§1. The construction of K^c

Let us fix a measurable cardinal, which we call Ω, for the remainder of this paper. We shall sometimes think of Ω as the class of all ordinals; we could have worked in 3rd order set theory + "OR is measurable", but opted for a little more room. Fix also a normal measure μ_0 on Ω.

We now define by induction on $\xi < \Omega$ premice $\mathcal{N}_\xi$. Having defined $\mathcal{N}_\xi$, we set

$$\mathcal{M}_\xi = \mathfrak{C}_\omega(\mathcal{N}_\xi),$$

the ωth core of $\mathcal{N}_\xi$. The $\mathcal{M}_\xi$'s will converge to the levels of K^c, the "background-certified" core model for one Woodin cardinal. That is, we shall define K^c by setting

$$\mathcal{J}_\beta^{K^c} = \text{ eventual value of } \mathcal{J}_\beta^{\mathcal{M}_\xi}, \quad \text{as } \xi \to \Omega,$$

for all $\beta < \Omega$. The construction of the $\mathcal{N}_\xi$'s follows closely the construction in §11 of [FSIT].

In this section, we shall simply assume that the $\mathcal{N}_\xi$'s are all "reliable", that is, that $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists and is k-iterable for all $k \leq \omega$. By Theorem 8.1 of [FSIT], this amounts to assuming that if $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists, then it is k-iterable (for all $k \leq \omega$). We shall sketch a proof of this iterability assumption in §2, and give a full proof in §9. We shall also assume here that certain bicephali and psuedo-premice associated to $\langle \mathcal{N}_\xi \mid \xi < \Omega \rangle$ are sufficiently iterable. We prove this in §9.

Iterability comes from the existence of background extenders.

Definition 1.1. *Let $\mathcal{M}$ be an active premouse, F the extender coded by $\dot{F}^{\mathcal{M}}$ (i.e. its last extender), $\kappa = crit(F)$, and $\nu = \dot{\nu}^{\mathcal{M}} =$ sup of the generators of F. Let $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$. Then an $\mathcal{A}$-certificate for $\mathcal{M}$ is a pair (N, G) such that*

(a) *N is a transitive, power admissible set, $V_\kappa \cup \mathcal{A} \subseteq N$, ${}^\omega N \subseteq N$, and G is an extender over N,*

(b) *$F \cap ([\nu]^{<\omega} \times \mathcal{A}) = G \cap ([\nu]^{<\omega} \times \mathcal{A})$,*

(c) *$V_{\nu+1} \subseteq Ult(N, G)$, and ${}^\omega Ult(N, G) \subseteq Ult(N, G)$,*

(d) *$\forall \gamma(\omega\gamma < OR^{\mathcal{M}} \Rightarrow \mathcal{J}_\gamma^{\mathcal{M}} = \mathcal{J}_\gamma^{i(\mathcal{J}_\kappa^{\mathcal{M}})})$, where $i : N \to Ult(N, G)$ is the canonical embedding.*

Definition 1.2. *Let $\mathcal{M}$ be an active premouse, and κ the critical point of its last extender. We say $\mathcal{M}$ is countably certified iff for every countable $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$, there is an $\mathcal{A}$-certificate for $\mathcal{M}$.*

In the situation described in 1.2 we shall typically have $|N| = \kappa$, so that $(\mathrm{OR} \cap N) < lhG$. We are therefore not thinking of (N, G) as a structure to be iterated; N simply provides a reasonably large collection of sets to be measured by G. The conditions $V_\kappa \subseteq N$ and $V_{\nu+1} \subseteq \mathrm{Ult}(N, G)$ are crucial; the closure of N and $\mathrm{Ult}(N, G)$ under ω-sequences could probably be dropped.

It might seem that certificates (N, G) as in 1.1 are too much to ask for, and in particular condition 1.1 (c) might seem too strong. But notice that we get such pairs by taking Skolem hulls: if $\pi : N \cong H \prec V_\eta$ inverts the collapse of H, where $V_\kappa \subseteq H$ but $\kappa \notin H$, then letting G be the length $\pi(\kappa)$ extender derived from π, G is an extender over N, $V_\kappa \subseteq N$, and $V_{\pi(\kappa)} \subseteq \mathrm{Ult}(N, G)$. We shall also see that the embedding associated to the measure μ_0 on Ω gives rise to certificates.

Definitions 1.1 and 1.2 were inspired in part by earlier attempts by Mitchell to formulate a background condition along these lines.

