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# GROUND STATE SOLUTIONS FOR ASYMPTOTICALLY PERIODIC LINEARLY COUPLED SCHRÖDINGER EQUATIONS WITH CRITICAL EXPONENT

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## Abstract

We consider the following system of coupled nonlinear Schrödinger equations

$$\begin{cases} -\triangle u + a(x)u = |u|^{p-2}u + \lambda(x)v, & x \in \mathbf{R}^N, \\ -\triangle v + b(x)v = |v|^{2^*-2}v + \lambda(x)u, & x \in \mathbf{R}^N, \\ u, v \in H^1(\mathbf{R}^N), \end{cases}$$

where $N \geq 3$, $2 < p < 2^*$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $a, b, \lambda \in C(\mathbf{R}^N, \mathbf{R}) \cap L^\infty(\mathbf{R}^N, \mathbf{R})$ and $a(x)$, $b(x)$ and $\lambda(x)$ are asymptotically periodic, and can be sign-changing. By using a new technique, we prove the existence of a ground state of Nehari type solution for the above system under some mild assumptions on $a$, $b$ and $\lambda$. In particular, the common condition that $|\lambda(x)| < \sqrt{a(x)b(x)}$ for all $x \in \mathbf{R}^N$ is not required.

## 1. Introduction

In this paper, we study the following system of coupled nonlinear Schrödinger equations

$$\begin{cases} -\triangle u + a(x)u = |u|^{p-2}u + \lambda(x)v, & x \in \mathbf{R}^N, \\ -\triangle v + b(x)v = |v|^{2^*-2}v + \lambda(x)u, & x \in \mathbf{R}^N, \\ u, v \in H^1(\mathbf{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $2 < p < 2^*$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent; $a, b, \lambda : \mathbf{R}^N \mapsto \mathbf{R}$. Systems of this type arise in nonlinear optics [4].

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During the last decades, many works have been orientated to the analysis of nonlinear Schrödinger equation of the form

$$-\triangle u + V(x)u = f(x,u), \quad u \in H^1(\mathbf{R}^N)$$

under various hypotheses on the potential and the nonlinearity (see [11–13, 17, 18, 22–25] and the references therein).

However, by Kaminow [15], we know that single-mode optical fibers are not really "single-mode", but actually bimodal due to the presence of birefringence. Recently, different authors focused their attention on coupled nonlinear Schrödinger systems (see [1, 2, 5–10, 16, 19, 21, 26, 28–30] and the references therein) which describe physical phenomena, such as nonlinear optics and Bose-Einstein condensates (see, e.g., [3, 4, 14]).

What's more, since critical exponent elliptic problems create some difficulties in using variational methods due to the lack of compactness, there are a few papers to investigate the system of Schrödinger equations with critical exponent (see [8, 10, 16] and so on).

It is well known that solutions of (1.1) correspond to the critical points of the functional $\Phi : H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) \to \mathbf{R}$ given by

$$\begin{aligned}\Phi(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ - \frac{1}{p}\int_{\mathbf{R}^N} |u|^p\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbf{R}^N} |v|^{2^*}\,\mathrm{d}x.\end{aligned} \tag{1.2}$$

Let $\gamma_p$ be the sharp constant of the Sobolev embedding $H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2)\,\mathrm{d}x \geq \gamma_p \left(\int_{\mathbf{R}^N} |u|^p\,\mathrm{d}x\right)^{2/p},$$

and $S$ be the sharp constant of $\mathscr{D}^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2^*}(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} |\nabla u|^2\,\mathrm{d}x \geq S\left(\int_{\mathbf{R}^N} |u|^{2^*}\,\mathrm{d}x\right)^{2/2^*}, \tag{1.3}$$

where $\mathscr{D}^{1,2}(\mathbf{R}^N) := \{u \in L^2(\mathbf{R}^N) : |\nabla u| \in L^2(\mathbf{R}^N)\}$ with norm

$$\|u\|_{\mathscr{D}^{1,2}} := \left(\int_{\mathbf{R}^N} |\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

As in [8], define

$$\mu_0 = \left[\frac{2p}{N(p-2)} S^{N/2} \gamma_p^{-p/(p-2)}\right]^{(p/(p-2)-N/2)^{-1}}.$$

Chen and Zou [8] studied System (1.1) with $a(x) \equiv a > 0$, $b(x) \equiv b > 0$ and $\lambda(x) \equiv \lambda > 0$. Under the assumptions that $0 < a \leq \mu_0$ and $\lambda^2 < ab$, they proved

that System (1.1) has a positive solution $(\bar{u}, \bar{v}) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ satisfying $\Phi(\bar{u}, \bar{v}) = \inf_{\mathscr{M}} \Phi$, where

$$\mathscr{M} = \{(u, v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) \backslash \{(0, 0)\} : \Phi'(u, v) = 0\}.$$

In fact, $\mathscr{M}$ is a set of nontrivial critical points of $\Phi$, it may contain only two elements, which is very small subset of the Nehari manifold

$$\mathscr{N} = \{(u, v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N) \backslash \{(0, 0)\} : \langle \Phi'(u, v), (u, v) \rangle = 0\}.$$

In contrast to $\mathscr{M}$, $\mathscr{N}$ contains infinitely many elements of $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. If $(u, v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ is a solution at which $\Phi$ has least "energy" in set $\mathscr{N}$, we shall call it a Nehari-type ground state solution. In general, it is much more difficult to find a solution $(u, v)$ for Problem (1.1) with a constraint condition $\Phi(u, v) = \inf_{\mathscr{N}} \Phi$ than with one $\Phi(u, v) = \inf_{\mathscr{M}} \Phi$.

