12 Boolean algebra of order ω_1

Now we use the Martin-Solovay technique to produce a countably generated ccc cBa with order ω_1. Before doing so we introduce a countable version of α-forcing which will be useful for other results also. It is similar to one used in Miller [74] to give a simple proof about generating sets in the category algebra.

Let T be a nice tree of rank α ($2 \leq \alpha < \omega_1$). Define

$$\mathbb{P}_\alpha = \{p : D \to \omega : D \in [\omega]^{<\omega}, \forall s, s\hat{\ }n \in D \;\; p(s) \neq p(s\hat{\ }n)\}.$$

This is ordered by $p \leq q$ iff $p \supseteq q$. For $p \in \mathbb{P}_\alpha$ define

$$\text{rank}(p) = \max\{r_T(s) : s \in \text{domain}(p)\}$$

where r_T is the rank function on T.

Lemma 12.1 rank : $\mathbb{P}_\alpha \to \alpha + 1$ *satisfies the Rank Lemma 7.4, i.e, for every $p \in \mathbb{P}_\alpha$ and $\beta \geq 1$ there exists $\hat{p} \in \mathbb{P}_\alpha$ such that*

1. *$\hat{p}$ is compatible with p,*
2. rank$(\hat{p}) \leq \beta$, *and*
3. *for any $q \in \mathbb{P}_\alpha$ if* rank$(q) < \beta$ *and $\hat{p}$ and q are compatible, then p and q are compatible.*

proof:

First let $p_0 \leq p$ be such that for every $s \in \text{domain}(p)$ and $n \in \omega$ if

$$r_T(s\hat{\ }n) < \beta < \lambda = r_T(s)$$

then there exists $m \in \omega$ with $p_0(s\hat{\ }n\hat{\ }m) = p(s)$. Note that

$$r_T(s\hat{\ }n) < \beta < \lambda = r_T(s)$$

can happen only when λ is a limit ordinal and for any such s there can be at most finitely many n (because T is a nice tree).

Now let

$$E = \{s \in \text{domain}(p_0) : r_T(s) \leq \beta\}$$

and define $\hat{p} = p_0 \restriction E$. It is compatible with p since p_0 is stronger than both. From its definition it has rank $\leq \beta$. So let $q \in \mathbb{P}_\alpha$ have rank$(q) < \beta$ and be incompatible with p. We need to show it is incompatible with $\hat{p}$. There are only three ways for q and p to be incompatible:

1. $\exists s \in \text{domain}(p) \cap \text{domain}(q) \;\; p(s) \neq q(s)$,
2. $\exists s \in \text{domain}(q) \; \exists s\hat{\ }n \in \text{domain}(p) \;\; q(s) = p(s\hat{\ }n)$, or
3. $\exists s \in \text{domain}(p) \; \exists s\hat{\ }n \in \text{domain}(q) \;\; p(s) = q(s\hat{\ }n)$.

For (1) since $\text{rank}(q) < \beta$ we know $r_T(s) < \beta$ and hence by construction s is in the domain of $\hat{p}$ and so q and $\hat{p}$ are incompatible. For (2) since

$$r_T(s\hat{\ }n) < r_T(s) < \beta$$

we get the same conclusion. For (3) since $s\hat{\ }n \in \text{domain}(q)$ we know

$$r_T(s\hat{\ }n) < \beta.$$

If $r_T(s) = \beta$, then $s \in \text{domain}(\hat{p})$ and so q and $\hat{p}$ are incompatible. Otherwise since T is a nice tree,

$$r_T(s\hat{\ }n) < \beta < r_T(s) = \lambda$$

a limit ordinal. In this case we have arranged $\hat{p}$ so that there exists m with $p(s) = \hat{p}(s\hat{\ }n\hat{\ }m)$ and so again q and $\hat{p}$ are incompatible.
■

Lemma 12.2 *There exists a countable family $\mathcal{D}$ of dense subsets of $\mathbb{P}_\alpha$ such that for every G a $\mathbb{P}_\alpha$-filter which meets each dense set in $\mathcal{D}$ the filter G determines a map $x : T \to \omega$ by $p \in G$ iff $p \subseteq x$. This map has the property that for every $s \in T^{>0}$ the value of $x(s)$ is the unique element of ω not in $\{x(s\hat{\ }n) : n \in \omega\}$.*

proof:
For each $s \in T$ the set

$$D_s = \{p : s \in \text{domain}(p)\}$$

is dense. Also for each $s \in T^{>0}$ and $k \in \omega$ the set

$$E_s^k = \{p : p(s) = k \text{ or } \exists n \;\; p(s\hat{\ }n) = k\}$$

is dense.
■

The poset $\mathbb{P}_\alpha$ is separative, since if $p \not\leq q$ then either p and q are incompatible or there exists $s \in \text{domain}(q) \setminus \text{domain}(p)$ in which case we can find $\hat{p} \leq p$ with $\hat{p}(s) \neq q(s)$.

Now if $\mathbb{P}_\alpha \subseteq \mathbb{B}$ is dense in the cBa $\mathbb{B}$, it follows that for each $p \in \mathbb{P}_\alpha$

$$p = [\![p \subseteq x]\!]$$

and for any $s \in T^{>0}$ and k

$$[\![x(s) = k]\!] = \prod_{m \in \omega} [\![x(s\hat{\ }m) \neq k]\!].$$

Consequently if

$$C = \{p \in \mathbb{P}_\alpha : \text{domain}(p) \subseteq T^0\}$$

then $C \subseteq \mathbb{B}$ has the property that $\text{ord}(C) = \alpha + 1$.

