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SOME REMARKS ON RIEMANNIAN MANIFOLDS WITH PARALLEL COTTON TENSOR

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Abstract

We give some sufficient conditions for stochastically complete Riemannian manifolds with parallel Cotton tensor to be either Einstein or of constant sectional curvature, and obtain an optimal pinching theorem. In particular, when $n = 4$, we give a full classification.

1. Introduction and main results

Constant sectional curvature manifolds and Einstein manifolds play an important role in global differential geometry (see [3, 29]). It is therefore a natural and interesting problem to explore sufficient (and possibly necessary) conditions to ensure that a given Riemannian manifold belongs to either one of the two classes. Thus it is one of the most important problem in the study of differential geometry, but is very difficult. Under various geometric conditions, many scholars have given some partial results to this problem [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 21, 22, 24, 26, 27, 28, 30]. In the compact case, for example, Tani [28] proved that any compact orientable conformally flat Riemannian space with constant scalar curvature and positive Ricci curvature must be a space of constant curvature, which had been improved by Goldberg [15]. Later it was proved by Tachibana [27] that any compact Riemannian manifold with positive curvature operator and harmonic curvature (i.e., the divergence of the Riemannian curvature tensor *Rm* vanishes, see [3]) must be a space of constant curvature. Moreover, under some (optimal) integral pinching conditions, it has been classified for conformally flat manifolds (e.g., [6, 11, 14, 17, 30]) and the manifolds with harmonic curvature (e.g., [5, 9, 10, 14, 21]). In the geodesically complete case, Pigola, Rigoli and Setti [24] obtained some characterizations of constant

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curvature spaces (see also [11, 12, 14, 30]). In 2013, Mastrolia, Monticelli and Rigoli [22] studied the pinching problem for stochastically complete manifolds (possibly not geodesically complete) with positive curvature operator with the aid of the weak maximum principle at infinity. Recently, Chu and Fang [8] proved a rigidity theorem for stochastically complete manifolds with parallel Cotton tensor which improves Theorems 1.5 and 1.7 in [22]. In this note, we improve the rigidity theorem given by Chu and Fang to the optimal pinching condition. In particular, when $n = 4$, we give a full classification. In order to state our conclusions, we need to make some preparations.

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let (M^n, g) be a Riemannian n-manifold. The Riemannian curvature tensor Rm of (M^n, g) is defined as in [3] by

$$Rm(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

and

$$R_{ijkl} = \langle Rm(e_i, e_j)e_k, e_l \rangle,$$

where $\{e_i\}$ is a local orthonormal frame field. The decomposition of the Riemannian curvature tensor Rm into irreducible components yields

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) \\ &\quad - \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &= W_{ijkl} + \frac{1}{n-2}(\mathring{R}_{ik}\delta_{jl} - \mathring{R}_{il}\delta_{jk} + \mathring{R}_{jl}\delta_{ik} - \mathring{R}_{jk}\delta_{il}) \\ &\quad + \frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &= W_{ijkl} + \frac{1}{n-2}(A_{ik}\delta_{jl} - A_{il}\delta_{jk} + A_{jl}\delta_{ik} - A_{jk}\delta_{il}), \end{aligned} \tag{1.1}$$

where R is the scalar curvature, and W_{ijkl}, R_{ij}, $\mathring{R}_{ij}$ and A_{ij} denote the components of the Weyl curvature tensor W, the Ricci tensor Ric, the trace-free Ricci tensor $\mathring{Ric} = Ric - \frac{R}{n}g$ and the Schouten tensor $A = Ric - \frac{R}{2(n-1)}g$, respectively. Then the Cotton tensor C is defined as

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik}, \tag{1.2}$$

which is related to the divergence of W by

$$W_{ijkl,l} = -\frac{n-3}{n-2} C_{ijk} \tag{1.3}$$

in view of the second Bianchi identity. (M^n, g) is said to has parallel Cotton tensor if the covariant derivative of C vanishes. It is easy to see that every Riemannian manifold with parallel Ricci tensor or harmonic Weyl curvature (i.e., the divergence of W vanishes, see [3]) has parallel Cotton tensor.