Subsequently, Li and Tang [16], based on the work of [8], established the existence of Nehari-type ground state solutions for System (1.1). Under assumptions:

(i) $a(x)$, $b(x)$ and $\lambda(x)$ are 1-periodic in each of $x_1, x_2, \ldots, x_N$;
(ii) $0 < a(x) < \mu_0$, $b(x) > 0$, $\forall x \in \mathbf{R}^N$;
(iii) $|\lambda(x)|^2 < a(x)b(x)$, $\forall x \in \mathbf{R}^N$,

they proved that (1.1) has a solution $(\bar{u}, \bar{v}) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ such that $\Phi(\bar{u}, \bar{v}) = \inf_{\mathscr{N}} \Phi > 0$.

In all the work mentioned above, it has been assumed that $\inf_{\mathbf{R}^N} a > 0$, $\inf_{\mathbf{R}^N} b > 0$, $a$, $b$ and $\lambda$ are periodic and $|\lambda(x)|^2 < a(x)b(x)$ for all $x \in \mathbf{R}^N$. Clearly, in this case,

$$\|(u, v)\| = \left\{ \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2 - 2\lambda(x)uv] \, \mathrm{d}x \right\}^{1/2}$$

is a norm defined on $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$, which is equivalent to $\|(\cdot, \cdot)\|_{H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)}$. Here we point out that their arguments become invalid if $a(x_0)b(x_0) \leq 0$ or $|\lambda(x_0)|^2 \geq a(x_0)b(x_0)$ for some $x_0 \in \mathbf{R}^N$. To the best of our knowledge, there seems no existence result for System (1.1) dealing with this case in the literature. In addition, the problem will become more complex by the dropping of periodicity of $a(x)$, $b(x)$ and $\lambda(x)$.

In the present paper, motivated by [8, 16, 18], considering the case that $a(x_0)b(x_0) \leq 0$ or $|\lambda(x_0)|^2 \geq a(x_0)b(x_0)$ for some $x_0 \in \mathbf{R}^N$, we shall investigate the existence of Nehari-type ground solutions for asymptotically periodic linearly coupled Schrödinger equations with critical exponent.

Before presenting the main results, we introduce the following assumptions.

(A0) $a(x) = a_0(x) + a_1(x)$, $a_0 \in C(\mathbf{R}^N, [0, \infty))$, $a_0(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, $a_1 \in C(\mathbf{R}^N, (-\infty, 0])$ and $\lim_{|x| \to \infty} a_1(x) = 0$;
(B0) $b(x) = b_0(x) + b_1(x)$, $b_0 \in C(\mathbf{R}^N, [0, \infty))$, $b_0(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, $b_1 \in C(\mathbf{R}^N, (-\infty, 0])$ and $\lim_{|x| \to \infty} b_1(x) = 0$;
(S0) $\lambda(x) = \lambda_0(x) + \lambda_1(x)$, $\lambda_0 \in C(\mathbf{R}^N, \mathbf{R})$, $\lambda_0(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, $\lambda_1 \in C(\mathbf{R}^N, \mathbf{R})$ and $\lim_{|x| \to \infty} \lambda_1(x) = 0$;

(S1) $a(x) \le \bar{a} \in (0,\infty)$, $b(x) \le \bar{b} \in (0,\infty)$, $\sup_{\mathbf{R}^N}|\lambda| < \sqrt{\bar{a}\bar{b}}$, and there exists a constant $\zeta_0 > 0$ such that

$$\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x$$
$$\ge \zeta_0 \int_{\mathbf{R}^N} ([\bar{a} - a(x)]u^2 + [\bar{b} - b(x)]v^2)\,\mathrm{d}x,$$
$$\forall (u,v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N);$$

(S2) $|\lambda_0(x)|^2 \le a_0(x)b_0(x)$ and

$$|\lambda_1(x)|^2 \begin{cases} < a_1(x)b_1(x), & |x| \le 1 + \sqrt{N}, \\ \le a_1(x)b_1(x), & |x| > 1 + \sqrt{N}; \end{cases}$$

