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p-HARMONIC FUNCTIONS ON COMPLETE MANIFOLDS WITH A WEIGHTED POINCARÉ INEQUALITY

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Abstract

In this paper, we consider p-harmonic functions on complete Riemannian manifolds and give different proofs of some main theorems by Chang-Chen-Wei, in [3]. Moreover, we are able to refine their results in case of weakly p-harmonic functions. Some applications to study the connectedness at infinity of stable minimal hypersurfaces are also given.

1. Introduction

Suppose that M is a complete noncompact oriented Riemannian manifold of dimension n. At a point $x \in M$, let $\{\omega_1, \dots, \omega_n\}$ be a positively oriented orthonormal coframe on $T_x^*(M)$, for $\ell \geq 1$, the Hodge star operator is given by

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-\ell}},$$

where $j_1, \dots, j_{n-\ell}$ are selected such that $\{\omega_{i_1}, \dots, \omega_{i_\ell}, \omega_{j_1}, \dots, \omega_{j_{n-\ell}}\}$ gives a positive orientation. Let d be the exterior differential operator, so its dual operator d^* is defined by

$$d^* = (-1)^{n(\ell+1)+1} * d * .$$

Then the Hogde-Laplace-Beltrami operator Δ acting on the space of smooth ℓ-forms $\Omega^\ell(M)$ is of form

$$\Delta = -(d^*d + dd^*).$$

Recall that an ℓ-form ω on M is said to be p-harmonic $(p > 1)$ if ω satisfies $d\omega = 0$ and $d^*(|\omega|^{p-2}\omega) = 0$. When $p = 2$, a p-harmonic 1-form is exactly a harmonic 1-form. Some vanishing propeties of the set of p-harmonic 1-forms are given by X. Zhang and Chang-Guo-Sung (see [21, 4]). The p-harmonic 1-forms have closed relationship to the set of p-harmonic functions. To describe p-harmonic functions, let us define the p-Laplace operator on a Riemannian

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manifold M by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

for any function $u \in W^{1,p}_{loc}(M)$ and $p > 1$. A function $u \in W^{1,p}_{loc}(M)$ is said to be p-harmonic if $\Delta_p u = 0$. Hence, if u is a smooth p-harmonic function then $\omega := du$ is a p-harmonic 1-form. Motivated by beautiful applications of the theory of harmonic functions to study geometric structures of Riemannian manifold, let us introduce a geometric notation which is related to the connectedness at infinity via the theory of p-harmonic functions. Let $E \subset M$ be an end of M, namely, E is an unbounded connected component of $M \backslash \Omega$ for a sufficiently large compact subset $\Omega \subset M$ with smooth boundary. As in usual harmonic function theory, we define the p-parabolicity and p-nonparabolicity of E as follows (see [1, 2, 5, 7, 16]):

Definition 1.1. An end E of the Riemannian manifold M is called *p-parabolic* if for every compact subset $K \subset \bar{E}$

$$\operatorname{cap}_p(K, E) := \inf \int_E |\nabla f|^p = 0$$

where the infimum is taken among all $f \in \mathscr{C}^\infty_c(\bar{E})$ such that $f \geq 1$ on K. Otherwise, the end E is called *p-nonparabolic* or *p-hyperbolic*.

In [1], the authors gave a characterization of p-hyperbolic ends on complete Riemannian manifolds which carry a Sobolev type inequality. In [16], Pigola et al. discussed potential theoretic properties of the ends of a manifold enjoying an $L^{p,q}$-Sobolev inequality. They proved that a complete manifold with more than one end never supports an $L^{p,q}$-Sobolev inequality provided the negative part of its Ricci tensor is small. Recently, in [3], Chang-Chen-Wei studied p-harmonic functions with finite L^q energy and proved some vanishing type theorems for p-harmonic functions with finite L^q-energy on Riemannian manifolds satisfying a weighted Poincaré inequality. Then such these vanishing theorems are applied to study properties of p-parabolic ends of the manifolds. Recall that M is said to have a weighted Poincaré inequality, if

$$\int_M \rho\varphi^2 \leq \int_M |\nabla\varphi|^2 \tag{1.1}$$

holds true for any smooth function $\varphi \in \mathscr{C}^\infty_0(M)$ with compact support in M. The positive function ρ is called the weighted function. Therefore, if the bottom of the spectrum of Laplacian $\lambda_1(M)$ is positive then M satisfies a weighted Poincaré inequality with $\rho \equiv \lambda_1$. Here $\lambda_1(M)$ can be characterized by variational principle

$$\lambda_1(M) = \inf\left\{\frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2} : \varphi \in \mathscr{C}^\infty_0(M)\right\}.$$

