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DE RHAM THEORY AND COCYCLES OF CUBICAL SETS FROM SMOOTH QUANDLES

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Abstract

We show a de Rham theorem for cubical manifolds, and study rational homotopy type of the classifying spaces of smooth quandles. We also show that secondary characteristic classes in [8, 9] produce cocycles of quandles.

1. Introduction

Characteristic classes in topology are interpreted as cohomology classes of the classifying space of a Lie group G. According to Chern-Weil theory, the classes are recovered from some invariant theory. Dupont [7] used simplicial manifolds to study the classifying spaces, and reformulate the Chern-Weil theory universally. Moreover, according to the enriched Chern-Weil theory [8, 9], the characteristic classes (with a condition) produce cocycles of G^{δ}, where G^{δ} is the Lie group G with descrete topology. This approach recovers some of secondary characteristic classes, including the Chern-Simon class.

Meanwhile, a quandle [17, 21] is a set with a certain binary operation; a typical example is a homogenous set as in symmetric space (see §§2–3 for the details). Furthermore, as an analog of the classifying space of a group, Fenn, Rourke, and Sanderson [12] defined a space BX from a quandle X, which is called the rack space, and is cubically constructed from a $\square$-set; cocycles in the cohomology provided applications to low-dimensional topology (see [2, 3]), e.g., including the Chern-Simon invariant [15] and K_2-invariant [24] of links. However, in most papers on quandles, X was assumed to be equipped with descrete topology.

In this paper, we focus on the situation where a quandle X has a manifold structure as a homogenous space, and we study the cohomology of BX. After Section 2 reviews quandles with manifold structure, Section 3 discusses differential forms on cubical manifolds, and shows a de Rham theorem on BX (Theorem

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3.5): This result is a cubical translation of [7]. As a corollary, Section 4 completely determines the rational cohomology of the rack space BX, where the cohomology of X satisfies some conditions. Furthermore, for such an X, Section 5 provides a formula of computing the rational homotopy type of BX, as in Milnor-Moore theorem; see Theorem 5.1.

In Sections 6–7, we will examine a contrast between the cohomology groups of BX and BX^{δ}, where X^{δ} means the discrete topology of X. First, we show (Theorem 6.4) that if X is compact and "semi-homogenous", every $\mathbf{R}$-value continuous cocycle of BX^{δ} is trivial (cf. the computation of second (co)-homology of BX^{δ}; see Appendix B). To obtain non-trivial cocycles, the last section 7 examines cocycles with the coefficient $\mathbf{C}/\mathbf{Z}$ modulo $\mathbf{Z}$, where we use a chain map of Inoue-Kabaya [15] to bridge the complex of BX^{δ} and the enriched Chern-Weil theory. As a result, we show (Proposition 7.3) that every secondary characteristic class in the sense of [8, 9] yields a $\mathbf{C}/\mathbf{Z}$-value cocycle of BX^{δ}. Hence, in doing so, we hope that this proposition produces many cocycles of a quandle BY, when Y is a subquandle of X.

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2. Preliminaries on smooth quandles

We start by reviewing quandles and smooth quandles. A *quandle* [17, 21] is a set Q with a binary operation $\lhd : Q^2 \to Q$ satisfying the following three:

(Q1) For any $x \in Q$, $x \lhd x = x$,

(Q2) For any $x, y \in Q$, there exists a unique element $z \in Q$ such that $z \lhd y = x$,

(Q3) For any $x, y, z \in Q$, $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$.

A *smooth quandle* is a C^{∞}-manifold Q with a C^{∞}-map $\lhd : Q^2 \to Q$ satisfying (Q1), (Q3) and that $(\bullet \lhd x) : Q \to Q$ is diffeomorphic for any $x \in Q$. Let $\mathrm{Inn}(Q)$ be the subgroup of $\mathrm{Diff}(Q)$ generated by $(\bullet \lhd y)$, where y runs over Q. We equip $\mathrm{Inn}(Q) \subset \mathrm{Diff}(Q)$ with the compact open topology. A quandle Q is said to be *transitive*, if the action of $\mathrm{Inn}(Q)$ on Q is transitive; see [17, 21]. A quandle Q is *of type n*, if there exists $n \in \mathbf{Z}$ which is the minimal number satisfying $x \lhd^n y = x$ for any $x, y \in Q$.

Example 2.1. Let X be a symmetric space, i.e., a C^{∞}-manifold equipped with a Riemannian metric such that each point $y \in X$ admits an isometry $s_y : X \to X$ that reverses every geodesic line $\gamma : (\mathbf{R}, 0) \to (X, y)$, meaning that $s_y \circ \gamma(t) = \gamma(-t)$. Then, X has a quandle structure of type 2 defined by $x \lhd y := s_y(x)$. In addition, similar Riemannian manifolds with quandle structure of type > 2 are studied in [18] as *generalized symmetric spaces*.

Example 2.2 ([17, 21]). As an important example in this paper, we will see that transitive quandle structures turn to be good operations defined on homogenous spaces. Let G be a Lie group, and H be a closed subgroup. If $z_0 \in G$ commutes with any $h \in H$, then the homogenous space $H \backslash G$ has a quandle structure given by

$$[x] \lhd [y] := [z_0^{-1} x y^{-1} z_0 y], \tag{1}$$

for representatives $x, y \in G$. In what follows, we write (G, H, z_0) for such a transitive quandle. We define $\kappa : H \backslash G \to G$ by the map which sends $[x]$ to $x^{-1} z_0 x$, which the reader should keep in mind.

Conversely, we will explain that if Q is a smooth quandle and is transitive, Q is reduced to some (G, H, z_0). For $x_0 \in Q$, let $\mathrm{Stab}(x_0) \subset G$ be the stabilizer subgroup of x_0. We equip the group $\mathrm{Inn}(Q)$ with a quandle operation given by (1). Then it is known [17, Theorem 7.1] that the natural map

$$\mathrm{Inn}(Q) \to Q \quad \text{given by } g \mapsto x_0 \cdot g \tag{2}$$

is a quandle homomorphism, which induces the quandle isomorphism $\mathrm{Stab}(x_0) \backslash \mathrm{Inn}(Q) \cong Q$. Moreover, Ishikawa [16, Theorem 2.4] showed that $\mathrm{Inn}(Q)$ is a Lie group. In conclusion, the structure of the smooth quandle Q is determined by the Lie groups $\mathrm{Stab}(x_0) \subset \mathrm{Inn}(Q)$.

Accordingly, throughout this paper, we mainly focus on such smooth quandles (G, H, z_0), which are transitive quandles.

Moreover, we now observe the situation that G is compact. Then G has the Haar measure dg. By taking the quotient of dg, the smooth quandle Q has a metric such that $(\bullet \lhd x) : Q \to Q$ is isometric for any $x \in Q$. In other words, such a smooth quandle Q is called a metrizable s-manifolds in the book [18]. Hence, the topological type of such a Q is restricted, and is classified in some cases. For example, if $\pi_1(Q) = 0$, the type is of finite order, and G is a simple Lie group, then Q is a formal space in the sense of the rational homotopy theory; see [19] and references therein.

