9 Borel order of a field of sets

In this section we use the Sikorski-Loomis representation theorem to transfer the abstract Borel hierarchy on a complete boolean algebra into a field of sets.

A family $F \subseteq P(X)$ is a *σ-field* iff it contains the empty set and is closed under countable unions and complements in X. $I \subseteq F$ is a *σ-ideal* in F iff

1. I contains the empty set,
2. I is closed under countable unions,
3. $A \subseteq B \in I$ and $A \in F$ implies $A \in I$, and
4. $X \notin I$.

F/I is the countably complete boolean algebra formed by taking F and modding out by I, i.e. $A \approx B$ iff $A \Delta B \in I$. For $A \in F$ we use $[A]$ or $[A]_I$ to denote the equivalence class of A modulo I.

Theorem 9.1 *(Sikorski,Loomis, see [98] section 29) For any countably complete boolean algebra B there exists a σ-field F and a σ-ideal I such that B is isomorphic to F/I.*

proof:

Recall that the Stone space of B, stone(B), is the space of ultrafilters u on B with the topology generated by the clopen sets of the form:

$$[b] = \{u \in \text{stone}(B) : b \in u\}.$$

This space is a compact Hausdorff space in which the field of clopen sets exactly corresponds to B. B is countably complete means that for any sequence $\{b_n : n < \omega\}$ in B there exists $b \in B$ such that $b = \sum_{n\in\omega} b_n$. This translates to the fact that given any countable family of clopen sets $\{C_n : n \in \omega\}$ in stone(B) there exists a clopen set C such that $\bigcup_{n\in\omega} C_n \subseteq C$ and the closed set $C \setminus \bigcup_{n\in\omega} C_n$ cannot contain a clopen set, hence it has no interior, so it is nowhere dense. Let F be the σ-field generated by the clopen subsets of stone(B). Let I be the σ-ideal generated by the closed nowhere dense subsets of F (i.e. the ideal of meager sets). The Baire category theorem implies that no nonempty open subset of a compact Hausdorff space is meager, so $st(B) \notin I$ and the same holds for any nonempty clopen subset of stone(B). Since the countable union of clopen sets is equivalent to a clopen set modulo I it follows that the map $C \mapsto [C]$ is an isomorphism taking the clopen algebra of stone(B) onto F/I.

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Shortly after I gave a talk about my boolean algebra result (Theorem 8.2), Kunen pointed out the following result.

Theorem 9.2 *(Kunen see [73]) For every $\alpha \leq \omega_1$ there exists a field of sets H such that* ord$(H) = \alpha$.

proof:

Clearly we only have to worry about α with $2 < \alpha < \omega_1$. Let $\mathbb{B}$ be the complete boolean algebra given by Theorem 8.2. Let $\mathbb{B} \simeq F/I$ where F is a σ-field of sets and I a σ-ideal. Let $C \subseteq F/I$ be a countable set of generators. Define

$$H = \{A \in F : [A]_I \in C\}.$$

By induction on β it is easy to prove that for any $Q \in F$:

$$Q \in \utilde{\Sigma}^0_\beta(H) \text{ iff } [Q]_I \in \utilde{\Sigma}^0_\beta(C).$$

From which it follows that $\mathrm{ord}(H) = \alpha$.

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Note that there is no claim that the family H is countable. In fact, it is consistent (Miller [73]) that for every countable H either $\mathrm{ord}(H) \leq 2$ or $\mathrm{ord}(H) = \omega_1$.