We proceed to the inductive definition of $\mathcal{N}_\xi$. As we define the $\mathcal{N}_\xi$'s we verify:

A_ξ : $\forall \alpha < \xi\ \forall \kappa$ (if $\rho_\omega(\mathcal{M}_\gamma) \geq \kappa$ for all γ s.t. $\alpha < \gamma \leq \xi$, then letting $\eta = (\kappa^+)^{\mathcal{M}_\alpha}$, $\mathcal{J}_\eta^{\mathcal{M}_\alpha} = \mathcal{J}_\eta^{\mathcal{M}_\xi}$).

(Here we let $\omega\eta = \mathrm{OR} \cap \mathcal{M}_\alpha$ in the case $\mathcal{M}_\alpha \models \kappa^+$ doesn't exist.)

We begin by setting $\mathcal{N}_0 = (V_\omega, \in)$. (So $\mathcal{M}_0 = \mathcal{N}_0$.) Now suppose $\mathcal{N}_\xi$, and hence $\mathcal{M}_\xi$, is given.

Case 1. $\mathcal{M}_\xi = (J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma)$ is a passive premouse, and there is an extender F over $\mathcal{M}_\xi$ such that

(1) $(J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma, \tilde{F})$ is a 1-small, countably certified premouse, and, letting $\kappa = \mathrm{crit}(F)$, we have

(2) κ is inaccessible,

(3) $(\kappa^+)^{\mathcal{M}_\xi} = \kappa^+ \Rightarrow$ for stationary many $\beta < \kappa$, β is inaccessible and $(\beta^+)^{\mathcal{M}_\xi} = \beta^+$.

In this case, we choose an F as above with $\nu(F)$, the sup of the generators of F, as small as possible, and we set

$$\mathcal{N}_{\xi+1} = (J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma, \tilde{F}).$$

As we mentioned above, the results of §9 and §8 of [FSIT] imply that $\mathcal{N}_{\xi+1}$ is reliable. Thus $\mathcal{M}_{\xi+1} = \mathfrak{C}_\omega(\mathcal{N}_{\xi+1})$ exists. We get $A_{\xi+1}$ from (the proof of) Theorem 8.1 of [FSIT].

Case 2. Otherwise.

In this case, let $\omega\gamma = \mathrm{OR} \cap \mathcal{M}_\xi$, and set

$$\mathcal{N}_{\xi+1} = (J_{\gamma+1}^{\dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}}, \in, \dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}).$$

Again, $\mathcal{N}_{\xi+1}$ is reliable by §9, and 8.1 of [FSIT] yields $A_{\xi+1}$.

So in Case 1 we add a countably certified extender to our extender sequence, while in Case 2 we take one step in the constructible closure of the sequence we have. In both cases we then form the ωth core of the structure we have.

Now suppose we have defined $\mathcal{N}_\xi$ for $\xi < \lambda$, where $\lambda < \Omega$ is a limit ordinal. Set

$$\eta = \liminf_{\xi \to \lambda} (\rho_\omega(\mathcal{M}_\xi)^{+^{\mathcal{M}_\xi}})$$

(where $\rho_\omega(\mathcal{M}_\xi)^{+^{\mathcal{M}_\xi}} = \mathrm{OR} \cap \mathcal{M}_\xi$ is possible. We set $\mathcal{N}_\lambda$ = unique passive premouse $\mathcal{P}$ s.t.

(a) $\forall \beta < \eta(\mathcal{J}_\beta^{\mathcal{P}}$ = eventual value of $\mathcal{J}_\beta^{\mathcal{M}_\xi}$ as $\xi \to \lambda)$ and (b) $\mathcal{J}_\eta^{\mathcal{P}} = \mathcal{P}$. Such a premouse exists as A_ξ holds for all $\xi < \lambda$. It is easy to verify A_λ.

This completes the inductive definition of the $\mathcal{N}_\xi$, for $\xi < \Omega$.

Theorem 8.1 of [FSIT] actually gives the following strengthening of our induction hypothesis A_η. (Cf. 11.2 of [FSIT].)