Recall that a Riemannian manifold (M^n, g) is said to be stochastically complete if the Brownian motion on (M^n, g) has the property that the total probability of the particle to be found in the state space is constantly equal to one. That is to say, stochastical completeness is the property of a stochastic process to have infinite life time (see [16]). We must pay particular attention to that the properties of stochastic completeness and geodesic completeness are independent of each other. In 1974, Azencott [2] gave the first examples of Riemannian manifolds which are geodesically complete but stochastically incomplete. In 1978, Yau [31] showed that a geodesically complete Riemannian manifold with a lower Ricci curvature bound is stochastically complete. In 2003, Pigola, Rigoli and Setti [23] proved that stochastic completeness is equivalent to the validity of a weak form of the Omori-Yau maximum principle for the Laplace-Beltrami operator Δ, i.e., for every function $u \in C^2(M^n)$ with $u^* = \sup_{M^n} u < \infty$, there exists a sequence $\{x_k\} \in M^n$ such that

$$\text{(i)}\ u(x_k) > u^* - \frac{1}{k}; \quad \text{(ii)}\ \triangle u(x_k) < \frac{1}{k}.$$

for all $k \in \mathbf{N}$. A Riemannian manifold (M^n, g) is called parabolic if every subharmonic function on (M^n, g) with an upper bound is a constant. In particular, every parabolic Riemannian manifold is stochastically complete.

In this note, we give some sufficient conditions for stochastically complete Riemannian manifolds with parallel Cotton tensor to be either Einstein or of constant sectional curvature and obtain the following rigidity theorems.

THEOREM 1.1. *Let (M^n, g) $(n \geq 3)$ be a stochastically complete Riemannian n-manifold with parallel Cotton tensor and positive constant scalar curvature R. If*

$$|W|^2 + \frac{2n}{(n-2)}|\mathring{Ric}|^2 < \frac{2}{(n-2)(n-1)}R^2, \tag{1.4}$$

then (M^n, g) is Einstein. In particular, (M^3, g) has positive constant sectional curvature. Moreover, if $n \geq 4$ and

$$|W|^2 + \frac{2n}{(n-2)}|\mathring{Ric}|^2 < \frac{4}{n^2 C(n)^2}R^2, \tag{1.5}$$

where $C(n)$ is defined in Lemma 2.3, then (M^n, g) has positive constant sectional curvature.

THEOREM 1.2. *Let (M^n, g) $(n \geq 3)$ be a parabolic Riemannian n-manifold with parallel Cotton tensor and positive constant scalar curvature R. If*

$$|W|^2 + \frac{2n}{(n-2)}|\mathring{Ric}|^2 \leq \frac{2}{(n-2)(n-1)}R^2, \tag{1.6}$$

then (M^n, g) is Ricci parallel. Furthermore, if at one point the strict inequality in (1.6) *holds, then (M^n, g) is Einstein.*

Remark 1.3. The pinching condition of Theorem 1.1 is optimal. The equality in (1.4) really holds on $\mathbf{R}^1 \times \mathbf{S}^{n-1}$ with the product metric. When $n = 4$, the equality in (1.5) really holds on $\mathbf{CP}^2$ with the Fubini-Study metric. It is easy to see that Theorems 1.1 and 1.2 improve Theorem 1.1 given by Chu and Fang in [8]. If (M^n, g) $(n \geq 4)$ is a compact Riemannian n-manifold with harmonic curvature and positive scalar curvature, then the first author gave a $L^{n/2}$ integral pinching condition in [9] which is close to (1.4).

Theorem 1.4. *Let (M^4, g) be a parabolic Riemannian 4-manifold with parallel Cotton tensor and positive constant scalar curvature R. Assume that (M^4, g) is geodesically complete. If*

$$|W|^2 + 4|\mathring{Ric}|^2 \leq \frac{1}{3}R^2, \tag{1.7}$$

then the universal covering Riemannian manifold of (M^4, g) is one of the following:

i) *a round $\mathbf{S}^4$;*
ii) *a $\mathbf{CP}^2$ with the Fubini-Study metric;*
iii) *a $\mathbf{S}^2 \times \mathbf{S}^2$ with product metric;*
iv) *a $\mathbf{R}^1 \times \mathbf{S}^3$ with product metric.*

Theorem 1.5. *Let (M^5, g) be a parabolic Riemannian 5-manifold with parallel Cotton tensor and positive constant scalar curvature R. If*

$$|W|^2 + \frac{10}{3}|\mathring{Ric}|^2 \leq \frac{1}{40}R^2, \tag{1.8}$$

then (M^5, g) is Einstein and locally symmetric. Moreover, if (M^5, g) is geodesically complete, then the universal covering Riemannian manifold of (M^5, g) is a round $\mathbf{S}^5$.