(S3) $\sup_{\mathbf{R}^N}|\lambda| < \sqrt{(\sup_{\mathbf{R}^N} a)\cdot(\sup_{\mathbf{R}^N} b)}$, and there exist a constant $\theta \in (0,1)$ and a bounded measurable set $\Omega \in \mathbf{R}^N$ such that

$$|\lambda(x)| \le \theta\sqrt{|a(x)b(x)|}, \quad \forall x \in \mathbf{R}^N \backslash \Omega,$$

$$\inf_{\mathbf{R}^N\backslash\Omega} a := a_* > 0, \quad \int_\Omega |(1-\theta)a_* + |\lambda(x)| - a(x)|^{N/2}\,\mathrm{d}x \le S^{N/2}$$

and

$$\inf_{\mathbf{R}^N\backslash\Omega} b := b_* > 0, \quad \int_\Omega |(1-\theta)b_* + |\lambda(x)| - b(x)|^{N/2}\,\mathrm{d}x \le S^{N/2}.$$

We will prove in Section 2 that condition (S3) implies condition (S1) (see Lemma 2.2). Here is an example where the assumptions apply.

*Example* 1.1. Let

$$a_0(x) = \alpha_0, \quad a_1(x) = -\frac{\alpha_1}{1+|x|^2}, \quad a(x) = \alpha_0 - \frac{\alpha_1}{1+|x|^2};$$
$$b_0(x) = \beta_0, \quad b_1(x) = -\frac{\beta_1}{1+|x|^2}, \quad b(x) = \beta_0 - \frac{\beta_1}{1+|x|^2};$$
$$\lambda_0(x) = \varrho_0 \cos^2 x_1, \quad \lambda_1(x) = \sqrt{|\alpha_1\beta_1|}e^{-|x|^2}, \quad \lambda(x) = \varrho_0 \cos^2 x_1 + \sqrt{|\alpha_1\beta_1|}e^{-|x|^2}.$$

Then there exist many constants $\alpha_0$, $\beta_0$, $\varrho_0$, $\alpha_1$, and $\beta_1$ such that above $a$, $b$ and $\lambda$ satisfy (A0), (B0), (S0), (S2) and (S3). For example:

$$\alpha_0 = \frac{1}{4}\omega_N^{-2/N}S, \quad \beta_0 = \frac{1}{9}\omega_N^{-2/N}S, \quad \varrho_0 = \frac{1}{3}\alpha_0 = \frac{1}{12}\omega_N^{-2/N}S,$$

$$\alpha_1 = \frac{1}{2}\alpha_0 = \frac{1}{8}\omega_N^{-2/N}S, \quad \beta_1 = \frac{1}{2}\beta_0 = \frac{1}{18}\omega_N^{-2/N}S,$$

where and in the sequel, $\omega_N = \mathrm{meas}(B_1(0)) = 2\pi^{N/2}/N\Gamma(N/2)$. It is easy to check that $a$, $b$ and $\lambda$ satisfy (A0), (B0), (S0) and (S2). Next we verify that (S3) holds. Set

$$\Omega := \{x \in \mathbf{R}^N : |x| \le 1\}.$$

Then, we have

$$a_* = \inf_{\mathbf{R}^N\setminus\Omega} a = \alpha_0 - \frac{1}{2}\alpha_1 = \frac{3}{4}\alpha_0 > 0, \quad b_* = \inf_{\mathbf{R}^N\setminus\Omega} b = \beta_0 - \frac{1}{2}\beta_1 = \frac{3}{4}\beta_0 > 0$$

and

$$|\lambda(x)|^2 \le (\varrho_0 + \sqrt{\alpha_1\beta_1}e^{-1})^2 = \frac{1}{4}(1+e^{-1})^2\alpha_0\beta_0, \quad \forall x \in \mathbf{R}^N\setminus\Omega.$$

Clearly,

$$|\lambda(x)|^2 \le \frac{8}{9}a(x)b(x), \quad \forall x \in \mathbf{R}^N\setminus\Omega,$$

which implies that the first inequality in (S3) holds with $\theta = 8/9$. Moreover, it is easy to see that

$$\begin{aligned}
&\int_\Omega |(1-\theta)a_* + |\lambda(x)| - a(x)|^{N/2}\,\mathrm{d}x \\
&\quad = \int_\Omega \left|\frac{\alpha_0}{12} + \frac{1}{2}\sqrt{\alpha_0\beta_0}(\cos x_1 + e^{-|x|^2}) - \alpha_0 + \frac{\alpha_0}{2(1+|x|^2)}\right|\mathrm{d}x \\
&\quad \le \left(\frac{2}{3}\alpha_0\right)^{N/2}\omega_N < S^{N/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |(1-\theta)b_* + |\lambda(x)| - b(x)|^{N/2}\,\mathrm{d}x \\
&\quad = \int_\Omega \left|\frac{\beta_0}{12} + \frac{1}{2}\sqrt{\alpha_0\beta_0}(\cos x_1 + e^{-|x|^2}) - \beta_0 + \frac{\beta_0}{2(1+|x|^2)}\right|\mathrm{d}x \\
&\quad \le \left(\frac{13}{12}\beta_0\right)^{N/2}\omega_N < S^{N/2}.
\end{aligned}$$

In particular, we mention that $\lambda^2(0) > a(0)b(0)$.