Lemma 1.3. *Suppose $\kappa \leq \rho_\omega(\mathcal{M}_\xi)$ for all $\xi \geq \alpha_0$, and let $\xi \geq \alpha_0$ be such that $\kappa = \rho_\omega(\mathcal{M}_\xi)$. Then $\mathcal{M}_\xi$ is an initial segment of $\mathcal{M}_\eta$, for all $\eta \geq \xi$; moreover $\mathcal{M}_{\xi+1}$ satisfies "every set has cardinality $\leq \kappa$".*

Lemma 1.3 implies $\liminf_{\xi \to \Omega} \rho_\omega(\mathcal{M}_\xi) = \Omega$, and therefore we can define a premouse K^c of ordinal height Ω by

$$\mathcal{J}_\beta^{K^c} = \text{eventual value of} \quad \mathcal{J}_\beta^{\mathcal{M}_\xi}, \quad \text{as} \quad \xi \to \Omega\,,$$

for all $\beta < \Omega$.

The following is a cheapo form of weak covering. It is crucial in what follows; it tells us that we haven't been too miserly about putting extenders on the K^c sequence.

Theorem 1.4. *Exactly one of the following holds*:

(a) $K^c \models$ *There is a Woodin cardinal,*

(b) *for* μ_0- *a.e.* $\alpha < \Omega$, $(\alpha^+)^{K^c} = \alpha^+$.

Proof. (a) $\Rightarrow$ $\neg$ (b): Every $\mathcal{J}_\beta^{K^c}$ is 1-small, so if $K^c \models \delta$ is Woodin, then $E_\gamma^{K^c} = \emptyset$ if $\gamma \geq \delta$. As Ω is measurable, there is a club class of indiscernibles for K^c, or equivalently, a countable mouse $\mathcal{N}$ which is not 1-small. Comparing $\mathcal{N}$ with K^c, we get that for μ_0- a.e. $\alpha < \Omega$, $(\alpha^+)^{K^c}$ has cofinality ω in V.

Remark. We have no use for this direction in what follows.

$\neg$ (b) $\Rightarrow$ (a):

Let $j : V \to M = \mathrm{Ult}(V, \mu_0)$ be the canonical embedding. We are assuming then

$$(\Omega^+)^{j(K^c)} < \Omega^+\,.$$

(Of course, $\Omega^{+^M} = \Omega^+$.) Let $\mathcal{A} = P(\Omega) \cap j(K^c)$, so that $\mathcal{A} \in M$ and $M \models |\mathcal{A}| = \Omega$. Let E_j be the $(\Omega, j(\Omega))$ extender derived from j. By an ancient argument due to Kunen, whenever $|\mathcal{B}| = \Omega$, $E_j \cap ([j(\Omega)]^{<\omega} \times \mathcal{B}) \in M$.

(Proof: if $\mathcal{B} = \{B_\alpha \mid \alpha < \Omega\}$, then notice $\langle j(B_\alpha) \mid \alpha < \Omega\rangle = j(\langle B_\alpha \mid \alpha < \Omega\rangle) \restriction \Omega$, so $\langle j(B_\alpha) \mid \alpha < \Omega\rangle \in M$. Also, $B_\alpha \in (E_j)_c$ iff $c \in j(B_\alpha)$.)

In particular, setting

$$F = E_j \cap ([j(\Omega)]^{<\omega} \times j(K^c))\,,$$

we have that $F \in M$. Now it cannot be that every proper initial segment $F \restriction \nu$, $\nu < j(\Omega)$, of F belongs to $j(K^c)$, as otherwise these initial segments witness that Ω is Shelah in $j(K^c)$. But if $K^c \models$ There are no Woodins, then as $\mathcal{J}_\Omega^{j(K^c)} = K^c$, 1-smallness is not a barrier to adding these $F \restriction \nu$ to the $j(K^c)$ sequence. The following claim asserts that there are no other barriers. Its statement and proof run parallel to those of Lemma 11.4 of [FSIT].

Claim. Suppose $K^c \models$ There are no Woodin cardinals. Let $(\Omega^+)^{j(K^c)} \leq \rho < j(\Omega)$ and suppose that either $\rho = (\Omega^+)^{j(K^c)}$, or $\rho - 1$ exists and is a generator of F, or ρ is a limit of generators of F. Let G be the trivial completion of $F \restriction \rho$, and $\gamma = lhG$. Let $\boldsymbol{E}$ be the extender sequence of $j(K^c)$. Then either

(a) $\rho = (\Omega^+)^{j(K^c)}$, $\gamma \in \operatorname{dom} \boldsymbol{E}$, and $E_\gamma = G = F \restriction \gamma$, or

(b) $\rho - 1$ exists, $\gamma \in \operatorname{dom} \boldsymbol{E}$, and $E_\gamma = G = F \restriction \gamma$, or

