ON INFINITESIMAL STREBEL POINTS

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Abstract

In this paper, we prove that if X is a Riemann surface of infinite analytic type and $[\mu]_T$ is any element of Teichmüller space, then there exists $\mu_1 \in [\mu]_T$ so that $[\mu_1]_B$ is an infinitesimal Strebel point.

1. Introduction

If X is a hyperbolic Riemann surface and $M(X)$ denotes the set of Beltrami differentials on X, then Teichmüller space $T(X)$ is defined by the set of equivalence classes in $M(X)$ under the Teichmüller equivalence relation. Given $\mu \in M(X)$, we denote by $[\mu]_T \in T(X)$ the corresponding point of Teichmüller space. In the case where X is a closed surface, every $[\mu]_T$ different from the basepoint $[0]_T$ contains a uniquely extremal representative of the form

$$(1.1) \qquad \mu = k|\varphi|/\varphi$$

for $0 < k < 1$ and $\varphi \in A^1(X)$, that is, φ is an integrable holomorphic quadratic differential on X.

In the setting of infinite type surfaces, not every Teichmüller class has a uniquely extremal representative. The first such example to be constructed was the Strebel chimney [15]. If $X = \{z \in \mathbf{C} : \mathrm{Im}(z) < 0 \text{ or } |\mathrm{Re}(z)| < 1\}$ and $K > 1$, then for every $L \in [1/K, K]$

$$f_L(x + iy) = \begin{cases} x + iKy, & y \geq 0, |x| < 1 \\ x + iLy, & y < 0 \end{cases}$$

is an extremal representative in its Teichmüller class $[\mu_{f_1}]_T$. Here $\mu_f \in M(X)$ denotes the complex dilatation of a quasiconformal mapping f.

Strebel points, introduced in [4, 13], are those points in an infinite dimensional Teichmüller space where the behaviour is, in some sense, tame. In

particular, Strebel's frame mapping criterion [6] states that every Strebel point can be represented by a uniquely extremal Beltrami differential of the form (1.1). It is known that the set of Strebel points is both open and dense in Teichmüller space.

We may also consider the infinitesimal class of $\mu \in M(X)$, namely those $v \in M(X)$ which induce the same linear functional $\Lambda_v(\varphi) = \int_X v\varphi$ on $A^1(X)$ as μ. The set of equivalence classes is denoted by $B(X)$ and a particular class is denoted $[\mu]_B$. We can consider extremality of representatives of an equivalence class in $B(X)$. If μ is extremal in its class in $B(X)$, it is called infinitesimally extremal. There is an analogous notion of an infinitesimal Strebel point, and the infinitesimal version of Strebel's frame mapping criterion (see [3]) states that for such a point, it can again be represented by a uniquely infinitesimally extremal Beltrami differential of the form (1.1).

Typically $[\mu]_T$ and $[\mu]_B$ do not coincide but they share similar properties. In particular, μ is extremal if and only if it is infinitesimally extremal, see [8, 12, 14]. Further, μ is uniquely extremal if and only if it is uniquely infinitesimally extremal, see [2]. A point to note here is that not every uniquely extremal Beltrami differential has the form (1.1).

In [9], it was shown that if $\|\mu\|_\infty$ is small then there exists $v_1 \in [\mu]_T$ such that $[v_1]_B$ is an infinitesimal Strebel point and $v_2 \in [\mu]_B$ such that $[v_2]_B$ is a Strebel point. Further, in [10] it was proved that for any $\mu \in M(\mathbf{D})$ there exists $v \in [\mu]_T$ such that $[v]_B$ is an infinitesimal Strebel point. In this paper, we extend this latter result to all surfaces of infinite type.

THEOREM 1.1. *Let X be a Riemann surface of infinite analytic type. Given any Beltrami differential $\mu \in M(X)$, there exists $\mu_1 \in [\mu]_T$ such that $[\mu_1]_B$ is an infinitesimal Strebel point.*

Before proving this theorem, we first recall some material from Teichmüller theory, see for example [5, 7] for more details.

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2. Preliminaries

2.1. Beltrami differentials. Let X be a Riemann surface of infinite analytic type, that is, we require the genus of X or the number of punctures to be infinite. Denote by $M(X)$ the set of Beltrami differentials on X, that is, $(-1, 1)$-differential forms given in local coordinates by

$$\mu(z)\frac{d\bar{z}}{dz},$$

and where $\|\mu\|_\infty \leq k$ for some $k < 1$.