Now let $\sum_{\alpha<\omega_1} \mathbb{P}_\alpha$ be the *direct sum*, i.e., $p = \langle p_\alpha : \alpha < \omega_1 \rangle$ with $p_\alpha \in \mathbb{P}_\alpha$ and $p_\alpha = \mathbf{1}_\alpha = \emptyset$ for all but finitely many α. This forcing is equivalent to adding ω_1 Cohen reals, so the usual delta-lemma argument shows that it is ccc. Let

$$X = \{x_{\alpha,s,n} \in 2^\omega : \alpha < \omega_1, s \in T^0_\alpha, n \in \omega\}$$

be distinct elements of 2^ω. For $G = \langle G_\alpha : \alpha < \omega_1 \rangle$ which is $\sum_{\alpha<\omega_1} \mathbb{P}_\alpha$-generic over V, use X and Silver forcing to code the rank zero parts of each G_α, i.e., define $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ by $(p, q) \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$
iff
$p \in \sum_{\alpha<\omega_1} \mathbb{P}_\alpha$ and q is a finite set of consistent sentences of the form:

1. "$x \notin \overset{\circ}{U}_n$ " where $x \in X$ or
2. "$B \subseteq \overset{\circ}{U}_n$ " where B is clopen and $n \in \omega$.

with the additional proviso that whenever "$x_{\alpha,s,n} \notin \overset{\circ}{U}_n$ " $\in q$ then s is in the domain of p_α and $p_\alpha(s) \neq n$. This is a little stronger than saying $p \Vdash \check{q} \in \mathbb{Q}$, but would be true for a dense set of conditions.

The rank function

$$\text{rank} : (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}} :\to \omega_1$$

is defined by

$$\text{rank}(\langle p_\alpha : \alpha < \omega_1 \rangle, q) = \max\{\text{rank}(p_\alpha) : \alpha < \omega_1\}$$

which means we ignore q entirely.

Lemma 12.3 *For every $p \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ and $\beta \geq 1$ there exists $\hat{p}$ in the poset $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ such that*

1. *$\hat{p}$ is compatible with p,*
2. $\text{rank}(\hat{p}) \leq \beta$, *and*
3. *for any $q \in (\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}}$ if* $\text{rank}(q) < \beta$ *and $\hat{p}$ and q are compatible, then p and q are compatible.*

proof:
Apply Lemma 12.1 to each p_α to obtain $\hat{p_\alpha}$ and then let

$$\hat{p} = (\langle \hat{p_\alpha} : \alpha < \omega_1 \rangle, q).$$

This is still a condition because $\hat{p_\alpha}$ retains all the rank zero part of p_α which is needed to force $q \in \overset{\circ}{\mathbb{Q}}$.
■

Let $(\sum_{\alpha<\omega_1} \mathbb{P}_\alpha) * \overset{\circ}{\mathbb{Q}} \subseteq \mathbb{B}$ be a dense subset of the ccc cBa $\mathbb{B}$. We show that $\mathbb{B}$ is countably generated and $\text{ord}(\mathbb{B}) = \omega_1$. A strange thing about ω_1 is that if one countable set of generators has order ω_1, then all countable sets of generators have order ω_1. This is because any countable set will be generated by a countable stage.

One set of generators for $\mathbb{B}$ is

$$C = \{[\![\check{B} \subseteq \overset{\circ}{U}_n]\!] : B \text{ clopen }, n \in \omega\}.$$

Note that

$$[\![x \in \cap_{n\in\omega} U_n]\!] = \prod_{n\in\omega} [\![x \in U_n]\!] = \prod_{n\in\omega} \sum \{[\![\check{B} \subseteq \overset{\circ}{U}_n]\!] : x \in B\}$$

and also each $\mathbb{P}_\alpha$ is generated by

$$\{p \in \mathbb{P}_\alpha : \text{domain}(p) \subseteq T^0_\alpha\}.$$

We know that for each $\alpha < \omega_1$, $s \in T^0_\alpha$ and $n \in \omega$ if $p = (\langle p_\alpha : \alpha < \omega_1 \rangle, q)$ is the condition for which p_α is the function with domain $\{s\}$, and $p_\alpha(s) = n$, and the rest of p is the trivial condition, then

$$p = [\![\check{x}_{\alpha,s,n} \in \bigcap_{n\in\omega} \overset{\circ}{U}_n]\!].$$

From these facts it follows that C generates $\mathbb{B}$.

It follows from Lemma 8.4 that the order of C is ω_1. For any $\beta < \omega_1$ let $b = (\langle p_\alpha : \alpha < \omega_1 \rangle, q)$ be the condition all of whose components are trivial except for p_β, and p_β any the function with domain $\langle\rangle$. Then $b \notin \underset{\sim}{\Sigma}^0_\beta(C)$. Otherwise by Lemma 8.4, there would be some $a \leq b$ with $\text{rank}(a, C) < \beta$, but then p^a_β would not have $\langle\rangle$ in its domain.

This proves the ω_1 case of Theorem 8.2.