3. Preliminaries on cubical manifolds and differential n-forms

We introduce cubical manifolds, modifying the concept of $\square$-sets of Fenn-Rourke-Sanderson [12]. The discussion in this section is a cubical analogy of simplicial manifolds [7, §2]. A *cubical manifold* is a sequence of C^∞-manifolds $\{X_p\}_{p \in \mathbf{N}}$ together with *face* C^∞*-maps* $\delta_i^\varepsilon : X_p \to X_{p-1}$, for $\varepsilon \in \{0, 1\}$ and $1 \le i \le p$, satisfying

$$\delta_{j-1}^\eta \circ \delta_i^\varepsilon = \delta_i^\varepsilon \circ \delta_j^\eta, \quad \text{for any } 1 \le i < j \le p \quad \text{and} \quad \varepsilon, \eta \in \{0, 1\}.$$

Let I be the interval $[0, 1] \subset \mathbf{R}$, and I^p be the p-cube. Dually, for $1 \le i \le p$ and $\varepsilon \in \{0, 1\}$, we consider the map

$$\delta_i^\varepsilon : I^{p-1} \to I^p \quad \text{defined by } \delta_i^\varepsilon(t_1, \ldots, t_{p-1}) = (t_1, \ldots, t_{i-1}, \varepsilon, t_i, \ldots, t_{p-1}).$$

Then the (*fat*) *realization* $\|X\|$ of a cubical manifold X is defined to be the quotient space of $\bigsqcup_p I^p \times X_p$ subject to the relation $(\delta_i^\varepsilon(\mathrm{t}), x) \sim (\mathrm{t}, \delta_i^\varepsilon(x))$, where $\mathrm{t} \in I^{p-1}$ and $x \in X_p$ with $i = \{0, \dots, p\}$ and $\varepsilon \in \{0, 1\}$.

Example 3.1 (Rack space). Fenn-Rourke-Sanderson [12] introduced a classifying space as a cubical set, which is called *the rack space*. We will give the rack space of manifold version. Fix a smooth quandle (G, H, z_0) as in Example 2.2, and a manifold Y which is acted on by G (possibly $Y = \{\mathrm{pt.}\}$, $Y = Q$ or $Y = G$). Then, we define X_p to be $Y \times Q^p$, and define δ_j^ε by

$$\delta_j^0(y, x_1, \dots, x_p) = (y, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_p),$$

$$\delta_j^1(y, x_1, \dots, x_p) = (y \cdot \kappa(x_j), x_1 \lhd x_j, \dots, x_{j-1} \lhd x_j, x_{j+1}, \dots, x_p).$$

Then, the pair $(X_*, \delta_*^\varepsilon)$ is a cubical manifold. Moreover, the realization $\|X\|$ is exactly the rack space defined in [12, 13]. We will denote $\|X\|$ by $B_Y Q$. If Y is a singleton, we write BQ for $B_Y Q$ for simplicity. We remark that the canonical projection $B_Y Q \to BQ$ is a fibration with fiber Y.

Next, we will establish terminology of C^∞-forms on cubical manifolds.

Definition 3.2. (1) Let $\mathscr{A}^n(I^p \times X_p)$ be the set of n-forms on $I^p \times X_p$ of C^∞-class which are extended to n-forms on $\mathbf{R}^p \times X_p$.

(2) Similarly, we define $\mathscr{A}^n(I^p)$ by the set of n-forms on I^p of C^∞-class which are extended to n-forms on $\mathbf{R}^p$, and define $\mathscr{A}^n(X_p)$ by the set of n-forms on X_p of C^∞-class.

(3) An *n-form* φ on a cubical manifold is a sequence of n-forms $\phi^{(p)} \in \mathscr{A}^n(I^p \times X_p)$ satisfying the conditions $(\delta_i^\varepsilon \times \mathrm{id})^* \phi^{(p)} = (\mathrm{id} \times \delta_i^\varepsilon)^* \phi^{(p-1)}$ for any $i \in \{1, \dots, p\}$ and $\varepsilon \in \{0, 1\}$.

(4) We denote by $A^n(X)$ the set of all n-forms on X.

Then, the exterior differential d and the wedge product on $\mathscr{A}^n(I^p \times X_p)$ can be extended to those on $A^n(X)$. Thus, $A^*(X)$ is made into a differential graded algebra.

Next, we give bigraded complexes. Let $q_1 : I^p \times X_p \to I^p$ and $q_2 : I^p \times X_p \to X_p$ be the natural projections. Given a cubical manifold X, we first decompose $A^*(X)$ into a direct sum $A^n(X) = \bigoplus_{n=k+\ell} A^{k,\ell}(X)$, where $A^{k,\ell}(X)$ is composed of the forms φ of type (k, ℓ), i.e., φ restricted to $I^p \times X_p$ is presented by $q_1^*(\phi_I^{(k)}) \times q_2^*(\phi_X^{(\ell)})$ for some $\phi_I^{(k)} \in \mathscr{A}^k(I^p)$ and $\phi_X^{(\ell)} \in \mathscr{A}^\ell(X_p)$. Also let $d_\square$ (resp. d_X) denote the pullback of exterior differential on $\mathscr{A}^*(I^p)$ (resp. on $\mathscr{A}^*(X_p)$). Thus, we have a double complex $(A^{k,\ell}(X), d_\square, d_X)$, and the total complex $(A^*(X), d_{\mathrm{tot}})$, where $d_{\mathrm{tot}} = d_\square + d_X$. Further, we can define another double complex $(\mathscr{A}^{k,\ell}(X), \delta, d_X)$, where $\mathscr{A}^{k,\ell}(X) = \mathscr{A}^\ell(X_k)$ and $\delta = \sum_{i=1}^p (-1)^i (\delta_i^0 - \delta_i^1)$.

Then, we later give an isomorphism between the (double) complexes

THEOREM 3.3 (A cubical version of [7, Theorem 2.3].) *Assume that each X_p is a paracompact Hausdorff space. For any $\ell \in \mathbf{N}$ the chain complexes $(A^{*,\ell}(X), d_\square)$ and $(\mathscr{A}^{*,\ell}(X), \delta)$ are naturally chain homotopy equivalent. To be precise, there is a map $\mathscr{J} : A^{k,\ell}(X) \to \mathscr{A}^{k,\ell}(X)$ which gives a homotopy equivalence.*

Instead of giving the proof later (see Appendix A), we mention a corollary from the spectral sequences associated with the two double complexes. Consider the filtering with respect to the first index of the double complexes $A^{**}(X)$ and $\mathscr{A}^{**}(X)$; we have the spectral sequences $I(A)_r^{**}$ and $I(\mathscr{A})_r^{**}$, respectively. In parallel, we have other spectral sequences $II(A)_r^{**}$ and $II(\mathscr{A})_r^{**}$ by filtering with respect to the second index. As a consequence of Theorem 3.3, as a de Rham theory of cubical sets, the de Rham cohomology of $A^*(X)$ is isomorphic to the ordinary cohomology $H^*(\|X\|; \mathbf{R})$ of the fat realization $\|X\|$. To be precise,

COROLLARY 3.4. *The map $\mathscr{J}$ induces natural isomorphisms $I(A)_r^{**} \cong I(\mathscr{A})_r^{**}$ for $r \geq 2$ and $II(A)_r^{**} \cong II(\mathscr{A})_r^{**}$ for $r \geq 1$. In particular, they induce a canonical isomorphism from the cohomology of the total complexes, $\mathscr{K}_X : H^*(A^*(X), d_{\mathrm{tot}}) \cong H^*(\|X\|; \mathbf{R})$.*

Moreover, we will show the multiplication, although we defer the proof into Appendix A.

THEOREM 3.5 (Cubical version of [7, Theorem 2.14].) *Suppose that each X_p is a paracompact Hausdorff space. Then the isomorphism $\mathscr{K}_X : H^*(A^*(X), d) \cong H^*(\|X\|; \mathbf{R})$ is multiplicative where the multiplication on the left (resp. right) hand side is induced by the wedge-product (resp. the cup-product).*

4. Note on rational cohomology of the rack spaces

In this section, we will compute the rational cohomology of the rack space BQ. For this, we consider the invariant part, $A^n(Q)^G$, of n-forms, where the action of G on Q is induced from the right actions in (2). We have the inclusion $A^n(Q)^G \hookrightarrow A^n(Q)$.

PROPOSITION 4.1. *Let Q be a smooth quandle of the form (G, H, z_0). Assume that the inclusion $A^n(Q)^G \hookrightarrow A^n(Q)$ yields an isomorphism on cohomology. Then, there are isomorphisms*

$$H^n(BQ; \mathbf{R}) \cong \bigoplus_{n=i+j} H^i(Q^j; \mathbf{R}), \quad H^n(B_G Q; \mathbf{R}) \cong \bigoplus_{n=i+j} H^i(G \times Q^j; \mathbf{R}).$$

Proof. We consider the spectral sequence $II(\mathscr{A})_r^{**}$ in §3, which strongly converges to $E_\infty^n \cong H^n(A^*(B_G Q)) \cong H^n(B_G Q; \mathbf{R})$.