There are two equivalence relations we may impose on $M(X)$. The Teichmüller equivalence relation is defined as follows. Since X is a hyperbolic surface, it can be realized as $\mathbf{D}/G$ for some group G of covering transformations. We may lift $\mu, \nu \in M(X)$ to elements $\tilde{\mu}, \tilde{\nu}$ in the unit ball of $L^\infty(\mathbf{D})$ which are G-invariant in the sense that

$$(\tilde{\mu} \circ g)\frac{\overline{g'}}{g'} = \tilde{\mu},$$

for all $g \in G$. Via the Measurable Riemann Mapping Theorem, there is a unique quasiconformal mapping $f^{\tilde{\mu}} : \mathbf{D} \to \mathbf{D}$ which solves the Beltrami differential equation $f_{\bar{z}} = \tilde{\mu}f_z$, extends continuously to the boundary $\partial\mathbf{D}$ and fixes 1, -1, i. We then say that $\mu \sim_T \nu$ if and only if

$$(2.1) \qquad\qquad f^{\tilde{\mu}}|_{\partial\mathbf{D}} = f^{\tilde{\nu}}|_{\partial\mathbf{D}},$$

and denote an equivalence class by $[\mu]_T$. The set of equivalence classes is the Teichmüller space $T(X)$ of X.

The second equivalence class is defined as follows. Let $A^1(X)$ be the Bergman space of integrable holomorphic quadratic differentials on X, that is, $(2,0)$-differential forms give in local coordinates by

$$\varphi(z)\, dz^2,$$

where φ is holomorphic and satisfies

$$\|\varphi\|_1 := \int_X |\varphi| < \infty.$$

We remark that every such φ lifts to a holomorphic function $\tilde{\varphi}$ on $\mathbf{D}$ satisfying $(\tilde{\varphi} \circ g)(g')^2 = \tilde{\varphi}$ for all $g \in G$. It is well-known that the cotangent space at the basepoint of Teichmüller space, $[0]_T$, is isomorphic to $A^1(X)$. Every $\mu \in M(X)$ induces a linear functional on $A^1(X)$ defined by

$$\Lambda_\mu(\varphi) = \int_X \mu\varphi.$$

Note that the expression $\mu\varphi$ in local coordinates is $\mu(z)\varphi(z)|dz|^2$ and so can be integrated. If $\Lambda_\mu = \Lambda_\nu$ on $A^1(X)$, then we say that μ and ν are infinitesimally equivalent and write $\mu \sim_B \nu$. An equivalence class is denoted by $[\mu]_B$ and the set of equivalence classes is denoted by $B(X)$, and is isomorphic to $A^1(X)^*$. The norm of Λ_μ is

$$\|\Lambda_\mu\| = \sup\left\{\left|\int_X \mu\varphi\right| : \|\varphi\|_1 = 1\right\}.$$

If μ is extremal, then $\|\Lambda_\mu\| = \|\mu\|_\infty$.

In either equivalence class, we have the notion of extremality and unique extremality. If $\sim$ is either $\sim_T$ or $\sim_B$, then we say that μ is extremal if $\|\mu\|_\infty \leq \|\nu\|_\infty$ for all $\nu \sim \mu$, and we say that μ is uniquely extremal if the

inequality is strict for all $v \sim \mu$ and $v \neq \mu$. We will use the language *extremal* for $\sim_T$ and *infinitesimally extremal* for $\sim_B$. From compactness considerations, there is always an extremal and infinitesimal extremal representative in each class.

2.2. Extremal distortion. The notation we use in this section is slightly non-standard but is intended to unify the ideas between the two equivalence classes. Writing $k(\mu) = \|\mu\|_\infty$ for $\mu \in M(X)$, the corresponding extremal version is

$$k_0([\mu]_T) = \inf\{k(v) : v \sim_T \mu\}$$

and the infinitesimally extremal version is

$$k_1([\mu]_B) = \inf\{k(v) : v \sim_B \mu\}.$$

It is known that

(2.2) $$k_1([\mu]_B) = \|\Lambda_\mu\| = \sup_{\|\varphi\|_1=1} \text{Re} \int_X \mu\varphi.$$

The boundary dilatation of $\mu \in M(X)$ is

$$h(\mu) = \inf\{\|\mu|_{X\setminus E}\|_\infty : E \text{ is a compact subset of } X\}.$$