We are now in a position to state the main result of this paper.

THEOREM 1.2. *Assume that* (A0), (B0), (S0), (S1), (S2) *are satisfied, and that* $a(x) < \mu_0$ *for all* $x \in \mathbf{R}^N$. *Then Problem* (1.1) *has a solution* $(\bar{u}, \bar{v}) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ *such that* $\Phi(\bar{u}, \bar{v}) = \inf_{\mathcal{N}} \Phi > 0$.

COROLLARY 1.3. *Assume that* (A0), (B0), (S0), (S2), (S3) *are satisfied, and that* $a(x) < \mu_0$ *for all* $x \in \mathbf{R}^N$. *Then Problem* (1.1) *has a solution* $(\bar{u}, \bar{v}) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ *such that* $\Phi(\bar{u}, \bar{v}) = \inf_{\mathcal{N}} \Phi > 0$.

*Remark* 1.4. It was pointed out in [8] that, by Pohozaev Identity, System (1.1) with $a(x) \equiv a > 0$ and $b(x) \equiv b > 0$ has no nontrivial solutions if $p = 2^*$. That is why we assume $2 < p < 2^*$ in this paper.

The rest of the paper is organized as below. In Section 2, we provide some lemmas. The proofs of Theorem 1.2 and Corollary 1.3 are given in Section 3.

## 2. Preliminaries

In this section, we present some lemmas.

LEMMA 2.1. *Assume that* (S1) *are satisfied. Then there exist two positive constants* $\zeta_1, \zeta_2 > 0$ *such that*

$$\begin{aligned}\zeta_1 \|z\|^2_{H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)} &\le \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ &\le \zeta_2 \|z\|^2_{H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)}, \quad \forall z = (u, v) \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N).\end{aligned}$$

*Proof.* Using (S1), one has

$$\begin{aligned}&\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ &\quad = \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + \bar{a}u^2 + \bar{b}v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ &\qquad - \int_{\mathbf{R}^N} ([\bar{a} - a(x)]u^2 + [\bar{b} - b(x)]v^2)\,\mathrm{d}x \\ &\quad \ge \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + \bar{a}u^2 + \bar{b}v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ &\qquad - \frac{1}{\zeta_0} \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x.\end{aligned} \tag{2.1}$$

Moreover, (S1) implies that there exists a constant $\theta_0 \in (0, 1)$ such that

$$|\lambda(x)| \le \theta_0 \sqrt{\bar{a}\bar{b}}, \quad \forall x \in \mathbf{R}^N. \tag{2.2}$$

Combining (2.1) with (2.2), we get

$$\begin{aligned}&\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\ &\quad \ge \frac{\zeta_0}{1 + \zeta_0} \int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + \bar{a}u^2 + \bar{b}v^2 - 2\lambda(x)uv]\,\mathrm{d}x\end{aligned}$$

$$\geq \frac{\zeta_0(1-\theta_0)}{1+\zeta_0}\int_{\mathbf{R}^N}[|\nabla u|^2+|\nabla v|^2+\bar{a}u^2+\bar{b}v^2]\,\mathrm{d}x$$
$$\geq \zeta_1\|z\|^2_{H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)}.$$

On the other hand, from (S1) and boundedness of $a$, $b$ it follows that

$$\int_{\mathbf{R}^N}[|\nabla u|^2+|\nabla v|^2+a(x)u^2+b(x)v^2-2\lambda(x)uv]\,\mathrm{d}x$$
$$\leq \int_{\mathbf{R}^N}[|\nabla u|^2+|\nabla v|^2+2\bar{a}u^2+2\bar{b}v^2]\,\mathrm{d}x \leq \zeta_2\|z\|^2_{H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)}.$$

This completes the proof. □

Let $E = H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$. Define

$$(z_1,z_2)=\int_{\mathbf{R}^N}[\nabla u_1\cdot\nabla u_2+\nabla v_1\cdot\nabla v_2+a(x)u_1u_2+b(x)v_1v_2-\lambda(x)(u_1v_2+u_2v_1)]\,\mathrm{d}x,$$
$$\forall z_i=(u_i,v_i)\in E,\ i=1,2$$

and

$$\|z\|^2=\int_{\mathbf{R}^N}[|\nabla u|^2+|\nabla v|^2+a(x)u^2+b(x)v^2-2\lambda(x)uv]\,\mathrm{d}x,\quad \forall z=(u,v)\in E.$$

By Lemma 2.1, $(\cdot,\cdot)$ is an inner product and $\|\cdot\|$ is the corresponding norm on $E$. Moreover, $E$ is a Hilbert space with this inner product. For any $s\in[2,2^*]$, there exists an imbedding constant $\tilde{\gamma}_s\in(0,\infty)$ such that

$$\|z\|_s\leq\tilde{\gamma}_s\|z\|,\quad \forall z\in E,$$

where and in the sequel, by $\|\cdot\|_s$ we denote the usual norm in space $L^s(\mathbf{R}^N)\times L^s(\mathbf{R}^N)$.