2. Proofs of Theorems

Let (M^n, g) $(n \geq 3)$ be a Riemannian n-manifold with parallel Cotton tensor and constant scalar curvature. Then by the Bianchi identities, from (1.2) and (1.3) we have

$$W_{ijkl,l} = -\frac{n-3}{n-2}C_{ijk} = -\frac{n-3}{n-2}(R_{jk,i} - R_{ik,j}) = \frac{n-3}{n-2}R_{ijkl,l}. \tag{2.1}$$

By (2.1), the condition that Cotton tensor is parallel implies

$$R_{ijkl,lp} = W_{ijkl,lp} = 0, \quad \forall p = 1, 2, \ldots, n. \tag{2.2}$$

From (1.1), by the Bianchi identities and (1.2), we obtain

$$W_{ijkl,mp} + W_{ijlm,kp} + W_{ijmk,lp} = 0. \tag{2.3}$$

LEMMA 2.1. *Let (M^n, g) $(n \geq 3)$ be a Riemannian n-manifold with parallel Cotton tensor and constant scalar curvature. Then*

$$\begin{aligned} \triangle|\mathring{Ric}|^2 \geq 2|\nabla\mathring{Ric}|^2 - 2\sqrt{\frac{n-2}{2(n-1)}}\left(|W|^2 + \frac{2n}{n-2}|\mathring{Ric}|^2\right)^{1/2}|\mathring{Ric}|^2 \\ + 2\frac{R}{n-1}|\mathring{Ric}|^2. \end{aligned} \tag{2.4}$$

Remark 2.2. Replacing the parallel Cotton tensor and constant scalar curvature with the harmonic curvature, the estimate (2.4) is obtained in [9]. There is no essential difference between the two proofs. For completeness, we write the proof of Lemma 2.1 out.

Proof. We compute

$$\triangle|\mathring{Ric}|^2 = 2|\nabla\mathring{Ric}|^2 + 2\langle\mathring{Ric}, \triangle\mathring{Ric}\rangle = 2|\nabla\mathring{Ric}|^2 + 2\mathring{R}_{ij}\mathring{R}_{ij,kk}. \tag{2.5}$$

Since the Cotton tensor is parallel, by the Ricci identities, we obtain

$$\begin{aligned} \mathring{R}_{ij,kk} &= \mathring{R}_{ik,jk} = \mathring{R}_{ki,kj} + \mathring{R}_{li}R_{lkjk} + \mathring{R}_{kl}R_{lijk} \\ &= \mathring{R}_{kk,ij} + \mathring{R}_{li}R_{lkjk} + \mathring{R}_{kl}R_{lijk} \\ &= \mathring{R}_{li}R_{lkjk} + \mathring{R}_{kl}R_{lijk}, \end{aligned} \tag{2.6}$$

which gives

$$\triangle|\mathring{Ric}|^2 = 2|\nabla\mathring{Ric}|^2 + 2\mathring{R}_{ij}\mathring{R}_{ij,kk} = 2|\nabla\mathring{Ric}|^2 + 2\mathring{R}_{ij}\mathring{R}_{li}R_{lkjk} + 2\mathring{R}_{ij}\mathring{R}_{kl}R_{lijk}. \tag{2.7}$$

We compute

$$\triangle|\mathring{Ric}|^2 = 2|\nabla\mathring{Ric}|^2 + 2W_{kijl}\mathring{R}_{ij}\mathring{R}_{kl} + 2\frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jl}\mathring{R}_{li} + 2\frac{R}{n-1}|\mathring{Ric}|^2. \tag{2.8}$$

By Lemma 2.5 in [9] (see also Proposition 2.1 in [7]), from (2.8) we get (2.4). □

LEMMA 2.3. *Let (M^n, g) $(n \geq 4)$ be a Riemannian n-manifold with parallel Cotton tensor and constant scalar curvature. Then*

$$\triangle|W|^2 \geq 2|\nabla W|^2 - 2C(n)|W|^3 - 4\sqrt{\frac{n-1}{n}}|W|^2|\mathring{Ric}| + \frac{4R}{n}|W|^2, \tag{2.9}$$

where

$$C(n) = \begin{cases} \dfrac{\sqrt{6}}{2}, & n = 4 \\ \dfrac{8}{\sqrt{10}}, & n = 5 \\ \dfrac{2(n-2)}{\sqrt{(n-1)n}} + \dfrac{n^2-n-4}{\sqrt{(n-2)(n-1)n(n+1)}}, & n \geq 6. \end{cases}$$

Remark 2.4. Replacing the parallel Cotton tensor and constant scalar curvature with the harmonic curvature, the estimate (2.9) is obtained in [9]. But, when $n \geq 6$, the estimate (2.9) is stronger than the one obtained in [9] for $C(n)$ is smaller. Note that by the inequality proved by Li and Zhao [21], similar estimates can be improved in [9, 12, 13, 14], which lead to improving some corresponding rigidity theorems in [9, 12, 13, 14].