When M satisfies a weighted Poincaré inequality then M has many interesting properties concerning topology and geometry. It is worth to notice that weighted Poincaré inequalities not only generalize the first eigenvalue of the Laplacian, but also appear naturally in other PDE and gemetric problems. For example, $\lambda_1(M)$ is related to the problem of finding the best constant in the inequality

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$$

obtained by the continuous embedding $W_0^{1,2} \to L^2(M)$. It is also well known that a stable minimal hypersurface satisfies a weighted Poincaré inequality with the weight function

$$\rho = |A|^2 + \mathrm{Ric}(v, v)$$

where A is the second fundamental form and $\mathrm{Ric}(v, v)$ is the Ricci curvature of the ambient space in the normal direction. For further discussion on this topic, we refer to [9, 10, 14] and the references there in. In this paper, we first prove the following theorem.

THEOREM 1.2. *Suppose that M^n is a complete noncompact Riemannian manifold satisfying a weighted Poincaré inequality (P_ρ) and the Ricci curvature is bounded by*

$$\mathrm{Ric}_M(x) \geq -a\rho(x)$$

for all $x \in M$. If $p \geq 2$ and

$$a < \frac{4(p-1+\kappa)}{p^2}$$

then there is no non trivial (weakly) p-harmonic function on M with finite L^p energy, where

$$\kappa = \min\left\{\frac{(p-1)^2}{n-1}, 1\right\}.$$

Note that our condition on the range of a is better than that in Chang-Chen-Wei's paper (see [3]). Therefore, this theorem can be considered as a refinement of Theorem 1.1 in [3] (See Remark 2.2).

The second main theorem in this paper is an application of Theorem 1.2 to prove a property regarding to connectedness at infinity of submanifolds immersed in a Riemannian manifold. The statement of the result is as follows.

THEOREM 1.3. *Suppose that M is an immersed stable minimal hypersurface in a Riemannian manifold N with sectional curvature K_N bounded from below by $-c$, where $c > 0$ is a fixed constant. If M has positive spectrum λ_1 then M has at* *most one p-non parabolic end provided that*

$$\lambda_1(M) > \frac{2(n-1)c}{\frac{4(p-1+\kappa)}{p^2} - \frac{n-1}{n}},$$

for any $2 \le p < \dfrac{2(n+\sqrt{2n})}{n-1}$.

The paper has three sections. In the section 2, we will give a proof of Theorem 1.2. The application of Theorem 1.2 is given in the section 3. In the section 3, we will prove Theorem 1.3.

2. Vanishing theorems

LEMMA 2.1. *Let* ω *be a p-harmonic* 1*-form on* $M(p > 1)$. *Let* η *be a compactly supported nonnegative smooth function in* $M_+ := M \backslash \mathscr{S}$, *where* $\mathscr{S} := \{x \in M : \omega(x) \equiv 0 \text{ on } T_x M\}$ *and* $\varphi = \eta \cdot |\omega|^{\bar{q}-1}$, *for* $\bar{q} \in \mathbf{R}$. *Then*

$$\int_M \varphi^2 \langle \Delta\omega, \omega \rangle = \frac{(p-2)(2\bar{q}-p)}{4} \int_M \eta^2 \cdot |\omega|^{2\bar{q}-6} \cdot \langle d|\omega|^2, \omega \rangle^2 + (p-2) \int_M \eta \cdot |\omega|^{2\bar{q}-4} \cdot \langle d|\omega|^2, \omega \rangle \cdot \langle d\eta, \omega \rangle \tag{2.2}$$

It is worth to notice that, in [21], Zhang proved Lemma 2.1 with different coefficient in the first term of the right hand side. But due to the proof of Lemma 2.1 in [21], this coefficient should be corrected as in (2.2). Here we will give a different proof of Lemma 2.1. We think that our proof is more simple than that in Zhang's paper.