In view of the definition of $\|\cdot\|$ and (1.2), we can rewrite $\Phi$ in the form

$$\Phi(z)=\frac{1}{2}\|z\|^2-\frac{1}{p}\int_{\mathbf{R}^N}|u|^p\,\mathrm{d}x-\frac{1}{2^*}\int_{\mathbf{R}^N}|v|^{2^*}\,\mathrm{d}x,\quad \forall z=(u,v)\in E. \tag{2.3}$$

It is easy to see that $\Phi$ is of class $C^1(E,\mathbf{R})$, and

$$\begin{aligned}\langle\Phi'(z),\tilde{z}\rangle=&\int_{\mathbf{R}^N}[\nabla u\cdot\nabla\tilde{u}+\nabla v\cdot\nabla\tilde{v}+a(x)u\tilde{u}+b(x)v\tilde{v}-\lambda(x)(u\tilde{v}+v\tilde{u})]\,\mathrm{d}x\\&-\int_{\mathbf{R}^N}|u|^{p-2}u\tilde{u}\,\mathrm{d}x-\int_{\mathbf{R}^N}|v|^{2^*-2}v\tilde{v}\,\mathrm{d}x,\\&\forall z=(u,v),\ \tilde{z}=(\tilde{u},\tilde{v})\in E.\end{aligned} \tag{2.4}$$

LEMMA 2.2. *If* (S3) *holds, then* (S1) *does.*

*Proof.* By (S3) and the Hölder inequality, we get

$$
\begin{aligned}
&\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\
&\quad = \int_{\mathbf{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x + \int_\Omega [a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\
&\qquad + \int_{\mathbf{R}^N\setminus\Omega} [a(x)u^2 + b(x)v^2 - 2\lambda(x)uv]\,\mathrm{d}x \\
&\quad \geq \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \int_\Omega [a(x) - |\lambda(x)|]u^2\,\mathrm{d}x + \int_\Omega [b(x) - |\lambda(x)|]v^2\,\mathrm{d}x \\
&\qquad + (1-\theta)\int_{\mathbf{R}^N\setminus\Omega} [a(x)u^2 + b(x)v^2]\,\mathrm{d}x \\
&\quad \geq \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \int_\Omega [a(x) - |\lambda(x)|]u^2\,\mathrm{d}x + \int_\Omega [b(x) - |\lambda(x)|]v^2\,\mathrm{d}x \\
&\qquad + (1-\theta)\int_{\mathbf{R}^N\setminus\Omega} [a_*u^2 + b_*v^2]\,\mathrm{d}x \\
&\quad = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \int_\Omega [a(x) - (1-\theta)a_* - |\lambda(x)|]u^2\,\mathrm{d}x \\
&\qquad + \int_\Omega [b(x) - (1-\theta)b_* - |\lambda(x)|]v^2\,\mathrm{d}x + (1-\theta)\int_{\mathbf{R}^N} (a_*u^2 + b_*v^2)\,\mathrm{d}x \\
&\quad \geq \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \left(\int_\Omega |(1-\theta)a_* + |\lambda(x)| - a(x)|^{N/2}\,\mathrm{d}x\right)^{2/N} \|u\|_{2^*}^2 \\
&\qquad - \left(\int_\Omega |(1-\theta)b_* + |\lambda(x)| - b(x)|^{N/2}\,\mathrm{d}x\right)^{2/N} \|v\|_{2^*}^2 \\
&\qquad + (1-\theta)\int_{\mathbf{R}^N} (a_*u^2 + b_*v^2)\,\mathrm{d}x \\
&\quad \geq (1-\theta)\int_{\mathbf{R}^N} (a_*u^2 + b_*v^2)\,\mathrm{d}x \\
&\qquad + \left[1 - \left(\int_\Omega |(1-\theta)a_* + |\lambda(x)| - a(x)|^{N/2}\,\mathrm{d}x\right)^{2/N} S\right] \|\nabla u\|_2^2 \\
&\qquad + \left[1 - \left(\int_\Omega |(1-\theta)b_* + |\lambda(x)| - b(x)|^{N/2}\,\mathrm{d}x\right)^{2/N} S\right] \|\nabla v\|_2^2 \\
&\quad \geq (1-\theta)\min\left\{\frac{a_*}{\bar{a} - \underline{a}}, \frac{b_*}{\bar{b} - \underline{b}}\right\} \int_{\mathbf{R}^N} ([\bar{a} - a(x)]u^2 + [\bar{b} - b(x)]v^2)\,\mathrm{d}x, \quad \forall (u,v) \in E,
\end{aligned}
$$

where $\underline{a} := \inf_{\mathbf{R}^N} a$, $\underline{b} := \inf_{\mathbf{R}^N} b$, $\bar{a} := \sup_{\mathbf{R}^N} a$ and $\bar{b} := \sup_{\mathbf{R}^N} b$. This shows that (S1) holds with $\zeta_0 = (1-\theta)\min\{a_*/(\bar{a}-\underline{a}), b_*/(\bar{b}-\underline{b})\}$. □