The extremal version is

$$h_0([\mu]_T) = \inf\{h(v) : v \sim_T \mu\}$$

and the infinitesimally extremal version is

$$h_1([\mu]_B) = \inf\{h(v) : v \sim_B \mu\}.$$

It is clear that $h_0([\mu]_T) \leq k_0([\mu]_T)$ and $h_1([\mu]_B) \leq k_1([\mu]_B)$. In the first case, if $h_0([\mu]_T) < k_0([\mu]_T)$ then $[\mu]_T \in T(X)$ is called a *Strebel point*, and otherwise a non-Strebel point. In the second case, if $h_1([\mu]_B) < k_1([\mu]_B)$, then $[\mu]_B \in B(X)$ is called an *infinitesimal Strebel point*, and otherwise an infinitesimal non-Strebel point.

The case $\mu = 0$ gives a non-Strebel point $[0]_T \in T(X)$ and an infinitesimal non-Strebel point $[0]_B \in B(X)$.

2.3. Quasiconformal gluing. We will require the following result on quasiconformal gluing.

THEOREM 2.1 ([11], Theorem 2). *Let f_1 and f_2 be two K-quasiconformal mappings defined on disjoint simply connected subdomains Ω_1 and Ω_2 of $\overline{\mathbf{C}}$ respectively and $f_1(\Omega_1) \cap f_2(\Omega_2) = \emptyset$. Then for any two Jordan domains D_1 and D_2 with $\overline{D_1} \subset \Omega_1$ and $\overline{D_2} \subset \Omega_2$, there exists a quasiconformal mapping g of $\overline{\mathbf{C}}$ such that*

$$g|_{D_1} = f_1|_{D_1} \quad and \quad g|_{D_2} = f_2|_{D_2}.$$

3. Proof of Theorem 1.1

In [10], where the case $X = \mathbf{D}$ is considered, a particular choice of $\varphi = 1/\pi \in A^1(\mathbf{D})$ is chosen to simplify estimates. In our situation, there is not an obvious choice of φ. To deal with this, we will choose any φ of norm 1, decompose our surface X into three subsets and modify a given μ on some of these subsets. The modified Beltrami differential will remain in the same Teichmüller class but will represent an infinitesimal Strebel point.

Fix a Riemann surface X of infinite analytic type and $\varphi \in A^1(X)$ with $\|\varphi\|_1 = 1$. Let $\mu \in M(X)$ with $\|\mu\|_\infty = k < 1$ and $\varepsilon > 0$. Since X is hyperbolic, there is a covering group G so that $X \simeq \mathbf{D}/G$.

Step 1: find an appropriate decomposition of X. Realize X as a union of pairs of pants, that is, topological three-holed spheres where some of the holes are allowed to be points, see for example [1]. Label the pants $P_1, P_2, \ldots$ where for convenience we may assume $\bigcup_{i=1}^n P_i$ is connected for each n. Choose N large enough that if $G_1 = \bigcup_{i=N+1}^\infty P_i$, then

$$\int_{G_1} |\varphi| < \frac{\varepsilon}{2}.$$

Each P_i is obtained by gluing two hypergolic geodesic hexagons together along three pairs of sides called seams. The three boundary components of P_i are called cuffs. For $i = 1, \ldots, N$, let U_i be an open neighbourhood of the cuffs and seams in P_i so that $P_i \backslash U_i$ consists of two components, one arising from each hexagon, and so that

$$\int_{U_i} |\varphi| < \frac{\varepsilon}{4N}.$$

Note there is no issue if a boundary component of P_i reduces to a point. In this case, U_i wil contain a neighbourhood of this puncture.

We next take closed neighbourhoods of each U_i, say F_i, which will consist of two components, each a topological annulus so that $E_i := P_i \backslash (U_i \cup F_i)$ consists of two topological disks. We further require that

$$\int_{F_i} |\varphi| < \frac{\varepsilon}{4N}.$$

Our disjoint decomposition of X is then defined by $G = G_1 \cup \bigcup_{i=1}^N U_i$, $F = \bigcup_{i=1}^N F_i$ and $E = \bigcup_{i=1}^N E_i$. By construction, E and G are open, F is closed and

$$(3.1) \quad \int_G |\varphi| = \int_{G_1} |\varphi| + \sum_{i=1}^N \int_{U_i} |\varphi| < \frac{3\varepsilon}{4}, \quad \int_F |\varphi| = \sum_{i=1}^N \int_{F_i} |\varphi| < \frac{\varepsilon}{4}, \quad \int_E |\varphi| > 1 - \varepsilon.$$