Proof. By the Ricci identities, we obtain from (2.2) and (2.3)

$$\begin{aligned}
(2.10) \qquad \triangle|W|^2 &= 2|\nabla W|^2 + 2\langle W, \triangle W\rangle = 2|\nabla W|^2 + 2W_{ijkl}W_{ijkl,mm} \\
&= 2|\nabla W|^2 + 2W_{ijkl}(W_{ijkm,lm} + W_{ijml,km}) \\
&= 2|\nabla W|^2 + 4W_{ijkl}W_{ijkm,lm} \\
&= 2|\nabla W|^2 + 4W_{ijkl}(W_{ijkm,ml} + W_{hjkm}R_{hilm} + W_{ihkm}R_{hjlm} \\
&\qquad + W_{ijhm}R_{hklm} + W_{ijkh}R_{hmlm}) \\
&= 2|\nabla W|^2 + 4W_{ijkl}(W_{hjkm}R_{hilm} + W_{ihkm}R_{hjlm} \\
&\qquad + W_{ijhm}R_{hklm} + W_{ijkh}R_{hmlm}) \\
&= 2|\nabla W|^2 + 4W_{ijkl}(W_{hjkm}W_{hilm} + W_{ihkm}W_{hjlm} \\
&\qquad + W_{ijhm}W_{hklm} + W_{ijkh}W_{hmlm}) \\
&\quad + \frac{4}{n-2}W_{ijkl}[W_{hjkm}(\mathring{R}_{hl}\delta_{im} - \mathring{R}_{hm}\delta_{il} + \mathring{R}_{im}\delta_{hl} - \mathring{R}_{il}\delta_{hm}) \\
&\qquad + W_{ihkm}(\mathring{R}_{hl}\delta_{jm} - \mathring{R}_{hm}\delta_{jl} + \mathring{R}_{jm}\delta_{hl} - \mathring{R}_{jl}\delta_{hm}) \\
&\qquad + W_{ijhm}(\mathring{R}_{hl}\delta_{km} - \mathring{R}_{hm}\delta_{kl} + \mathring{R}_{km}\delta_{hl} - \mathring{R}_{kl}\delta_{hm}) \\
&\qquad + W_{ijkh}(\mathring{R}_{hl}\delta_{mm} - \mathring{R}_{hm}\delta_{ml} + \mathring{R}_{mm}\delta_{hl} - \mathring{R}_{ml}\delta_{hm})] \\
&\quad + \frac{4R}{n(n-1)}W_{ijkl}(W_{ljki} + W_{ilkj} + W_{ijlk}) + \frac{4R}{n}|W|^2 \\
&= 2|\nabla W|^2 + 4W_{ijkl}\left(2W_{hjkm}W_{hilm} - \frac{1}{2}W_{ijhm}W_{klhm}\right) \\
&\quad + \frac{4R}{n}|W|^2 + 4W_{ijkl}W_{ijkh}\mathring{R}_{hl}
\end{aligned}$$

$$\geq 2|\nabla W|^2 - 4\left(2W_{ijlk}W_{jhkm}W_{himl} + \frac{1}{2}W_{ijkl}W_{hmij}W_{klhm}\right)$$
$$+ \frac{4R}{n}|W|^2 - 4\sqrt{\frac{n-1}{n}}|W|^2|\mathring{Ric}|,$$

in which we use Lemma 2.4 of [18] for the inequality.