LEMMA 2.3 ([18, Lemma 8]). *Let $X$ be a Banach space. Let $M_0$ be a closed subspace of the metric space $M$ and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

*If $\Psi \in C^1(X, \mathbf{R})$ satisfies*

$$\infty > d := \inf_{\gamma\in\Gamma} \sup_{t\in M} \Psi(\gamma(t)) > d_0 := \sup_{\gamma_0\in\Gamma_0} \sup_{t\in M_0} \Psi(\gamma_0(t)),$$

*then there exists a sequence $\{u_n\} \subset X$ satisfying*

$$\Psi(u_n) \to d, \quad \|\Psi'(u_n)\|(1+\|u_n\|) \to 0.$$

LEMMA 2.4. *Assume that* (S1) *is satisfied. Then for any $z = (u, v) \in E\backslash\{(0,0)\}$, there exists $t(z) > 0$ such that $t(z)z \in \mathcal{N}$.*

*Proof.* Let $z = (u, v) \in E\backslash\{(0,0)\}$ be fixed and define the function $g(t) := \Phi(tz)$ on $[0, \infty)$. Obviously, we have

$$g'(t) = 0 \Leftrightarrow tz \in \mathcal{N} \Leftrightarrow \|z\|^2 = t^{p-2}\|u\|_p^p + t^{2^*-2}\|v\|_{2^*}^{2^*}.$$

Using (S1), it is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t$ large. Therefore, $\max_{t\in[0,\infty)} g(t)$ is achieved at $t_0 = t(z)$ so that $g'(t_0) = 0$ and $t(z)z \in \mathcal{N}$. □

LEMMA 2.5. *Assume that* (S1) *is satisfied. For $z = (u, v) \in \mathcal{N}$, there holds*

$$\Phi(z) \geq \Phi(tz), \quad \forall t \geq 0. \tag{2.5}$$

*Proof.* For $z = (u, v) \in \mathcal{N}$, one has

$$\Phi(z) - \Phi(tz) = \left(\frac{1-t^2}{2} + \frac{t^p - 1}{p}\right)\|u\|_p^p + \left(\frac{1-t^2}{2} + \frac{t^{2^*}-1}{2^*}\right)\|v\|_{2^*}^{2^*}.$$

It is easy to verify that

$$\frac{1-t^2}{2} + \frac{t^s - 1}{s} \geq 0, \quad \forall t \geq 0,\ s \in (2, 2^*].$$

Then (2.5) holds. □

Define

$$c_1 := \inf_{\mathcal{N}} \Phi, \quad c_2 := \inf_{z\in E\backslash\{(0,0)\}} \max_{t\geq 0} \Phi(tz), \quad c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = (0,0), \Phi(\gamma(1)) < 0\}.$$

Employing Lemmas 2.3–2.5, we can show the following lemma in the same way as [16, 27].

LEMMA 2.6. *Assume that* (S1) *is satisfied. Then* $c_1 = c_2 = c > 0$ *and there exists a sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\Phi(z_n) \to c, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0. \tag{2.6}$$

LEMMA 2.7. *Assume that* (S1) *is satisfied. Then any sequence* $\{z_n\} \subset E$ *satisfying* (2.6) *is bounded in* $E$.

*Proof.* By virtue of (2.3) and (2.4), we obtain that

$$\begin{aligned} d + o(1) &= \Phi(z_n) - \frac{1}{p}\langle \Phi'(z_n), z_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|z_n\|^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right)\|v_n\|_{2^*}^{2^*} \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|z_n\|^2. \end{aligned}$$

Hence, the sequence $\{z_n\}$ is bounded in $E$. □

Analogous to the proof of [8, Lemma 2.4], we have the following lemma.

LEMMA 2.8. *Assume that* (S1) *is satisfied, and that* $a(x) < \mu_0$. *Then there holds* $c_2 < \frac{1}{N} S^{N/2}$.

## 3. Proofs of the results

In this section, we give the proofs of Theorem 1.2 and Corollary 1.3.

Define a functional $\Phi_0$ as follows:

$$\begin{aligned} \Phi_0(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} [|\nabla u|^2 + |\nabla v|^2 + a_0(x)u^2 + b_0(x)v^2 - 2\lambda_0(x)uv]\,\mathrm{d}x \\ - \frac{1}{p}\int_{\mathbf{R}^N} |u|^p\,\mathrm{d}x - \frac{1}{2^*}\int_{\mathbf{R}^N} |v|^{2^*}\,\mathrm{d}x, \quad \forall (u,v) \in E. \end{aligned} \tag{3.1}$$