Step 2: Given a Beltrami differential μ in $M(X)$, we modify it on E and F. Let $f : X \to Y$ be a quasiconformal map with complex dilatation μ. Then

there is a covering group G' so that $Y \simeq \mathbf{D}/G'$. We may lift f to a quasi-conformal map $\tilde{f} : \mathbf{D} \to \mathbf{D}$ with the following property: for every $g \in G$, there is a unique $g' \in G'$ so that

$$(3.2) \qquad\qquad \tilde{f} \circ g = g' \circ \tilde{f}.$$

Denote by π_1, π_2 the projections from $\mathbf{D}$ to X, Y respectively. We then have $\pi_2 \circ \tilde{f} = f \circ \pi_1$. Since $\tilde{f}$ extends continuously to $\partial \mathbf{D}$, we may extend $\tilde{f}$ to a quasiconformal map $\tilde{f} : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ via reflection in $\partial \mathbf{D}$.

Given a pair of pants P_i with $i \in \{1, \dots, N\}$, consider one of the two hexagons σ that are glued to give P_i. Consider a component H of $\pi_1^{-1}(\sigma)$, which is a geodesic hexagon in $\mathbf{D}$. Let $H' = \tilde{f}(H) \subset \mathbf{D}$. If we intersect H with $\pi_1^{-1}(E)$ and $\pi_1^{-1}(F)$ then we obtain an open topological disk D and a closed topological annulus A respectively. Let $\Omega_1 \subset \overline{\mathbf{C}}$ be an open neighbourhood of $\overline{\mathbf{C}} \backslash (A \cup D)$ and let $\Omega_2 \subset \overline{\mathbf{C}}$ be an open neighbourhood of $\overline{D}$ so that $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$ and both Ω_1 and Ω_2 are simply connected in $\overline{\mathbf{C}}$. Note that $\partial \Omega_1$ and $\partial \Omega_2$ are both contained in the interior of A.

By the Measurable Riemann Mapping Theorem, we may find a quasicon-formal map $g : \Omega_2 \to f(\Omega_2)$ with complex dilatation

$$(3.3) \qquad\qquad \mu_1 = \frac{k_1 \overline{\tilde{\varphi}}}{|\tilde{\varphi}|},$$

where k_1 is to be determined and $\tilde{\varphi}$ is a lift to $\mathbf{D}$ of the Beltrami differential φ on X. We may then apply Theorem 2.1 with Ω_1, Ω_2 as above, $f_1 = \tilde{f}$, $f_2 = g$ and $D_1 = \overline{\mathbf{C}} \backslash (A \cup D)$, $D_2 = D$. Hence there is a quasiconformal map $h : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ satisfying

$$h|_D = g, \quad h|_{\overline{\mathbf{C}} \backslash (A \cup D)} = \tilde{f}.$$

We then replace $\tilde{f}$ on H by this new map h to obtain a quasiconformal map on H we call $\widetilde{f_1}$. We repeat this construction for each pre-image of a hexagon in a fundamental region D_X for X in $\mathbf{D}$ to obtain a quasiconformal map $\widetilde{f_1}$ on D_X. On $\pi_1^{-1}(E) \cap D_X$, the new quasiconformal map $\widetilde{f_1}$ has complex dilatation (3.3).

To extend this construction to the rest of $\mathbf{D}$, if $g \in G$, we define $\widetilde{f_1}$ via (3.2), that is, for $z \in g(D_X)$, we let

$$(3.4) \qquad\qquad \widetilde{f_1}(z) = g' \circ \widetilde{f_1} \circ g^{-1}.$$

For $z \in g(D_X)$, we have by the formula for the complex dilatation of a compo-sition, (3.3) and the fact $\tilde{\varphi}$ is a G-invariant quadratic differential that

$$\mu_{\widetilde{f_1}}(z) = \mu_{g' \circ \widetilde{f_1} \circ g^{-1}}(z)$$

$$= \mu_{\widetilde{f_1}}(g^{-1}(z)) \frac{\overline{(g^{-1})'(z)}}{(g^{-1})'(z)}$$

$$= k_1 \frac{\overline{\tilde{\varphi}(g^{-1}(z))}}{|\tilde{\varphi}(g^{-1}(z))|} \frac{\overline{(g^{-1})'(z)}}{(g^{-1})'(z)}$$