Proof. Observe that for any smooth function f and smooth 1-form ω, we have

$$d^*(f\omega) = f d^*\omega + \langle df, \omega \rangle.$$

Since ω is p-harmonic, the above equation implies

$$|\omega|^{p-2} d^*\omega = -\langle d|\omega|^{p-2}, \omega \rangle,$$

$$d^*(\eta^2 |\omega|^{2(\bar{q}-1)}\omega) = \langle d(\eta^2 |\omega|^{2\bar{q}-p}), |\omega|^{p-2}\omega \rangle,$$

on M_+. In fact, we will use that η has compact support in M_+ to differentiate $|\omega|^{p-2}$ and $|\omega|^{2\bar{q}-p}$. Therefore,

$$\begin{aligned}\int_M \varphi^2 \langle \Delta\omega, \omega \rangle &= -\int_M \langle \Delta\omega, \eta^2 |\omega|^{2(\bar{q}-1)}\omega \rangle \\ &= -\int_M \langle dd^*\omega, \eta^2 |\omega|^{2(\bar{q}-1)}\omega \rangle\end{aligned}$$

$$
\begin{aligned}
&= -\int_M d^*\omega \cdot d^*(\eta^2|\omega|^{2(\bar{q}-1)}\omega) \\
&= -\int_M d^*\omega \cdot \langle d(\eta^2|\omega|^{2\bar{q}-p}), |\omega|^{p-2}\omega\rangle \\
&= -\int_M |\omega|^{p-2} d^*\omega \cdot \langle d(\eta^2|\omega|^{2\bar{q}-p}), \omega\rangle \\
&= \int_M \langle d|\omega|^{p-2}, \omega\rangle \cdot \langle d(\eta^2|\omega|^{2\bar{q}-p}), \omega\rangle \\
&= (p-2)(2\bar{q}-p)\int_M \eta^2|\omega|^{2\bar{q}-4}\langle d|\omega|, \omega\rangle^2 \\
&\quad + 2(p-2)\int_M \eta|\omega|^{2\bar{q}-3}\langle d\eta, \omega\rangle \cdot \langle d|\omega|, \omega\rangle \\
&= \frac{(p-2)(2\bar{q}-p)}{4}\int_M \eta^2|\omega|^{2\bar{q}-6}\langle d|\omega|^2, \omega\rangle^2 \\
&\quad + (p-2)\int_M \eta|\omega|^{2\bar{q}-4}\langle d\eta, \omega\rangle \cdot \langle d|\omega|^2, \omega\rangle.
\end{aligned}
$$

The proof is complete. □

For smooth p-harmonic 1-form, we have the following Kato inequality proved by Chang et al., in [3].

LEMMA 2.2. *Let ω be a smooth p-harmonic 1-form on a complete manifold M^n, $p > 1$ and $\kappa = \min\left\{\frac{(p-1)^2}{n-1}, 1\right\}$. Then at any $x \in M$ with $\omega(x) \neq 0$, we have*

$$|\nabla\omega|^2 \geq (1+\kappa)|\nabla|\omega||^2. \tag{2.3}$$

In fact, in [3], the authors proved Kato inequalities for p-harmonic functions $u \in \mathscr{C}^3(M)$ but their proof is still valid for smooth p-harmonic 1-forms.

Using the Kato inequality for p-harmonic 1-forms and Lemma 2.1, we obtain the following vanishing result which can be considered as a refinement of Theorem 1.2 in [3]. Our proof is different from that in [3].

THEOREM 2.3. *Suppose that M^n is a complete noncompact Riemannian manifold satisfying a weighted Poincaré inequality (P_ρ) and the Ricci curvature is bounded by*

$$\mathrm{Ric}_M(x) \geq -a\rho(x)$$

for all $x \in M$. If $q \geq 2$, $q - 1 + \kappa + b > 0$ and

$$a < \frac{4(q-1+\kappa+b)}{q^2},$$

where $b = \min\{0, (p-2)(q-p)\}$. *Let* ω *is a smooth* p*-harmonic* 1*-form on* M *with finite* L^q *energy. Then*

1. ω *is trivial if* $p \geq 2$.
2. ω *does not exist if* $1 < p < 2$.