(c) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$, ρ is not a generator of F, and $\gamma \in \operatorname{dom} \boldsymbol{E}$ and $E_\gamma = G = F \restriction \gamma$, or

(d) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$ ρ is itself a generator of F, $\rho \notin \operatorname{dom} \boldsymbol{E}$, and $\gamma \in \operatorname{dom} \boldsymbol{E}$ and $E_\gamma = G$ (but $E_\gamma \neq F \restriction \gamma$), or

(e) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$, ρ is itself a generator of F, $\rho \notin \operatorname{dom} \boldsymbol{E}$, and $\pi(\boldsymbol{E})_\gamma = G$, where π is the canonical embedding from $J_\rho^{\boldsymbol{E}}$ to $\operatorname{Ult}_0(J_\rho^{\boldsymbol{E}}, E_\rho)$.

Proof. By induction on ρ. As the proof is rather convoluted and follows closely the proof of Lemma 11.4 of [FSIT], we shall not give it here. The idea is as follows: we show that G satisfies, in M, all the conditions for being added to the $j(K^c)$ sequence as E_γ. We have 1-smallness by hypothesis, and coherence because F is a restriction of E_j. (If ρ is a generator of F, then we have to appeal to the condensation Theorem 8.2 of [FSIT] here.) We have the initial segment condition by induction. For our background certificates we can take (N, H), where $V_\Omega \cup \mathcal{A} \subseteq N$, $|N| = \Omega$, and $H = E_j \cap ([j(\Omega)]^{<\omega} \times N)$. (As we remarked earlier, $H \in M$.) Since G can be added to $j(K^c)$ in M as E_j, the construction of $j(K^c)$ guarantees $\gamma \in \operatorname{dom} \boldsymbol{E}$. A "bicephalus" or "Doddage" argument guarantees $G = E_\gamma$.

The foregoing is more or less a proof in the case (a) or (c) of the conclusion holds. If (b) holds, we run into technical problems with mixed type bicephali. If (d) or (e) holds, we also run into the problem that $\mathcal{J}_\gamma^{j(K^c)}$ may not be a stage $\mathcal{M}_\xi$ of the construction of $j(K^c)$ done within M. We refer the reader to §11 of [FSIT] for a full proof.

We should note that this argument requires the iterability of the bicephali and psuedo-premice which arise. (See §11 of [FSIT] for more detail on how they arise.) We shall prove this in §9. □

We define $A_0 \subseteq \Omega$ by

$$\alpha \in A_0 \Leftrightarrow \alpha \text{ is inaccessible and } (\alpha^+)^{K^c} = \alpha^+ \text{ and}$$
$$\{\beta < \alpha \mid \beta \text{ is inaccessible and } (\beta^+)^{K^c} = \beta^+\} \text{ is not stationary in } \alpha\,.$$

One can easily check that if $K^c \models$ there are no Woodin cardinals, then

(i) A_0 is stationary in Ω,

(ii) A_0 has μ_0-measure 0,
(iii) $\alpha \in A_0 \Rightarrow ((\alpha^+)^{K^c} = \alpha^+ \wedge \alpha$ is inaccessible),
(iv) $\alpha \in A_0 \Rightarrow \alpha$ is not the critical point of any total-on-K^c extender from the K^c sequence.

It was in order to insure the existence of a set with the properties of A_0 that we included condition (3) in Case 1 of the construction of K^c. (Condition (2) could be dropped, but it does no harm.)

The definition of K^c which we have given in this section is unnatural in one respect: its requirement that the $\mathcal{N}_\xi$'s, and hence K^c itself, be 1-small. We believe that, were this restriction simply dropped, the resulting $\mathcal{N}'_\xi$'s would converge to a model $(K^c)'$ of height Ω, and one could show that either $(\alpha^+)^{(K^c)'} = \alpha^+$ for μ_0 a.e. $\alpha < \Omega$, or $(K^c)' \models$ there is a superstrong cardinal. What is missing at the moment is a proof that if M is a countable elementary submodel of one of the $\mathcal{N}'_\xi$'s or their associated psuedo-premice and bicephali, then M is $\omega_1 + 1$ iterable (in the sense of definition 2.8 of this paper). At the moment we can only prove this iterability result for mice which are "tame" (do not have extenders overlapping Woodin cardinals), and thus it is only to such mice that we can extend the theory presented here. (See [CMWC] and [TM].)