We will study the $E_1^{p,q}$-term $H^p(\mathscr{A}^*(Q^q))$ in detail. We let $\mathscr{A}^*(Q^q)^{G^q}$ be the set of G^q-invariant forms on Q^q, where G^q acts on Q^q componentwise. By assumption, the inclusion $\mathscr{A}^*(Q^q)^{G^q} \hookrightarrow \mathscr{A}^*(Q^q)$ is a quasi-isomorphism for any q. For any G^q invariant p-form $\psi \in \mathscr{A}^p(Q^q)^{G^q}$, we note $(\delta_i^0 - \delta_i^1)^*(\psi) = 0$ by definition; therefore, $\delta^*(\psi) = (\sum_{i=1}^q (-1)^i(\delta_i^0 - \delta_i^1))^*(\psi) = 0$. Thus, this spectral sequence collapses at $E_2^{p,q}$, i.e., $E_2 = E_\infty$. Hence, we can get the conclusion:

$$H^n(BQ;\mathbf{R}) \cong H^n(A^*(BQ)) \cong E_\infty^n \cong \bigoplus_{n=i+j} E_2^{i,j} \cong \bigoplus_{n=i+j} H^i(Q^j;\mathbf{R}).$$

Next, we will show the second isomorphism in a similar way. Consider the spectral sequence $II(\mathscr{A})_r^{**}$ in §3, where $X_p = G \times X^p$. Then, we can readily see that this spectral sequence $E_2^{p,q}$ abuts to $E_\infty^{p,q}$. To conclude, we have the second claim as follows:

$$H^n(B_GQ;\mathbf{R}) \cong H^n(A^*(B_GQ)) \cong E_\infty^n \cong \bigoplus_{n=i+j} E_2^{i,j} \cong \bigoplus_{n=i+j} H^i(G \times Q^j;\mathbf{R}). \qquad \square$$

Although the assumption in this proposition seems strong, there are many examples.

Example 4.2. If Q is the $2m$-sphere, and G is the orthogonal group $O(2m+1)$, then the generator of $H^{2m}(S^{2m}) \cong \mathbf{R}$ is represented by the $O(2m+1)$-invariant volume form. Thus, $A^*(Q)^G \hookrightarrow A^*(Q)$ is quasi-isomorphic.

As another example, consider the unitary group $G = U(m)$ and the Grassmann manifold $\mathrm{Gr}(m,n)$ over $\mathbf{C}$, where $m, n \in \mathbf{N}$ with $n < m$. The cohomology is generated by the Chern classes. Chern-Weil theory implies that the Chern classes are invariant with respect to the action of $U(m)$. Hence, this situation satisfies the assumption.

In general, if G is compact, the Cartan algebra of G/H enables us to compute $H^n(G/H;\mathbf{R})$ with generators from some information of $\bigwedge^* \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G; see [25] and references therein for the details. Thus, we can check whether G/H satisfies the assumption or not.

Remark 4.3. As seen in the proof, for $Q = (G, H, z_0)$, the inclusion $A^n(Q)^G \hookrightarrow A^n(Q)$ gives rise to a ring homomorphism $H^*(BQ;\mathbf{R}) \to \bigoplus_{n=i+j} H^i(Q^j;\mathbf{R})$. However, in general, it seems far from an isomorphism.

For example, if $Q = S^{2n-1}$ and $G = O(2n-1)$, Q does not satisfy the assumption. Moreover, as a private communication, Ishikawa pointed out that the cohomology of BQ is far from the result of Proposition 4.1.

We give an example of computing $H_*(BQ)$ where Q is the $2m$-sphere:

Example 4.4. Let Q be the $2m$-sphere, S^{2m}, as a symmetric space, i.e., a quandle of type 2. Then, $H_{dR}^k(Q) \cong \mathbf{R}$ if and only if $k = 0$ and $k = 2m$.

Therefore, for $k, j \geq 0$, the dimension of $H^{2mj}(Q^k)$ is equal to $\binom{k}{j}$. Hence, the Poincaré series $\sum_k \dim H^k(BQ;\mathbf{R})s^k$ is

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s^{2mj}\binom{k}{j}s^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} s^{2mj+k}\binom{k}{j} = \sum_{k=0}^{\infty}(1+s^{2mk})s^k \tag{3}$$
$$= \frac{1}{1-s-s^{2m+1}} \in \mathbf{Z}[[s]].$$

5. Rational homotopy group of the rack spaces

We will show Theorem 5.1 of computing the rational homology of BQ.

THEOREM 5.1. *Let Q be a smooth quandle of the form (G, H, z_0). Suppose that G is connected and compact, and satisfies the same assumption in Proposition 4.1. Let $u_i = \dim \pi_i(BQ) \otimes \mathbf{Q}$. Then, the following equality holds*:

$$\sum_{k\geq 0}\dim(H^k(BQ;\mathbf{R}))s^k = \prod_{i=0}^{\infty}\frac{(1+s^{2i+1})^{u_{2i+1}}}{(1-s^{2i})^{u_{2i}}} \in \mathbf{Z}[[s]]. \tag{4}$$

Remark 5.2. The homotopy group $\pi_i(BQ)$ contains $\pi_*(\Omega S^2)$ as a direct summand. Indeed, letting P be the quandle on the single point, any maps $Q \to P$ and $P \to Q$ are quandle homomorphisms, and $BP \simeq \Omega S^2$ is shown [13, 14].

To prove the theorem, we review a monoid structure on $B_G Q$, following [4]. For any $n, m \in \mathbf{N}$, we take a map $\mu : (I^n \times G \times Q^n) \times (I^m \times G \times Q^m) \to I^{n+m} \times G \times Q^{n+m}$ defined by

$$\mu([t_1, \dots, t_n, g, x_1 \dots, x_n], [t'_1, \dots, t'_m, h, x'_1 \dots, x'_m])$$
$$:= [t_1, \dots, t_n, t'_1, \dots, t'_m, gh, x_1 h, \dots x_n h, x'_1 \dots, x'_m].$$

Regarding $B_G Q$ as a quotient of $\bigsqcup_p (I^p \times G \times Q^p)$, this μ passes to a binary operation $B_G Q \times B_G Q \to B_G Q$, which makes $B_G Q$ into an associative topological monoid with unit [4, §2.5]. Recall a well-known fact that there exists a simplicial set Z such that $B_G Q$ is weak equivalent to a (based) loop space ΩZ as an H-space.

Next, we will observe the equality (5) below from Milnor-Moore theorem. Here, since Q and G are compact, $B_G Q$ is a CW-complex of finite type; hence, so is Z (see [11] for more detail). Since the space $B_G Q$ is connected by assumption, we notice $\pi_0(Z) \cong 0$ and $\pi_1(Z) \cong \pi_0(B_G Q) \cong 0$, that is, the space Z is simply connected. Since the cohomology group $H^*(B_G Q;\mathbf{R})$ is made into a Hopf algebra, Milnor-Moore theorem (see [11, §21]) immediately implies the isomorphisms

$$\mathrm{Prim}(H^*(B_G Q;\mathbf{Q})) \cong \mathrm{Prim}(H^*(\Omega Z;\mathbf{Q})) \cong \pi_*(\Omega Z) \otimes \mathbf{Q} \cong \pi_*(B_G Q) \otimes \mathbf{Q},$$

where $\mathrm{Prim}(H^*(B_GQ;\mathbf{Q}))$ means the subspace consisting of primitive elements of $H^*(B_GQ;\mathbf{Q})$. Then, the Poincaré-Birkoff-Witt theorem (see [11, §33(c)]) directly leads to

$$\sum_{k\geq 0} \dim(H^k(B_GQ;\mathbf{R}))s^k = \prod_{i=0}^{\infty} \frac{(1+s^{2i+1})^{r_{2i+1}}}{(1-s^{2i})^{r_{2i}}} \in \mathbf{Z}[[s]], \tag{5}$$

where $r_i = \dim \pi_i(B_GQ) \otimes \mathbf{Q}$.