Then (A0), (B0) and (S0) imply that $\Phi_0 \in C^1(E, \mathbf{R})$, and

$$\begin{aligned}(3.2)\quad \langle \Phi_0'(z), \tilde{z}\rangle = &\int_{\mathbf{R}^N} [\nabla u \cdot \nabla \tilde{u} + \nabla v \cdot \nabla \tilde{v} + a_0(x) u\tilde{u} + b_0(x) v\tilde{v} - \lambda_0(x)(u\tilde{v} + v\tilde{u})]\, \mathrm{d}x \\ &- \int_{\mathbf{R}^N} |u|^{p-2} u\tilde{u}\, \mathrm{d}x - \int_{\mathbf{R}^N} |v|^{2^*-2} v\tilde{v}\, \mathrm{d}x, \\ &\forall z = (u, v),\ \tilde{z} = (\tilde{u}, \tilde{v}) \in E.\end{aligned}$$

*Proof of Theorem* 1.2. Lemma 2.6 yields the existence of a sequence $\{z_n\} = \{(u_n, v_n)\} \subset E$ satisfying (2.6). This, together with Lemma 2.8, implies

$$(3.3)\qquad \Phi(z_n) \to c < \frac{1}{N} S^{N/2}, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0.$$

Employing Lemma 2.7, $\{z_n\}$ is bounded in $E$. Passing to a subsequence, we can assume that

$$\begin{aligned}(3.4)\qquad & z_n = (u_n, v_n) \rightharpoonup \bar{z} = (\bar{u}, \bar{v}) \quad \text{in } E, \quad z_n \to \bar{z} \quad \text{in } L^s_{\mathrm{loc}}(\mathbf{R}^N) \times L^s_{\mathrm{loc}}(\mathbf{R}^N), \\ & 2 \le s < 2^*, \quad z_n \to \bar{z} \quad \text{a.e. on } \mathbf{R}^N.\end{aligned}$$

Next, we prove that $\bar{z} \neq (0, 0)$. Arguing by contradiction, suppose that $\bar{z} = (0, 0)$, i.e. $z_n \rightharpoonup (0, 0)$ in $E$, and so $z_n \to (0, 0)$ in $L^s_{\mathrm{loc}}(\mathbf{R}^N) \times L^s_{\mathrm{loc}}(\mathbf{R}^N)$, $2 \le s < 2^*$ and $z_n \to (0, 0)$ a.e. on $\mathbf{R}^N$. Hence, using the Hölder inequality, it is easy to get that

$$(3.5)\qquad \lim_{|x|\to\infty} \int_{\mathbf{R}^N} a_1(x) u_n^2\, \mathrm{d}x = 0, \quad \lim_{|x|\to\infty} \int_{\mathbf{R}^N} b_1(x) v_n^2\, \mathrm{d}x = 0,$$

and

$$(3.6)\qquad \lim_{|x|\to\infty} \int_{\mathbf{R}^N} \lambda_1(x) u_n v_n\, \mathrm{d}x = 0.$$

Note that

$$(3.7)\qquad \Phi_0(z_n) = \Phi(z_n) - \frac{1}{2} \int_{\mathbf{R}^N} [a_1(x) u_n^2 + b_1(x) v_n^2 - 2\lambda_1(x) u_n v_n]\, \mathrm{d}x$$

and

$$\begin{aligned}(3.8)\qquad \langle \Phi_0'(z_n), \tilde{z}_n\rangle = \langle \Phi'(z_n), \tilde{z}_n\rangle - &\int_{\mathbf{R}^N} [a_1(x) u_n \tilde{u}_n + b_1(x) v_n \tilde{v}_n \\ &- 2\lambda_1(x)(u_n \tilde{v}_n + \tilde{u}_n v_n)]\, \mathrm{d}x, \quad \forall \tilde{z}_n = (\tilde{u}_n, \tilde{v}_n) \in E.\end{aligned}$$

By a standard argument, we can prove that

$$(3.9)\qquad \Phi_0(z_n) \to c, \quad \|\Phi_0'(z_n)\|(1 + \|z_n\|) \to 0.$$

If $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbf{R}} \int_{B_1(y)} |z_n|^2 \, dx = 0$, then using the Lions' concentration compactness principle [20, Lemma 1.21] we have

$$\|z_n\|_s^s \to 0 \Rightarrow \|u_n\|_s^s \to 0 \quad \text{and} \quad \|v_n\|_s^s \to 0, \quad \forall 2 < s < 2^*. \tag{3.10}$$

Hence, it follows from (2.3), (2.4), (3.3) and (3.10) that

$$\|v_n\|_{2^*}^{2^*} = N\left[\Phi(z_n) - \frac{1}{2}\langle \Phi'(z_n), z_n\rangle - \frac{p-2}{2p}\|u_n\|_p^p\right] = Nc + o(1). \tag{3.11}$$

The combination of (2.4), (3.3), (3.10) and (3.11) implies that

$$\|z_n\|^2 = \langle \Phi'(z_n), z_n\rangle + \|u_n\|_p^p + \|v_n\|_{2^*}^{2^*} = Nc + o(1),$$

which, together with (3.5) and (3.6), yields that

$$\int_{\mathbf{R}^N} [|\nabla u_n|_2^2 + |\nabla v_n|_2^2 + a_0(x)u_n^2 + b_0(x)v_n^2 - 2\lambda_0(x)u_n v_n] \, dx = Nc + o(1). \tag{3.12}$$