Case 1. When $n = 4$, it was proved in Lemma 3.5 of [18] that

$$\left|2W_{ijlk}W_{jhkm}W_{himl} + \frac{1}{2}W_{ijkl}W_{hmij}W_{klhm}\right| \leq \frac{\sqrt{6}}{4}|W|^3.$$

Case 2. When $n = 5$, Jack and Parker [20] have proved that $W_{ijkl}W_{hmij}W_{klhm} = 2W_{ijlk}W_{jhkm}W_{himl}$. By Lemma 2.4 of [18], we consider W as a self adjoint operator on $\wedge^2 V$, and obtain

$$\left|2W_{ijlk}W_{jhkm}W_{himl} + \frac{1}{2}W_{ijkl}W_{hmij}W_{klhm}\right| = \frac{3}{2}|W_{ijkl}W_{hmij}W_{klhm}| \leq \frac{4}{\sqrt{10}}|W|^3.$$

Case 3. When $n \geq 6$, using the inequality (14) of [21] and Lemma 2.4 of [18] to estimate the first term and the second term in the right-hand side of the following first formula respectively, we have

$$\left|2W_{ijlk}W_{jhkm}W_{himl} + \frac{1}{2}W_{ijkl}W_{hmij}W_{klhm}\right|$$
$$\leq 2|W_{ijlk}W_{jhkm}W_{himl}| + \frac{1}{2}|W_{ijkl}W_{hmij}W_{klhm}|$$
$$\leq \left[\frac{(n-2)}{\sqrt{(n-1)n}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}\right]|W|^3.$$

Combining (2.10) with Cases 1, 2 and 3, we get (2.9). □

Proof of Theorem 1.1. Note that the stochastically completeness of (M^n, g) implies that the weak maximum principle for the Laplace-Beltrami operator $\triangle$ holds on (M^n, g). Since R is a positive constant, from (1.4) we know that $|\mathring{Ric}|^* = \sup_{M^n}|\mathring{Ric}| < \infty$. Writing $|W|(x_\infty) = \lim_{k\to\infty}|W|(x_k)$ for $x_k \in M^n$ $(k = 1, 2, \ldots)$ satisfying the weak maximum principle for $\triangle$ on M^n and applying the weak maximum principle to (2.4), we have

$$0 \geq -\sqrt{\frac{n-2}{2(n-1)}}\left(|W|^2(x_\infty) + \frac{2n}{n-2}(|\mathring{Ric}|^*)^2\right)^{1/2}(|\mathring{Ric}|^*)^2 + \frac{R}{n-1}(|\mathring{Ric}|^*)^2$$
$$= \sqrt{\frac{n-2}{2(n-1)}}(|\mathring{Ric}|^*)^2\left[\sqrt{\frac{2}{(n-2)(n-1)}}R - \left(|W|^2(x_\infty) + \frac{2n}{n-2}(|\mathring{Ric}|^*)^2\right)^{1/2}\right]$$
$$\geq 0.$$

From the above, by (1.4), we have that $|\mathring{Ric}|^* = 0$, i.e., (M^n, g) is Einstein. When $n = 3$, (M^n, g) has positive constant sectional curvature.

It is easy to see that (1.5) implies (1.4). Under the assumption (1.5), by the above result we get that (M^n, g) is Einstein. When $n \geq 4$, by Lemma 2.3, we have

$$\triangle|W|^2 \geq 2|\nabla W|^2 - 2C(n)|W|^3 + \frac{4R}{n}|W|^2. \tag{2.11}$$

Based on (2.11), using the same argument as in the proof of the first part of Theorem 1.1, we get $W = 0$, i.e., (M^n, g) is conformally flat. Hence we obtain that (M^n, g) has positive constant sectional curvature. □

Proof of Theorem 1.2. From (1.6) and (2.4), we get $|\mathring{Ric}| < \infty$ and

$$\begin{aligned}\triangle|\mathring{Ric}|^2 &\geq 2|\nabla \mathring{Ric}|^2 + 2\sqrt{\frac{n-2}{2(n-1)}}|\mathring{Ric}|^2 \\ &\quad \times \left[\sqrt{\frac{2}{(n-2)(n-1)}}R - \left(|W|^2 + \frac{2n}{n-2}|\mathring{Ric}|^2\right)^{1/2}\right] \\ &\geq 0.\end{aligned} \tag{2.12}$$

Hence $|\mathring{Ric}|$ is a bounded above subharmonic function on (M^n, g). Since (M^n, g) is parabolic, $|\mathring{Ric}|$ is constant. By (2.12) and M^n has constant scalar curvature, we deduce that (M^n, g) is Ricci parallel. If at one point the strict inequality in (1.6) holds, from (2.12) we get that at one point $\mathring{Ric} = 0$. Since (M^n, g) is Ricci parallel, we obtain $\mathring{Ric} = 0$. i.e., (M^n, g) is Einstein. □

Proof of Theorem 1.4. By Theorem 1.2, (M^4, g) is Ricci parallel. Thus we divide into two cases: (M^4, g) is Einstein or not.