Proof of Theorem 5.1. First, notice that the natural projection $B_GQ \to BQ$ is a principal (topological) G-bundle (see [12, §3] or [4, Proposition 6]). Let $\iota : G \to B_GQ$ be the fiber inclusion. Then, we have the long exact sequence of homotopy groups

$$\begin{aligned} \cdots \to \pi_n(G) \otimes \mathbf{Q} \xrightarrow{\iota_*} \pi_n(B_GQ) \otimes \mathbf{Q} \to \pi_n(BQ) \otimes \mathbf{Q} \\ \to \pi_{n-1}(G) \otimes \mathbf{Q} \to \cdots \quad \text{(exact)}. \end{aligned}$$

Notice that B_GQ includes the Lie group G as a topological submonoid by definitions, and ι is a monoid homomorphism. The induced map $\iota : H_*(G;\mathbf{R}) \to H_*(B_GQ;\mathbf{R})$ is injective by Proposition 4.1. An observation of the primitive elements implies the injectivity of $\iota_* : \pi_n(G) \otimes \mathbf{Q} \to \pi_n(B_GQ) \otimes \mathbf{Q}$. Thus, (5) is divisible by $\sum_k \dim(H^k(G;\mathbf{R}))s^k$. Hence, dividing (5) by the Milnor-Moore theorem on G, we have the conclusion (4). □

Example 5.3. If Q is S^{2m} and $G = SO(2m+1)$ as in Example 4.4, we can compute the rational homotopy from the Poincaré series (3). We focus only on the cases of $m = 1, 2, 3$, and give a list of rank $\pi_k(BS^2)$ as follows.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
rank $\pi_k(BS^2)$	1	1	1	1	1	2	2	2	3	5	6	7	11	27	47	85	151
rank $\pi_k(BS^4)$	1	1	0	0	1	1	1	1	1	2	2	2	3	7	11	16	23
rank $\pi_k(BS^6)$	1	1	0	0	0	0	1	1	1	1	1	1	1	3	5	7	10

6. Continuous R-value rack cocycles

In Sections 6–7, we focus on the rack space BX^δ, where X^δ means a smooth quandle with descrete topology. The cohomology of BX^δ coincides with the rack cohomology [12, 13, 14], and has applications to low-dimensional topology; see, e.g., [2, 3, 15, 24].

For this, let us briefly review rack cohomology [12, 13, 14]. Let X be a quandle. Then, $C_n^R(X)$ is defined to be the free right $\mathbf{Z}$-module generated by X^n. For $(x_1, \dots, x_n) \in X^n$, we define $\partial_n^R(x_1, \dots, x_n)$ by

$$\sum_{1 \le i \le n} (-1)^i((x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - (x_1 \lhd x_i, \dots, x_{i-1} \lhd x_i, x_{i+1}, \dots, x_n))$$
$$\in C^R_{n-1}(X).$$

This yields a homomorphism $\partial^R_n : C^R_n(X) \to C^R_{n-1}(X)$ such that $\partial^R_n \circ \partial^R_{n+1} = 0$. Dually, for an abelian group A, we have the cochain complex $C^n_R(X; A)$ defined by $\mathrm{Hom}(C^R_n(X), A)$ with the dual operation of ∂^R_n. As seen in, e.g., [2, 3, 15], for applications to low-dimensional topology, it is important to concretely describe an n-cocycle as a map $X^n \to A$ with $n \le 4$.

In this section, we will restrict on the continuous subcochain group. Let Q be a smooth quandle of the form (G, H, z_0). That is, we consider the subcomplex of $C^n_R(Q; \mathbf{R})$ defined by

$$C^n_{\mathrm{cont}}(Q) := \{f : Q^n \to \mathbf{R} \mid f \text{ is continuous}\},$$

which was first studied in [10], and the cohomology called *the continuous cohomology*. Furthermore, we introduce a class of Q:

Definition 6.1 (cf. homogeneousness in [20]). Fix $m \in \mathbf{Z}$. The smooth quandle Q is said to be *semi-homogenous* (of level m), if for any $a \in Q$ there is a zero measure set O_a such that the C^∞-map $Q \backslash O_a \to Q \backslash (a \lhd O_a)$ which sends x to $a \lhd x$ is a covering of degree m.

Example 6.2. For example, the quandle on the m-sphere S^m is semi-homogenous of level 2. Indeed, letting $q \in S^m$ be the antipodal point against a, and O_a be the equator between a and q, we can easily show the map $Q \backslash O_a \to Q \backslash \{q\}$ is a covering of degree 2. In parallel, since the projective spaces $\mathbf{R}P^m$, $\mathbf{C}P^m$ are quotients of some spheres, we can easily see that $\mathbf{R}P^m$ and $\mathbf{C}P^m$ are semi-homogenous.

More generally, we conjecture that, if X is the smooth quandle from every compact symmetric space (explained in Example 2.1), X may be semi-homogenous. In fact, T. Nagano [22] introduced the concept of "centrosome", and he and M. S. Tanaka gave many examples of centrosome, which indicate the existnece of zero-measure sets O_a satisfying Definition 6.1.

Example 6.3. We will consider the case where Q is semi-homogenous and of finite order. Then, O_q must be the empty set; thus, the covering $Q \to Q$ which sends x to $a \lhd x$ must be bijective. Namely $m = 1$. This bijectivity was called homogenous property in [20].

We will show a theorem, as a continuous version of [20, Theorem 1.1], which assumes semi-homogeneousness.

Theorem 6.4. *If a transitive smooth quandle $Q = G/H$ is semi-homogenous and compact, every cocycle in $C^n_{\mathrm{cont}}(Q)$ is cohomologous to a constant map. In particular, the cohomology $H^n_{\mathrm{cont}}(Q)$ is $\mathbf{R}$.*

In conclusion, in order to obtain non-trivial rack cocycles of Q, we should assume neither compactness of G nor the continuous $\mathbf{R}$-value cochain. For example, the quandle on $Q = \mathbf{R}^2$ with $x \lhd y = 2y - x$ has a non-trivial continuous 2-cocycle $X^2 \to \mathbf{R}$: see Corollary B.2. On the other hand, if $Q = \mathbf{R}/\mathbf{Z} = S^1$ is a quandle with $x \lhd y = 2y - x$, then Proposition B.1 implies that the universal 2-cocycle from $C_2^R(Q; \mathbf{Z})$ is not continuous.

To prove Theorem 6.4, we need several lemmas. Hereafter we assume that Q is semi-homogenous in this section. Using the Haar measure of G, we can choose a metric dy on Q which is invariant with respect to the action of G. We may assume $\int_Q dy = 1$.

