xE_\alpha y$ and $y \not\!E_\alpha x$. But by the Claim there exist $\beta < \lambda$ such that $x \not\!E_\beta y$ and hence $x \not\!E_\lambda y$, a contradiction. A similar argument using the second Claim works for transitivity.

■

Let G be any generic filter over V with the property that it collapses ω_1 but not ω_2. For example, Levy forcing with finite partial functions from ω to ω_1 (see Kunen [54] or Jech [43]). Then $\omega_1^{V[G]} = \omega_2^V$. By absoluteness, E is still an equivalence relation and for any α if E_α was an equivalence relation in V, then it still is one in $V[G]$. Since

$$E_{\omega_1^V} = \bigcap_{\alpha<\omega_1^V} E_\alpha$$

and the intersection of equivalence relations is an equivalence relation, it follows that the Borel relation $E_{\omega_1^V}$ is an equivalence relation. So now suppose that E had more than ω_2 equivalence classes in V. Let Q be a set of size ω_2 in V of pairwise E-inequivalent reals. Then Q has cardinality ω_1 in $V[G]$ and for every $x \neq y \in Q$ there exists $\alpha < \omega_1^V$ with $x \not\!E_\alpha y$. Hence it must be that the elements of Q are in different $E_{\omega_1^V}$ equivalence classes. Consequently, by Silver's Theorem 30.1 there exists a perfect set P of $E_{\omega_1^V}$-inequivalent reals. Since in $V[G]$ the equivalence relation E refines $E_{\omega_1^V}$, it must be that the elements of P are pairwise E-inequivalent also. The following is a Σ^1_2 statement:

$$V[G] \models \exists P \text{ perfect } \forall x \forall y\ \ (x, y \in P \text{ and } x \neq y) \rightarrow x \not\!E y.$$

Hence, by Shoenfield Absoluteness 20.2, V must think that there is a perfect set of E-inequivalent reals.

A way to avoid taking a generic extension of the universe is to suppose Burgess's Theorem is false. Then let M be the transitive collapse of an elementary substructure of some sufficiently large V_κ (at least large enough to know about absoluteness and Silver's Theorem). Let $M[G]$ be obtained as in the above proof by Levy collapsing ω_1^M. Then we can conclude as above that M thinks E has a perfect set of inequivalent elements, which contradicts the assumption that M thought Burgess's Theorem was false.
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By Harrington's Theorem 25.1 it is consistent to have $\underset{\sim}{\Pi}^1_2$ sets of arbitrary cardinality, e.g it is possible to have $\mathfrak{c} = \omega_{23}$ and there exists a $\underset{\sim}{\Pi}^1_2$ set B with $|B| = \omega_{17}$. Hence, if we define

$$xEy \text{ iff } x = y \text{ or } x, y \notin B$$

then we get $\underset{\sim}{\Sigma}^1_2$ equivalence relation with exactly ω_{17} equivalence classes, but since the continuum is ω_{23} there is no perfect set of E-inequivalent reals.

See Burgess [15] [16] and Hjorth [41] for more results on analytic equivalence relations. For further results concerning projective equivalence relations see Harrington and Sami [37], Sami [94], Stern [107] [108], Kechris [49], Harrington and Shelah [38], Shelah [95], and Harrington, Marker, and Shelah [39].