The Core Model Iterability Problem

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These notes develop a method for constructing *core models*, that is, canonical inner models of the form $L[\boldsymbol{E}]$, where $\boldsymbol{E}$ is a coherent sequence of extenders. They extend the earlier work in this area of Dodd and Jensen ([DJ1], [DJ2], [DJ3]) and Mitchell ([M1], [M?]). The Dodd-Jensen theory produces models having measurable cardinals, and Mitchell's extension of it produces models having measurable cardinals κ of order κ^{++}. Here we shall extend this theory so that it can produce core models having Woodin cardinals.

The extent of our debt to Dodd, Jensen, and Mitchell will become apparent; nevertheless, we shall not assume that the reader is familiar with their work. We shall, however, assume that he is familiar with the fine structure theory for core models having Woodin cardinals which is developed in [FSIT].

Our work here goes beyond [FSIT] in that it involves a construction of $L[\boldsymbol{E}]$ models which makes no use of extenders over V. Many of the applications of core model theory require such a construction. The authors of [FSIT] use extenders over V in order to show that the inner model $L[\boldsymbol{E}]$ they construct is sufficiently iterable: roughly, they demand that the extenders put onto $\boldsymbol{E}$ be the restrictions of background extenders over V, then use this fact to embed iteration trees on $L[\boldsymbol{E}]$ into iteration trees on V, and then quote the results of [IT] concerning iteration trees on V. Here we shall describe a weakened background condition on the extenders put onto $\boldsymbol{E}$ which does not require full extenders over V, and yet suffices to carry out something like the old proof of the iterability of $L[\boldsymbol{E}]$. The result is a solution to what is called the "core model iterability problem" in [FSIT].

The notes are organized as follows. As in Mitchell's work on the core model for sequences of measures ([M1], [M?]), we construct the model K in which we are ultimately most interested in two steps. In §1 we construct a model, which we call K^c, whose extenders have "background certificates". These background certificates guarantee the iterability of K^c, its levels, and various associated structures. (In Mitchell's work, the background condition is countable completeness, but here we seem to need more.) In order to show that K^c and the K we derive from it are large enough to be useful, we seem to need something like the existence of a measurable cardinal. We fix a normal measure μ_0 on a measurable cardinal Ω throughout this paper; we shall have $\mathrm{OR} \cap K^c = \mathrm{OR} \cap K = \Omega$. We use the measurability of Ω to show in §1 that either $K^c \models$ there is a Woodin cardinal, or $(\alpha^+)^{K^c} = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$. Since in applications we are seeking an inner model with a Woodin cardinal, we assume through most of the paper that there is no such model, and thus we have $(\alpha^+)^{K^c} = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$. As in Mitchell's work, this "weak covering property" of K^c is crucial.

We also use the measurability of Ω in a different way in §4. Further, we use the tree property of Ω to show that iteration trees of length Ω are well behaved. The fact that we do not develop the basic theory of K within ZFC may be a defect in our work.

In §2 we sketch the main new ideas in the proof that K^c is iterable. In §9 we give a full proof of a general iterability theorem which covers iteration trees and psuedo-iteration trees on K^c, its levels, and the associated bicephali and psuedo-premice.

As in Mitchell's work, the "true core model" K is a Skolem hull of K^c. In §3 and §4 we develop some concepts, derived from Mitchell's work, which are useful in the construction of this hull. In §5 we do the construction: given a stationary $S \subseteq \Omega$ with certain properties, we construct a model $K(S) \preceq K^c$. We show $(\alpha^+)^{K(S)} = \alpha^+$ for μ_0- a.e. α. We also show that $K(S)$ is invariant under small forcing; that is, $K(S) = K(S)^{V[G]}$ whenever G is V generic over some $\mathbb{P} \in V_\Omega$. Finally, we show that $K(S)$ is independent of S, and define K to be the common value of $K(S)$ for all S. We have then that $(\alpha^+)^K = \alpha^+$ for μ_0 - a.e. α, and that K is invariant under set forcing.