Lemma 6.5 (cf. Lemmas 3.1 and 3.2 in [20]). *For any $x, w \in Q$ and any continuous function $K : Q \to \mathbf{R}$, the following equalities hold.*

$$\int_Q K(x \lhd y)\, dy = \int_Q K((x \lhd w) \lhd y)\, dy = \int_Q K((x \lhd y) \lhd w)\, dy. \tag{6}$$

Proof. We begin by computing the first term as

$$\int_Q K(x \lhd y)\, dy = \int_{Q \backslash x \lhd O_x} K(x \lhd y)\, dy = m \int_{Q \backslash O_x} K(y')\, dy' = m \int_Q K(y')\, dy'.$$

By replacing x by $x \lhd w$, we similarly have $\int_Q K((x \lhd w) \lhd y)\, dy = m \int_Q K(y')\, dy'$, which deduces the first equality in (6). By the right invariance of dy, replacing y to $y \lhd^{-1} w$ implies

$$\int_Q K((x \lhd y) \lhd w)\, dy = \int_Q K((x \lhd (y \lhd^{-1} w)) \lhd w)\, dy = \int_Q K((x \lhd w) \lhd y)\, dy.$$

This is the second equality in (6) exactly. □

Next, we will prepare some maps. We introduce two maps ∂_n^0 and ∂_n^1 from $C^n_{\mathrm{cont}}(Q)$ to $C^{n+1}_{\mathrm{cont}}(Q)$ by setting

$$\partial_i^0(h)(x_1, \dots, x_{n+1}) = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}),$$
$$\partial_i^1(h)(x_1, \dots, x_{n+1}) = h(x_1 \lhd x_i, \dots, x_{i-1} \lhd x_i, x_{i+1}, \dots, x_{n+1}).$$

By definition, we should notice $\partial_n^R(h) = \sum_{i=1}^{n+1} (-1)^i (\partial_i^0(h) - \partial_i^1(h))$. In addition, for $j \le n$, we define $\phi_n^j : C^n_{\mathrm{cont}}(Q) \to C^n_{\mathrm{cont}}(Q)$ by

$$\phi_n^j(h)(x_1, \dots, x_n) := \int_{Q^j} h(x_1 \lhd y_1, \dots, x_j \lhd y_j, x_{j+1}, \dots, x_n)\, dy_1 \cdots dy_j,$$

ϕ_n^0 by the identity map, and ϕ_n^{n+1} by ϕ_n^n. Furthermore, we define $D_n^j : C^n_{\mathrm{cont}}(Q) \to C^{n-1}_{\mathrm{cont}}(Q)$ by

$$\begin{aligned} &D_n^j(k)(x_1, \ldots, x_{n-1}) \\ &\quad := \int_{Q^j} k(x_1 \lhd y_1, \ldots, x_{j-1} \lhd y_{j-1}, x_j, y_j, x_{j+1} \ldots, x_{n-1})\, dy_1 \cdots dy_j, \end{aligned}$$

for $j < n$, and D_n^n by the zero map. Here, we should compare [20]; Precisely, if Q is of finite order, the maps ϕ_n^j and D_n^j coincide with the maps defined in [20, §3]. In addition, we give lemmas as relation among the above maps:

LEMMA 6.6 (cf. Lemmas 3.3–3.8 in [20]). *The following equalities hold.*

$$\begin{aligned} &\partial_i^0 \circ D_n^j(h) = \partial_i^1 \circ D_n^j(h) && \textit{for } 1 \le i \le j \le n, \\ &D_{n+1}^j \circ \partial_i^0(h) = D_{n+1}^j \circ \partial_i^1(h) && \textit{for } 1 \le i \le j \le n, \\ &\partial_{j+1}^0 \circ D_n^j(h) = \phi_n^{j-1}(h) && \textit{for } 1 \le j < n, \\ &\partial_{j+1}^1 \circ D_n^j(h) = \phi_n^j(h) && \textit{for } 1 \le j < n, \\ &D_{n+1}^j \circ \partial_i^0(h) = \partial_{i+1}^0 \circ D_n^j(h) && \textit{for } 1 \le j < i \le n+1, \\ &D_{n+1}^j \circ \partial_i^1(h) = \partial_{i+1}^1 \circ D_n^j(h) && \textit{for } 1 \le j < i \le n+1. \end{aligned}$$

Proof. The proofs are almost the same as those of Lemmas 3.3–3.8 in [20], respectively. Thus, we show only the first equality. We now denote $a \lhd b$ by a^b for simplicity. For $i < j$, we can easily show that $\partial_i^0 \circ D_n^j(h)(x_1, \ldots, x_n)$ is equal to

$$\int_{Q^j} h(x_1^{y_1}, \ldots, x_{i-1}^{y_{i-1}}, x_{i+1}^{y_{i+1}}, \ldots, x_{j-1}^{y_{j-1}}, x_j, y_j, x_{j+1}, \ldots, x_n)\, dy_1 \cdots dy_j,$$

and, that $\partial_i^1 \circ D_n^j(h)(x_1, \ldots, x_n)$ is equal to

$$\int_{Q^j} h({x_1^{y_1}}^{x_i \lhd y_i}, \ldots, {x_{i-1}^{y_{i-1}}}^{x_i \lhd y_i}, x_{i+1}^{y_{i+1}}, \ldots, x_{j-1}^{y_{j-1}}, x_j, y_j, x_{j+1}, \ldots, x_n)\, dy_1 \cdots dy_j.$$

In addition, if $i = j$, we similarly have

$$\begin{aligned} \partial_j^0 \circ D_n^j(h) &= \int_{Q^j} h(x_1^{y_1}, \ldots, x_{j-1}^{y_{j-1}}, y_j, x_{j+1}, \ldots, x_n)\, dy_1 \cdots dy_j, \\ \partial_j^0 \circ D_n^j(h) &= \int_{Q^j} h(x_1^{y_1 \lhd x_j}, \ldots, x_{j-1}^{y_{j-1} \lhd x_j}, y_j, x_{j+1}, \ldots, x_n)\, dy_1 \cdots dy_j. \end{aligned}$$

Applying Lemma 6.5 $i-1$ times and Fubini theorem to the integrals, we obtain the equality $\partial_i^0 \circ D_n^j(h) = \partial_i^1 \circ D_n^j(h)$ as required. $\square$

Putting all this together, we have

PROPOSITION 6.7 (cf. Proposition 3.1 in [20]). *For $j > 1$, $D_n^j : C^*_{\mathrm{cont}}(Q) \to C^{*+1}_{\mathrm{cont}}(Q)$ is a chain homotopy from ϕ_n^j to ϕ_n^{j-1}.*

Proof. The computation in the proof is same as that of Proposition 3.1 in [20], by using Lemma 6.6. Thus we may omit the detailed computation. □

Proof of Theorem 6.4. This proposition implies that every cocycle in $C^n_{\mathrm{cont}}(Q)$ is cohomologous to the map $\phi^n_n(h)$. By the proof of Lemma 6.5, we notice

$$\int_{Q^n} h(x_1 \lhd y_1, \dots, x_n \lhd y_n)\, dy_1 \cdots dy_n = m^n \int_{Q^n} h(y'_1, \dots, y'_n)\, dy'_1 \cdots dy'_n.$$

Namely, this $\phi^n_n(h)$ does not depend on $x_1, \dots, x_n$, that is, a constant map. To summarize, every cocycle in $C^n_{\mathrm{cont}}(Q)$ is cohomologous to a constant map, as required. □

7. Rack cocycles from secondary characteristic classes

In order to get non-trivial rack cocycles of quandles, we will introduce an algorithm to obtain $\mathbf{C}/\mathbf{Z}$-value rack cocycles from the secondary characteristic classes.

Our approach in this section is based on the works of Dupont and Kamber [7, 8, 9]. Thus, §7.1 reviews the works, and §7.2 describes the algorithm.

7.1. Review of Dupont [7, 8] on presentations of group cocycles

First, we prepare some homogenous complexes. Given a set X acted on by a group G, let $C^\Delta_n(X)$ be the free $\mathbf{Z}$-module generated by $(n+1)$-tuples of X, that is, $C^\Delta_n(X) = \mathbf{Z}\langle X^{n+1}\rangle$. This $C^\Delta_n(X)$ has a differential operator ∂^Δ_* defined by

$$\partial^\Delta_n(x_0, \dots, x_n) = \sum_{i:0 \le i \le n} (-1)^i (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

The action of G on X gives rise to the diagonal action on $C^\Delta_n(X)$. Denote by $C^\Delta_n(X)_G$ the coinvariant $C^\Delta_n(X) \otimes_{\mathbf{Z}[G]} \mathbf{Z}$. For example, if $X = G$ with natural action of G, the complex $C^\Delta_*(G)$ gives a $\mathbf{Z}[G]$-free resolution of the augmentation $\mathbf{Z}[G] \to \mathbf{Z}$. Therefore, the homology of $C^\Delta_*(G)_G$ is isomorphic to the ordinary group homology of G.