From (1.3), (3.11), (3.12) and (S2), one can get

$$\begin{aligned} Nc + o(1) = \|v_n\|_{2^*}^{2^*} &\le S^{-N/(N-2)}(\|\nabla v_n\|_2^2)^{N/(N-2)} \\ &\le S^{-N/(N-2)}\Bigg\{\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 + \int_{\mathbf{R}^N} [a_0(x)u_n^2 \\ &\qquad + b_0(x)v_n^2 - 2\lambda_0(x)u_n v_n] \, dx\Bigg\}^{N/(N-2)} \\ &= \left(\frac{Nc}{S}\right)^{N/(N-2)} + o(1). \end{aligned} \tag{3.13}$$

Applying Lemma 2.6, (3.13) implies that

$$c_2 = c \ge \frac{1}{N} S^{N/2},$$

which contradicts (3.3). Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbf{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |z_n|^2 \, dx > \frac{\delta}{2}$. Define $\tilde{z}_n(x) = (\tilde{u}_n(x), \tilde{v}_n(x)) = z_n(x + k_n)$. Then $\|\tilde{z}_n\|_{H^1(\mathbf{R})\times H^1(\mathbf{R})} = \|z_n\|_{H^1(\mathbf{R})\times H^1(\mathbf{R})}$ and

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{z}_n|^2 \, dx > \frac{\delta}{2}. \tag{3.14}$$

Since $a_0(x)$, $b_0(x)$ and $\lambda_0(x)$ are periodic, by (3.9), we have

$$\Phi_0(\tilde{z}_n) \to c, \quad \|\Phi_0'(\tilde{z}_n)\|(1 + \|\tilde{z}_n\|) \to 0. \tag{3.15}$$

Note that $\{\tilde{z}_n\}$ is bounded in $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$, then passing to a subsequence, we can assume that $\tilde{z}_n \rightharpoonup \tilde{z} = (\tilde{u}, \tilde{v})$ in $E$, $\tilde{z}_n \to \tilde{z}$ in $L^s_{\mathrm{loc}}(\mathbf{R}^N) \times L^s_{\mathrm{loc}}(\mathbf{R}^N)$, $2 \le s < 2^*$ and $\tilde{z}_n \to \tilde{z}$ a.e. on $\mathbf{R}^N$. Obviously, (3.14) implies that $\tilde{z} \ne (0,0)$ on $B_{1+\sqrt{N}}(0)$. By virtue of (3.7), (3.8), (3.15) and Fatou's lemma, we obtain that

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left[\Phi_0(\tilde{z}_n) - \frac{1}{2}\langle \Phi_0'(\tilde{z}_n), \tilde{z}_n\rangle\right] \\
&= \lim_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|\tilde{u}_n\|_p^p + \left(\frac{1}{2}-\frac{1}{2^*}\right)\|\tilde{v}_n\|_{2^*}^{2^*}\right] \\
&\ge \left(\frac{1}{2}-\frac{1}{p}\right)\|\tilde{u}\|_p^p + \left(\frac{1}{2}-\frac{1}{2^*}\right)\|\tilde{v}\|_{2^*}^{2^*} \\
&= \Phi_0(\tilde{z}) - \frac{1}{2}\langle \Phi_0'(\tilde{z}), \tilde{z}\rangle = \Phi_0(\tilde{z}).
\end{aligned}
\tag{3.16}
$$

For $\tilde{z} \ne (0,0)$, according to Lemma 2.4, there exists $t_0 = t(\tilde{z}) > 0$ such that $t_0\tilde{z} \in \mathcal{N}$, and so $\Phi(t_0\tilde{z}) \ge c$. Since $\langle \Phi_0'(\tilde{z}), \tilde{z}\rangle = 0$, by the same way as the proof of Lemma 2.5 we can verify that $\Phi_0(\tilde{z}) \ge \Phi_0(t\tilde{z})$ for all $t \ge 0$. Thus, it follows from (3.1), (3.7), (3.16) and (S2) that

$$
\begin{aligned}
c &\ge \Phi_0(\tilde{z}) \ge \Phi_0(t_0\tilde{z}) \\
&= \Phi(t_0\tilde{z}) + \frac{t_0^2}{2}\int_{\mathbf{R}^N}[2\lambda_1(x)\tilde{u}\tilde{v} - a_1(x)\tilde{u}^2 - b_1(x)\tilde{v}^2]\,\mathrm{d}x \\
&> \Phi(t_0\tilde{z}) \ge c.
\end{aligned}
$$

This contradiction implies that that $\bar{z} \ne (0,0)$ given by (3.4). By a standard argument, we can verify that $\Phi'(\bar{z}) = 0$ and $\Phi(\bar{z}) = c = \inf_{\mathcal{N}} \Phi > 0$. This shows that $\bar{z} \in H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ is a solution for Problem (1.1). $\square$

Obviously, Corollary 1.3 is the direct consequence of Lemma 2.2 and Theorem 1.2.

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