$$= k_1 \frac{\overline{\tilde{\varphi}(z)}}{|\tilde{\varphi}(z)|} \frac{\overline{g'(g^{-1}(z))}^2}{|g'(g^{-1}(z))^2|} \frac{\overline{(g^{-1})'(z)}}{(g^{-1})'(z)}$$

$$= \frac{k_1 \overline{\tilde{\varphi}(z)}}{|\tilde{\varphi}(z)|}.$$

Hence on $\pi_1^{-1}(E)$, $\tilde{f_1}$ has complex dilatation given by $\dfrac{k_1 \overline{\tilde{\varphi}(z)}}{|\tilde{\varphi}(z)|}$. By (3.4), $\tilde{f_1}$ is a G-invariant quasiconformal mapping which projects to a quasiconformal map $f_1 : X \to Y$ with Beltrami differential μ_1. The map f_1 agrees with f outside a compact subset of X and on a neighbourhood of seams and collars, and agrees with a quasiconformal map with complex dilatation $k_1 \bar{\varphi}/|\varphi|$ on a large subset.

Step 3: μ_1 is in the same Teichmüller class as μ. Since $\tilde{f_1}$ extends continuously to the boundary, we just need to show that the boundary map of $\tilde{f_1}$ agrees with that of $\tilde{f}$. To see this, let $w_0 \in \partial \mathbf{D}$. Since the components of $\pi_1^{-1}(X \backslash G)$ in $\mathbf{D}$ are pairwise disjoint and have uniformly bounded diameter, we can find a sequence of points z_n contained in $\pi_1^{-1}(G)$ which converges to w_0. Since $\tilde{f_1}$ agrees with $\tilde{f}$ on $\pi_1^{-1}(G)$ and both maps extend continuously to $\partial \mathbf{D}$, we see that $\tilde{f_1}(w_0) = \tilde{f}(w_0)$. Since w_0 is arbitrary, by (2.1) it follows that these two maps determine the same two points of Teichmüller space.

Step 4: μ_1 is infinitesimally extremal for an appropriate choice of k_1. Choose $\varepsilon > 0$ small enough and let

$$(3.5) \qquad k_1 > \frac{(1+\varepsilon)k + \varepsilon}{1-\varepsilon}.$$

Note that since the function $p(x) = ((1+x)k + x)/(1-x)$ satisfies $p(0) = k$ and $p'(0) > 0$, this can be achieved.

We have by (2.2)

$$\|\Lambda_{\mu_1}\| = \sup\left\{ \left| \mathrm{Re} \int_X \mu_1 \psi \right| : \psi \in A^1(X), \|\psi\|_1 = 1 \right\}$$

$$\geq \left| \mathrm{Re} \int_X \mu_1 \varphi \right|$$

$$\geq \left| \mathrm{Re} \int_E \mu_1 \varphi + \mathrm{Re} \int_F \mu_1 \varphi + \mathrm{Re} \int_G \mu_1 \varphi \right|$$

$$\geq \left| \mathrm{Re} \int_E \mu_1 \varphi \right| - \left| \mathrm{Re} \int_F \mu_1 \varphi \right| - \left| \mathrm{Re} \int_G \mu_1 \varphi \right|.$$

By (3.1), we have

$$\mathrm{Re} \int_E \mu_1 \varphi = \int_E \mu_1 \varphi = \int_E k_1 |\varphi| > k_1 (1 - \varepsilon),$$

and

$$\left| \operatorname{Re} \int_F \mu_1 \varphi \right| \leq \|\mu_1\|_\infty \int_F |\varphi| < \varepsilon.$$

The contribution on G is, again by (3.1)

$$\left| \operatorname{Re} \int_G \mu_1 \varphi \right| = \left| \operatorname{Re} \int_G \mu \varphi \right| \leq \|\mu\|_\infty \int_G |\varphi| < k\varepsilon.$$

Combining these estimates, we obtain

$$\|\Lambda_{\mu_1}\| > k_1(1 - \varepsilon) - (k + 1)\varepsilon.$$

Finally, by (3.5), we obtain

$$k_1([\mu_1]_B) = \|\Lambda_{\mu_1}\| > k_1(1 - \varepsilon) - (k + 1)\varepsilon > k = \|\mu\|_\infty \geq \|\mu_1|_G\|_\infty \geq h_1([\mu_1]_B),$$

and so $[\mu_1]_B$ is an infinitesimal Strebel point.

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