Next, we will explain Proposition 7.1 below. Let V be a manifold which is $(q-1)$-connected for some $q \in \mathbf{Z}$, and G be a Lie group with transitive action on V. Let $C^{\mathrm{sing}}_*(V)$ be the chain complex of smooth singular simplicies in V. This chain complex is naturally made into a right $\mathbf{Z}[G]$-module, and is acyclic of length $q-1$. Then, we can find a chain transformation σ of G-modules, which ensures the following commutative diagram:

$$(7)\qquad \begin{array}{ccccccccc} \mathbf{Z} & \xleftarrow{\partial^\Delta_0} & C^\Delta_0(G) & \xleftarrow{\partial^\Delta_1} & C^\Delta_1(G) & \xleftarrow{\partial^\Delta_2} \cdots \xleftarrow{\partial^\Delta_q} & C^\Delta_q(G) \\ \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ \mathbf{Z} & \xleftarrow{\partial_0} & C^{\mathrm{sing}}_0(V) & \xleftarrow{\partial_1} & C^{\mathrm{sing}}_1(V) & \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_q} & C^{\mathrm{sing}}_q(V). \end{array}$$

As is known as the comparison theorem, this σ is unique up to homotopy. Furthermore, for a G-invariant complex value q-form ω, we define a cochain $\mathscr{C}(\omega) \in \mathrm{Hom}(C_q^\Delta(G)_G, \mathbf{C})$ by

$$\mathscr{C}(\omega)(g_0, g_1, \ldots, g_q) := \int_{\sigma(g_0, g_1, \ldots, g_q)} \omega, \tag{8}$$

for $g_0, g_1, \ldots, g_q \in G$. The following is due to Stokes theorem.

PROPOSITION 7.1 ([8, Proposition 10.4]). *Suppose that ω is closed, and that the integral $\int_z \omega$ lies in $\mathbf{Z}$ for any $z \in C_q^{\mathrm{sing}}(V; \mathbf{Z})$.*

Then, the cochain $\mathscr{C}(\omega)$ is a q-cocycle mod $\mathbf{Z}$. Furthermore, it is nullcohomologous if $\omega = d\omega'$ for some G-invariant $(q-1)$-form ω'.

As an insightful result, Dupont-Kamber [9] showed that this formulation includes Chern-Simons classes as follows:

Example 7.2 ([9]). Let G be $GL_k(\mathbf{C})$, and V be $GL_k(\mathbf{C})/GL_{k-1}(\mathbf{C})$. By Bott periodicity, V is $(2k-2)$-connected, and has $H^{2k-1}(V; \mathbf{Z}) \cong \mathbf{Z}$. Since V is the complexification of the compact homogeneous space $U(k)/U(k-1)$, the generator of the $(2k-1)$-th cohomology group can be represented by a complex value G-invariant $(2k-1)$-form ω_k. Then, the group cocycle $\mathscr{C}(\omega_k) \in H^{2k-1}(GL_n(\mathbf{C}); \mathbf{C}/\mathbf{Z})$ is shown to be equal to the k-th Chern-Simons class.

7.2. Relation to secondary characteristic classes

Under the condition in the previous subsection, we will show that every secondary characteristic class in the sense of [8, 9] produces an n-cocycle in the rack complex.

For this, we review Inoue-Kabaya map [15]. Let Q be a smooth quandle of the form (G, H, z_0). For $n \in \mathbf{Z}_{n \geq 2}$, consider the following set composed of maps:

$$I_n := \{\iota : \{2, 3, \ldots, n\} \to \{0, 1\}\}. \tag{9}$$

For a tuple $(x_0, \ldots, x_n) \in Q^{n+1}$ and for each $\iota \in I_n$, we define $x(\iota, i) \in Q$ by

$$x(\iota, i) := (\cdots((x_i \lhd^{\iota(i+1)} x_{i+1}) \lhd^{\iota(i+2)} x_{i+2}) \cdots) \lhd^{\iota(n)} x_n.$$

Here $x \lhd^0 y = y$. Choose $p \in Q$. If $n \geq 2$, we define a homomorphism

$$\varphi_n : C_n^R(Q; \mathbf{Z}) \to C_n^\Delta(Q)_G,$$

by setting

$$\varphi_n(x_1, \ldots, x_n) := \sum_{\iota \in I_n} (-1)^{\iota(2)+\iota(3)+\cdots+\iota(n)} (p, x(\iota, 1), \ldots, x(\iota, n)).$$

If $n = 1$, we define $\varphi_1(a) = (p, a)$. This φ_n is shown to be a chain map. Namely, $\partial_n^\Delta \circ \varphi_n = \varphi_{n-1} \circ \partial_n^R$.

Next, we review a (G, H)-projectivity of the complex $C_n^\Delta(Q)$ from [1, §3]. To this aim, an exact sequence $N \xrightarrow{i} M \xrightarrow{j} L$ of right $\mathbf{Z}[G]$-module homomorphisms is (G,H)-*exact*, if the kernel of j is a direct $\mathbf{Z}[H]$-module summand of M. A right $\mathbf{Z}[G]$-module A is said to be (G,H)-*projective* if, for every (G,H)-exact sequence $0 \to N \xrightarrow{i} M \xrightarrow{j} L \to 0$, and every $\mathbf{Z}[G]$-homomorphism $\psi : A \to L$, there is a $\mathbf{Z}[G]$-homomorphism $\psi' : A \to M$ such that $q \circ \psi' = \psi$. Then, it is shown [1, Proposition 3.10] that the above module $C_n^{\Delta}(Q)$ is (G,H)-projective, and the following sequence is (G,H)-exact:

$$\cdots \xrightarrow{\partial_{n+1}^{\Delta}} C_n^{\Delta}(Q) \xrightarrow{\partial_n^{\Delta}} \cdots \to C_1^{\Delta}(Q) \xrightarrow{\partial_1^{\Delta}} C_0^{\Delta}(Q) \to \mathbf{Z} \to 0.$$

Moreover, we can easily verify that the bottom sequence in (7) is also (G,H)-exact. Thus, by (G,H)-projectivity (see [1, Proposition 3.11]), the chain map σ factors through a chain $\mathbf{Z}[G]$-map $\tau : C_n^{\Delta}(Q) \to C_n^{\mathrm{sing}}(Q)$ for $n \leq q$. Here, the choice of τ is unique up to homotopy. Hence, similarly, for any G-invariant q-form ω such that $\int_z \omega$ lies in $\mathbf{Z}$ for any $z \in C_q^{\mathrm{sing}}(Q;\mathbf{Z})$, it can be easily shown that the following map is a q-cocycle modulo $\mathbf{Z}$.

$$\mathscr{T}(\omega) : Q^{q+1} \to \mathbf{C}/\mathbf{Z}; \quad (x_0, x_1, \ldots, x_q) \mapsto \int_{\tau(x_0, x_1, \ldots, x_q)} \omega. \tag{10}$$

On the other hand, since $C_q^{\Delta}(Q)$ is a $\mathbf{Z}[G]$-module, the above chain map τ in (7) factors through $C_q^{\Delta}(Q)$. In conclusion, we have

Proposition 7.3. *Let ω be the q-cocycle satisfying the assumption in Proposition* 7.1. *Then, the pullback $\varphi_q^*(\mathscr{T}(\omega)) \in C_R^q(Q;\mathbf{C}/\mathbf{Z})$ is a rack q-cocycle.*

As mentioned in Example 7.2, the class of cocycles presented by $\mathscr{T}(\omega)$ contains a class of generalized Chern-Simons classes. In summary, such generalized classes can be represented as rack cocycles. Hence, it is reasonable to hope that this proposition produces many rack cocycles of X, when X is a subquandle of V.