Case 1. (M^4, g) is Einstein. Since (M^4, g) is geodesically complete and the scalar curvature $R > 0$, we see from Myers' Theorem that the universal covering Riemannian manifold $\tilde{M}^4$ of (M^4, g) is compact. If $W = 0$, $\tilde{M}^4$ is a round $\mathbf{S}^4$. By the Chern-Gauss-Bonnet formula (see Equation 6.31 of [3])

$$\int_{\tilde{M}^4} |W|^2 - 2\int_{\tilde{M}^4} |\mathring{Ric}|^2 + \frac{1}{6}\int_{\tilde{M}^4} R^2 = 32\pi^2\chi(\tilde{M}^4),$$

where $\chi(\tilde{M}^4)$ is the Euler-Poincaré characteristic of $\tilde{M}^4$, we get from (1.7) that

$$\int_{\tilde{M}^4} |W|^2 \leq \frac{64\pi^2\chi(\tilde{M}^4)}{3}.$$

It is easy to see that an Einstein manifold has harmonic Weyl curvature. Hence by Theorems 1.4 and 1.5 in [10], $\tilde{M}^4$ is either a $\mathbf{CP}^2$ with the Fubini-Study metric or a $\mathbf{S}^2 \times \mathbf{S}^2$ with product metric for $W \neq 0$.

Case 2. (M^4, g) is not Einstein. By Theorem 1 in [25], we have that $\tilde{M}^4$ is either a $\mathbf{R}^1 \times \mathbf{S}^3$ with product metric or a $(\Sigma_1, g_1) \times (\Sigma_2, g_2)$ with product metric, where the surface (Σ_i, g_i) has constant Gauss curvature k_i, $k_1 \neq k_2$ and $k_1 + k_2 > 0$. If (M^4, g) is the latter, by computing (1.1), we get that $W_{1212} = W_{3434} = \dfrac{k_1 + k_2}{3}$, $W_{1313} = W_{1414} = W_{2323} = W_{2424} = -\dfrac{k_1 + k_2}{6}$, $\mathring{R}_{11} = \mathring{R}_{22} = -\mathring{R}_{33} = -\mathring{R}_{44} = \dfrac{k_1 - k_2}{2}$. Thus we have

$$
\begin{aligned}
(2.13) \qquad |W|^2 + 4|\mathring{Ric}|^2 &= 8\frac{(k_1 + k_2)^2}{9} + 16\frac{(k_1 + k_2)^2}{36} + 4(k_1 - k_2)^2 \\
&= 4\frac{(k_1 + k_2)^2}{3} + 4(k_1 - k_2)^2,
\end{aligned}
$$

and $\dfrac{1}{3}R^2 = 4\dfrac{(k_1 + k_2)^2}{3}$. Hence by (1.7) we get $k_1 = k_2$, i.e., (M^4, g) is Einstein. This is a contradiction. □

Proof of Theorem 1.5. By Theorem 1.1, (M^5, g) is Einstein. By Lemma 2.3, we have

$$
(2.14) \qquad \triangle|W|^2 \geq 2|\nabla W|^2 + \left(\frac{4R}{5} - \frac{16}{\sqrt{10}}|W|\right)|W|^2.
$$

Hence $|W|$ is a bounded above subharmonic function on (M^5, g). Since (M^5, g) is parabolic, $|W|$ is constant. From (2.14) we deduce that $\nabla W = 0$, and $\nabla Rm = 0$. Hence M^5 is locally symmetric. If M^5 is complete, since the scalar curvature $R > 0$, the universal covering Riemannian manifold $\tilde{M}^5$ of (M^5, g) is compact. Since $\tilde{M}^5$ is Einstein and $|W|^2 \leq \frac{1}{40}R^2$, by Theorem 2.1 and Lemma 3.1 in [4] ($|W|^2$ in [4] has a $\frac{1}{4}$ difference to ours), the first Betti number and the second Betti number are both zero. By the Smale's work, $\tilde{M}^5$ is homeomorphic to a round $\mathbf{S}^5$. Combining with the classification of homogeneous Einstein 5-manifolds given by [1], $\tilde{M}^5$ is a round $\mathbf{S}^5$. □

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