Proof. Let ω be an arbitrary smooth p-harmonic 1-form with finite L^q energy. To simply notations, let us denote p-harmonic 1-form and its dual p-harmonic vector field by ω. On $M_+ = M \backslash \mathscr{S}$, the Bochner formula implies that

$$\begin{aligned}\frac{1}{2}\Delta(|\omega|^2) &= \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + \mathrm{Ric}_M(\omega, \omega)\\ &\geq \langle \Delta\omega, \omega\rangle + (1+\kappa)|\nabla|\omega|\,|^2 - a\rho|\omega|^2.\end{aligned}$$

Here we used Kato inequality (2.3) and the assumption on Ricci curvature in the last inequality. Let η be a compactly supported nonnegative smooth function in M_+ and let $\varphi = \eta|\omega|^{\bar{q}-1}$, where $\bar{q} = \dfrac{q}{2} \in \mathbf{R}$. Multiplying both sides of the above Bochner type inequality by φ^2 and integrating over M gives

$$\frac{1}{2}\int_M \varphi^2\Delta(|\omega|^2) \geq \int_M \varphi^2\langle\Delta\omega, \omega\rangle + (1+\kappa)\int_M \varphi^2|\nabla|\omega|\,|^2 - a\int_M \rho\varphi^2|\omega|^2.$$

Therefore, integration by part implies

$$2\int_M \varphi|\omega|\langle\nabla\varphi, \nabla|\omega|\rangle + \int_M \varphi^2\langle\Delta\omega, \omega\rangle + (1+\kappa)\int_M \varphi^2|\nabla|\omega|\,|^2 - a\int_M \rho\varphi^2|\omega|^2 \leq 0.$$

Using Lemma 2.1, we have

$$\begin{aligned}\int_{M^+} \varphi^2\langle\Delta\omega, \omega\rangle \geq \min\{0, (p-2)(2\bar{q}-p)\}&\int_{M^+} \eta^2 \cdot |\omega|^{2\bar{q}-2} \cdot |d|\omega|\,|^2\\ &- 2(p-2)\int_{M^+} \eta \cdot |\omega|^{2\bar{q}-1} \cdot |d|\omega|\,| \cdot |d\eta|.\end{aligned}$$

Note that $q = 2\bar{q}$, two above inequalities imply

$$\begin{aligned}&a\int_M \rho\eta^2|\omega|^q + 2(p-2)\int_M \eta|\omega|^{q-1}|\,d\eta| \cdot |\nabla|\omega|\,|\\ &\quad\geq 2\int_M \eta|\omega|^{\bar{q}}\langle\nabla(\eta|\omega|^{\bar{q}-1}), \nabla|\omega|\rangle + (1+\kappa+b)\int_M \eta^2|\omega|^{2(\bar{q}-1)} \cdot |\nabla|\omega|\,|^2\\ &\quad= 2\int_M \eta|\omega|^{q-1}\langle\nabla\eta, \nabla|\omega|\rangle + (q-2)\int_M \eta^2|\omega|^{q-2}|\nabla|\omega|\,|^2\\ &\qquad+ (1+\kappa+b)\int_M \eta^2|\omega|^{q-2} \cdot |\nabla|\omega|\,|^2.\end{aligned}$$

Consequently,

$$a\int_M \rho\eta^2|\omega|^q + 2(p-1)\int_M \eta|\omega|^{q-1}|d\eta|\cdot|\nabla|\omega|\,| \geq (q-1+\kappa+b)\int_M \eta^2|\omega|^{q-2}|\nabla|\omega|\,|^2. \tag{2.4}$$

Since M satisfies a weighted Poincaré inequality, for $\varepsilon > 0$, we have

$$\begin{aligned}\int_M \rho\eta^2|\omega|^q &\leq \int_M |\nabla(\eta|\omega|^{\bar{q}})|^2 \\ &\leq (1+\varepsilon)\bar{q}^2\int_M \eta^2|\omega|^{q-2}|\nabla|\omega|\,|^2 + \left(1+\frac{1}{\varepsilon}\right)\int_M |\omega|^q|\nabla\eta|^2.\end{aligned} \tag{2.5}$$

Moreover, using the fundamental inequality $2AB \leq \varepsilon A^2 + \dfrac{B^2}{\varepsilon}$, we infer

$$2\eta|\omega|^{q-1}|d\eta|\cdot|\nabla|\omega|| \leq \varepsilon|\omega|^{q-2}|\nabla|\omega|\,|^2 + \frac{1}{\varepsilon}|\omega|^q|\nabla\eta|^2 \tag{2.6}$$