Example 7.4. In the paper of Inoue-Kabaya [15], they consider the case $(PSL_2(\mathbf{C}), H, z_0)$, where H is the unipotent subgroup $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbf{C} \right\}$ and $z_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We remark that G/H is bijective to $(\mathbf{C} \times \mathbf{C} \setminus \{(0,0)\})/\pm$. In this case, the Chern-Simons 3-class $\hat{C}_3$ is well-understood (see, e.g., [7, Charters 7–11] or [15, §7]), and is represented by a map $\hat{C}_3 : V^4 \to \mathbf{C}/4\pi^2\mathbf{Z}$ with a cocycle expression. Furthermore, $\hat{C}_3$ has a close relation to complex volume of hyperbolic 3-manifolds. For this reason, the paper [15] presented $\hat{C}_3$ as a rack 3-cocycle, and gave a result on the complex volume; see [15, Theorem 7.3].

A. Appendix; Proofs of Theorems 3.3 and 3.5

We will prove Theorems 3.3 and 3.5. The outline of the proofs is based on [7, 4]: precisely, Dupont [7] showed a de Rham theory of simplicial manifolds, and Clauwens [4] constructed a triangulation of $\square$-sets, which induced a ring isomorphism on cohomology; As such, we give a bridge between their results, and give the proof of the theorems.

For this purpose, we first set up notation on simplicial manifolds from [7]. *A simplicial manifold* Y is defined as a sequence of manifolds Y_n for $n \in \mathbf{N}$ together with face maps $\delta_i : Y_n \to Y_{n-1}$ for $0 \le i \le n$ such that

$$\delta_{j-1}\delta_i = \delta_i\delta_j \quad \text{for any } 0 \le i < j \le n.$$

Let $\Delta^p \subset \mathbf{R}^{p+1}$ be the standard simplex

$$\Delta^p := \left\{ t = (t_0, \dots, t_p) \in \mathbf{R}^{p+1} \mid t_i \ge 0, \sum_{0 \le i \le p} t_i = 1 \right\},$$

and let $\epsilon^i : \Delta^{p-1} \to \Delta^p$ be the i-th face map. Then, the fat realization $\|Y\|_\Delta$ of Y is the quotient space of $\bigsqcup_{p \ge 0} \Delta^p \times Y$, with the identifications

$$(\epsilon^i(t), y) \sim (t, \delta_i y), \quad t \in \Delta^{p-1},\ s \in Y_p,\ i = 0, 1, \dots, p.$$

Then, we denote $\mathscr{A}_\Delta^*(Y_p)$ by the DGA consisting of n-forms on $\Delta^p \times Y_p$ which are extended to C^∞ forms on $(\sum_i t_i = 1) \times Y_p$. Moreover, an *$n$-form* φ on Y is a sequence of n-forms $\phi^{(p)} \in \mathscr{A}_\Delta^n(Y_p)$ of C^∞-class satisfying $(\epsilon^i \times \mathrm{id})^* \phi^{(p)} = (\mathrm{id} \times \delta_i)^* \phi^{(p-1)}$ for any $i \in \{1, \dots, p\}$. Then, we can define the de Rham cohomology of $\mathscr{A}_\Delta^*(Y_*)$. Furthermore, we decompose $A_\Delta^*(Y)$ into a sum $A_\Delta^n(Y) = \bigoplus_{n=k+\ell} \mathscr{A}_\Delta^{k,\ell}(Y)$, where $A_\Delta^{k,\ell}(Y)$ is composed of the forms φ of type (k, ℓ), i.e., φ restricted to $\Delta^p \times Y_p$ is $q_1^*(\phi_I^{(k)}) \times q_2^*(\phi_Y^{(\ell)})$ for some $\phi_I^{(k)} \in \mathscr{A}_\Delta^k(I^p)$ and $\phi_Y^{(\ell)} \in \mathscr{A}_\Delta^\ell(Y_p)$. Here $q_1 : \Delta^p \times Y_p \to \Delta^p$ and $q_2 : \Delta^p \times Y_p \to Y_p$ are the projections. Also let d_Δ (resp. d_Y) denote the pullback of exterior differential on $\mathscr{A}_\Delta^*(\Delta^p)$ (resp. on $\mathscr{A}_\Delta^*(Y_p)$). Thus, we have a double complex $(A_\Delta^{k,\ell}(Y), d_\Delta, d_Y)$, and the total complex $(A_\Delta^*(Y), d)$, where $d = d_\Delta + d_Y$. Further, we consider another double complex $(\mathscr{A}_\Delta^{k,\ell}(Y), \delta, d_Y)$ where $\delta = \sum_{i=1}^p (-1)^i \delta_i$.

Following [4, §3.2], we give a triangulation from a $\square$-set. For $n \in \mathbf{N}$, let $[n]$ denote the set $\{1, 2, \dots, n\}$. A k-partition of $[n]$ is a sequence $S = (S_1; S_2; \dots; S_k)$ of nonempty subsets of $[n]$ which are mutually disjoint and satisfy $[n] = S_1 \cup \dots \cup S_k$.

Given a cubical manifold X, we define a simplicial manifold $T(X)$, as a manifold analogy of [4, §3]. The set of k-simplicies $T(X)_k$ consists of the pairs $(x; S)$, where $x \in X_n$ and S is a k-partition of $[n]$. The boundary maps are given by

$$\delta_0(x; S_1; \dots; S_k) = (\delta_{S_1}^1 x; \theta_{S_1}(S_2); \dots; \theta_{S_1}(S_k)), \tag{11}$$

$$\delta_i(x; S_1; \dots; S_k) = (x; S_1; \dots; S_{i-1}; S_i \cup S_{i+1}; S_{i+2}; \dots; S_k) \quad \text{for } 0 < i < k, \tag{12}$$

$$\delta_k(x; S_1; \dots; S_k) = (\delta_{S_k}^0(x); \theta_{S_k}(S_1); \dots; \theta_{S_k}(S_{k-1})). \tag{13}$$

Here, for $S \subset [n]$, we write θ_S for the unique order-preserving map from $[n] - S$ to $[n - \#(S)]$. Then, it is not so hard to check that $T(X)$ is a simplicial mani-

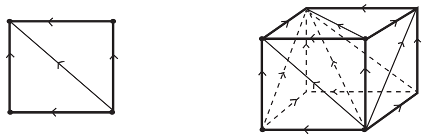

Figure 1. The canonically triangular decompositions of the square and the cube.

fold by definitions. Although the definition of $T(X)$ seems complicated, here is Figure 1 on a triangulation with $k = 2$ and $k = 3$.

Next, we will show Lemma A.1. Given $\mathrm{t} = (t_1, \dots, t_k) \in [0,1]^k$, we may choose a sequence $S_{\mathrm{t}} := (i_1, \dots, i_k) \in \{1, \dots, k\}^k$ with $1 \geq t_{i_1} \geq \cdots \geq t_{i_k} \geq 0$ such that $i_1, \dots, i_k$ are mutually distinct. Then, for $x \in X_n$, we make a correspondence from $\Phi(t_1, \dots t_k, x)$ to

$$((1 - t_{i_1}, t_{i_1} - t_{i_2}, \dots, t_{i_{k-1}} - t_{i_k}, t_{i_k}), (x; i_1; \dots; i_k)) \in \Delta^k \times T(X)_k.$$

Then, we can verify, by (11) and (13), that the correspondence descends to a continuous map $\Phi : \|X\| \to \|T(X)\|_\Delta$ on geometric realizations. Furthermore,

Lemma A.1. *For any cubical manifold X, the map $\Phi : \|X\| \to \|T(X)\|_\Delta$ is a homeomorphism.*

Proof. To construct the inverse mapping Ψ, we prepare notations. Suppose $(x; S_1; \dots; S_k)$ with $x \in X_n$ and $n \geq k$. We take the composite map

$$\mu_{S_1;\dots;S_k} := \epsilon^{|S_1|+|S_2|+\cdots+|S_k|} \circ \cdots \circ \epsilon^{|S_1|+|S_2|} \circ \epsilon^{|S_1|} : \Delta^k \to \Delta^n.$$