In §6 we give an optimally simple inductive definition of K: it turns out that $K \cap HC$ is $\Sigma_1(L_{\omega_1}(\mathbb{R}))$. (Woodin has shown that no simpler definition is possible in general. Mitchell showed in [M?] that if no initial segment of K satisfies $(\exists\kappa)(o(\kappa) = \kappa^{++})$, then $K \cap HC$ is Σ^1_5 in the codes.) In §7 we use the machinery we have developed to obtain the consistency strength lower bound of one Woodin cardinal for various propositions. In §8 we return to the pure theory, and obtain some information concerning embeddings of K. We show, for example, that if there is no inner model with a Woodin cardinal, then there is no nontrivial elementary $j : K \to K$. In contrast to the situation for "smaller K's", however, we show that there may be nontrivial elementary $j : K \to M$ which are not iteration maps.

Among the applications of the theory developed in these notes are the following theorems.

Theorem 0.1. *Let Ω be measurable, and suppose there is a presaturated ideal on ω_1; then there is a transitive set $M \subseteq V_\Omega$ such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

Corollary 0.2. *If Martin's Maximum holds, then there is a transitive set M such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

(It is known from [FMS] that Martin's Maximum implies that there is an inner model with a measurable cardinal and a saturated ideal on ω_1; by applying 0.1 inside this model we get 0.2. H. Woodin pointed this out to the author.)

Further work of Mitchell, Schimmerling, and the author on the weak covering property for K (cf. [WCP]) together with his work on Jensen's $\square$ principle in K (cf. [Sch]), led Schimmerling to the following improvement of 0.2.

Theorem 0.3. (Schimmerling, cf. [Sch]) *If PFA holds, then there is a transitive set M such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

(Theorem 0.3 also relies on an improvement, due to Magidor, of Todorčević's result that PFA implies $\forall\kappa$ ($\square_\kappa$ fails). (See [To].)

In a different vein, we have the following, more immediate applications of the theory presented here.

Theorem 0.4. *Suppose that every set of reals which is definable over $L_{\omega_1}(\mathbb{R})$ is weakly homogeneous; then there is a transitive set M such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

(H. Woodin supplied a crucial step in the proof of 0.4.) Since Woodin (unpublished) has shown that the existence of a strongly compact cardinal implies the hypothesis of 0.4, we have

Corollary 0.5. *Suppose there is a strongly compact cardinal; then there is a transitive set M such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

(We shall give a different proof of 0.5, one which avoids Woodin's unpublished work, in §8.)

We can also improve the lower bound of [IT] on the strength of the failure of UBH.

Theorem 0.6. *Let Ω be measurable, and suppose there is an iteration tree $\mathcal{T}$ on V_Ω such that $\mathcal{T} \in V_\Omega$ and $\mathcal{T}$ has distinct cofinal wellfounded branches; then there is a transitive set $M \subseteq V_\Omega$ such that*

$$M \models ZFC + \text{“There are two Woodin cardinals”}.$$

We can use the methods presented here to re-prove Woodin's result that $\forall x \in^\omega \omega (x^\sharp$ exists$) + \Delta^1_2$ determinacy implies that there is an inner model with a Woodin cardinal. Finally, using an idea of G. Hjorth, the author has recently (at least partially) generalized Jensen's Σ^1_3 correctness theorem to the core model constructed here. This yields a positive answer to a conjecture of A. S. Kechris.

Theorem 0.7. *Assume $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists and $\Sigma^1_3(x)$ has the separation property); then there is a transitive set M such that*

$$M \models ZFC + \text{“There is a Woodin cardinal”}.$$

Each of the hypotheses of the theorems above is known to be consistent under some large cardinal hypothesis or other. We shall not attempt a scholarly discussion of the history of or context for these theorems, as our focus here is the basic theory which produces them. We shall prove 0.1, 0.4, 0.6, and 0.7 in §7.

Historical note. We did most of the work described here in the Spring of 1990, and informally circulated it in a set of handwritten notes. Our main advance, which was isolating K^c and proving the results of §1 and §2 concerning it, was inspired in part by some ideas of Mitchell. The work in §8 was done somewhat later, and the Σ^1_3 correctness theorem of §7D was not proved until the Spring of 1993.