Decompose $(S_1; \dots; S_k) \subset [n]$ as $(s_1, \dots, s_n) \in \mathbf{N}^n$ pointwise. Furthermore, we regard this $(s_1, \dots, s_n)$ as a permutation $\sigma \in \mathfrak{S}_n$ and set up another map defined by

$$\Upsilon : \Delta^n \to I^n; \quad (t'_0, \dots, t'_n) \mapsto (t'_1 + \cdots + t'_n, t'_2 + \cdots + t'_n, \dots, t'_{n-1} + t'_n, t'_n).$$

Denote by $P_{n,k}$ the set of k-partitions of $[n]$ with discrete topology. Then, we define a map $\Psi : \Delta^k \times X^n \times P_{n,k} \to I^n \times X^n$ by

$$\Psi(t'_0, \dots, t_k, x; S_1; \dots; S_k) := (\Upsilon \circ \sigma^{-1} \circ \mu_{S_1;\dots;S_k}(t'_0, \dots, t'_k), x) \in I^n \times X^n$$

Then, by (11)–(13), this Ψ defines a continuous map $\|T(X)\|_\Delta \to \|X\|$. Moreover, it is not hard to check that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identities by construction. This completes the proof. $\square$

Following the proof, we can define the pullback $\Psi^*(\phi) \in A^{k,\ell}(T(X))$ of any form $\phi \in A^{k,\ell}(X)$. Moreover, we can similarly verify that

Lemma A.2. *The maps $\Phi^* : A_\Delta^{*,*}(T(X)) \to A_\Delta^{*,*}(X)$ and $\mathscr{A}_\Delta^{*,*}(T(X)) \to \mathscr{A}^{*,*}(X)$ are bigraded ring isomorphisms. Here, the inverse mappings are constructed from the pullback Ψ^*.*

We now use the above lemmas to prove the theorems.

Proof of Theorem 3.3. For any simplicial manifold Y, Dupont [7, Theorem 2.3] constructed a chain map $\mathscr{T} : A_{\Delta}^{k,\ell}(Y) \to \mathscr{A}_{\Delta}^{k,\ell}(Y)$ which gives a homotopy equivalence. Hence, when $Y = T(X)$, the composite $\Phi^* \circ \mathscr{T} \circ \Psi^* : A^{*,*}(X) \to \mathscr{A}^{*,*}(X)$ plays a role of the desired chain-map. □

Proof of Theorem 3.5. Dupont [7, Theorem 2.14] considered the map in the E_∞-term induced from $\mathscr{T} : A_{\Delta}^{k,\ell}(Y) \to \mathscr{A}_{\Delta}^{k,\ell}(Y)$, and the induced map $A_{\Delta}^*(Y) \to A_{\Delta}^*(\|Y\|)$ is multiplicative. The above maps Φ and Ψ are multiplicative by definitions. Thus, the map in the E_∞-term induced from $\Phi^* \circ \mathscr{T} \circ \Psi^*$ is also multiplicative. This completes the proof. □

B. Some computation of quandle homology of smooth quandles

In this section, we will compute rack quandle homology of "linear" quandles. Fix $\omega \in \mathbf{R}\backslash\{0,1\}$ and $n \in \mathbf{N}$. Let us assume that X is either a quandle on $\mathbf{R}^n$ with $x \lhd y = \omega x + (1-\omega) y$ or a quandle on $(\mathbf{R}/\mathbf{Z})^n$ with $x \lhd y = 2y - x$. (cf. the classification of smooth homogenous manifolds of dimension ≤ 2; see Ishikawa [16, §6]).

Proposition B.1. *If $\omega \neq \pm 1$ and $\omega \in \mathbf{Q}$, $H_2^R(X;\mathbf{Z})$ is $\mathbf{Z}$. On the other hand, if $\omega = -1$, $H_2^R(X;\mathbf{Z})$ is isomorphic to $(\mathbf{R}^n \wedge_{\mathbf{Q}} \mathbf{R}^n) \oplus \mathbf{Z}$.*

If $X = (\mathbf{R}/\mathbf{Z})^n$ with $x \lhd y = 2y - x$, then $H_2^R(X;\mathbf{Z})$ is isomorphic to $(\mathbf{R}/\mathbf{Q})^n \wedge_{\mathbf{Q}} (\mathbf{R}/\mathbf{Q})^n \oplus \mathbf{Z}$.

The key for the proof is the result of Clauwens [5]. Precisely, the paper computed the rack homology from the isomorphism

$$(14) \quad H_2^R(X;\mathbf{Z}) \cong \mathbf{Z} \oplus \frac{X \otimes_{\mathbf{Z}} X}{\{x \otimes y - \omega y \otimes x\}_{x,y \in X}}; \quad n(a,b) \mapsto (n, [(a-b) \otimes b]).$$

Proof. We will compute the right hand side in details. Recall elementary computations

$$(15) \qquad \mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}/\mathbf{Z} = 0, \quad \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}, \quad \text{and} \quad \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}.$$

Hence, if $X = \mathbf{R}^n$ with $\omega \neq \pm 1$, one has $H_2^R(X;\mathbf{Z}) \cong \mathbf{Z}$, because $x \otimes y = \omega x \otimes \omega y = \omega^2 x \otimes y$ in (14). On the other hand, if $\omega = -1$, the right hand side of (14) turns out to be the exterior product as stated above.

Finally, we consider $X = (\mathbf{R}/\mathbf{Z})^n$ with $x \lhd y = 2y - x$. Notice $\mathbf{R}/\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z} \oplus (\bigoplus_\lambda \mathbf{Q})$ as a $\mathbf{Z}$-module, where λ runs over an uncountable index set. Thus, $\mathbf{R}/\mathbf{Q} \cong \bigoplus_\lambda \mathbf{Q}$. Thus, the computation of $H_2^R(X;\mathbf{Z})$ immediately follows from (14) and (15). □

COROLLARY B.2. *Let $Q = \mathbf{R}^2$ be the quandle with $x \lhd y = 2y - x$. Then, the map $\mathscr{C} : Q^2 \to \mathbf{R}$ which takes $((x_1, y_1), (x_2, y_2))$ to $x_1 y_2 - x_2 y_1$ is a continuous 2-cocycle and is not null-cohomologous.*

Proof. Consider the $\mathbf{Q}$-linear map $q : \mathbf{R}^2 \wedge_{\mathbf{Q}} \mathbf{R}^2 \to \mathbf{R}$ which takes $(x, y) \wedge (z, w)$ to $xw - yz$. According to (14), the map $\mathscr{C}' : Q^2 \to \mathbf{R}^2 \wedge_{\mathbf{Q}} \mathbf{R}^2$ which sends (a, b) to $(a - b) \wedge b$ gives a universal 2-cocycle. Thus, the composite $q \circ \mathscr{C}'$ is not null-cohomologous. Noticing $\mathscr{C} = q \circ \mathscr{C}'$ completes the proof. □

Moreover, for a field F, we give a comment on the second cohomology of X_F (cf. Example 7.4). Here, this X_F is the quandle on the homogenous set G/H obteind from $G = PSL_2(F)$, $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}_{a \in F}$ and $z_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In addition, we recall from [24] the Milnor K_2-group $K_2(F)$ which is isomorphic to $F^\times \otimes_{\mathbf{Z}} F^\times / \{a \otimes (1 - a)\}_{a \in F \setminus \{0,1\}}$. If $F = \mathbf{C}$, $K_2(F)$ is known to be uniquely divisible, i.e., a direct sum of $\mathbf{Q}$'s,

PROPOSITION B.3 (A special result of [24, Corollary 8.5]). *If $F = \mathbf{C}$, then $H_2^R(X_F; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{C} \oplus K_2(\mathbf{C})$.*
Furthermore, if $F = \mathbf{R}$, then $H_2^R(X_F; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{R} \oplus K_2(\mathbf{C})^+$, where $K_2(\mathbf{C})^+$ is the invariant part of $K_2(\mathbf{C})$ with respect to the conjugate operation $\bar{\ } : \mathbf{C} \to \mathbf{C}$.

Concerning quandles on the spheres, W. E. Clark and M. Saito [6] studied some phenomena of quandle 2-cocycles, together with a relation to knot invariants.

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