Combining (2.4), (2.5), (2.6), we conclude that there exist two constants A_ε, B_ε such that

$$A_\varepsilon\int_M |\omega|^q|\nabla\eta|^2 \geq B_\varepsilon\int_M \eta^2|\omega|^{q-2}|\nabla|\omega|\,|^2,$$

where

$$A_\varepsilon = a\left(1+\frac{1}{\varepsilon}\right) + \frac{p-1}{\varepsilon}$$

$$B_\varepsilon = q-1+\kappa+b-a(1+\varepsilon)\bar{q}^2-(p-1)\varepsilon.$$

Suppose that $q = 2\bar{q}$ satisfies

$$q-1+\kappa+b-a\frac{q^2}{4} > 0$$

then we can choose $\varepsilon > 0$ small enough such that $B_\varepsilon > 0$. This implies that there exists a constant $C_\varepsilon > 0$ depending only on ε satisfying

$$\int_M \eta^2|\omega|^{q-2}|\nabla|\omega|\,|^2 \leq C_\varepsilon\int_M |\omega|^q|\nabla\eta|^2 \tag{2.7}$$

provided that

$$a < \frac{4(q-1+\kappa+b)}{q^2}.$$

We now want to show that (2.7) holds true for any $\psi \in \mathscr{C}_0^\infty(M)$. To do this, we will use a version of the Duzaar-Fuchs cut-off technique (see also [6, 13, 19]).

Indeed, we define

$$\varphi_\varepsilon = \min\left\{\frac{|\omega|}{\varepsilon}, 1\right\}$$

for $\varepsilon > 0$, then set $\eta_\varepsilon = \psi^2\varphi_\varepsilon$. Note that, when $\varepsilon \to 0$, $\varphi_\varepsilon \to 1$ pointwisely in M. It is easy to see that η_ε is a compactly supported continuous function and $\eta_\varepsilon = 0$ on $M \backslash M_+$. Since $q \geq 2$ and $|\omega|^q \in L^1(M)$, using an argument of Veronelli (see [19], page 22), we can replace η by η_ε in (2.11) and get

$$\begin{aligned} &\int_M \psi^4(\varphi_\varepsilon)^2|\omega|^{q-2}|\nabla|\omega||^2 \\ &\quad \leq 6C\int_M |\omega|^q|\nabla\psi|^2\psi^2(\varphi_\varepsilon)^2 + 3C\int_M |\omega|^q|\nabla\varphi_\varepsilon|^2\psi^4. \end{aligned} \tag{2.8}$$

Observe that

$$\int_M |\omega|^q|\nabla\eta_\varepsilon|^2\psi^4 \leq \varepsilon^{q-2}\int_M |\nabla|\omega||^2\psi^4\chi_{\{|\omega|\leq\varepsilon\}} \tag{2.9}$$

and the right hand side vanishes by dominated convergence as $\varepsilon \to 0$, because $|\omega| \in \mathscr{C}^\infty(M)$. Now, letting $\varepsilon \to 0$ and applying Fatou Lemma to the integral on the left hand side and dominated convergence to the first integral in the right hand side of (2.8), we obtain

$$\int_M \psi^4|\omega|^{q-2}|\nabla|\omega||^2 \leq 6C\int_M |\omega|^q|\nabla\psi|^2\psi^2, \tag{2.10}$$

where $\psi \in \mathscr{C}_0^\infty(M)$.

Now, choose the test function ψ such that $0 \leq \psi \leq 1$ on M

$$\psi = \begin{cases} 1 & \text{on } B(R) \\ 0 & \text{on } M\backslash B(R) \end{cases} \quad \text{and} \quad |\nabla\psi| \leq \frac{2}{R}.$$

Letting R tends to infinity in (2.10), we conclude that $|\omega|$ is constant. Note that M satisfies a weighted Poincaré inequality, hence M must have infinite volume. Since ω has finite L^q energy, this implies that ω is trivial.

On the other hand, if $1 < p < 2$, since the trivial 1-form is not a p-harmonic form, then such form does not exist. The proof is complete. □

Recall that a p-harmonic function u is said to be strong if $u \in \mathscr{C}^3(M)$. Hence, if u is a strongly p-harmonic function then $\omega = du$ is a $\mathscr{C}^2$ p-harmonic 1-form. By the proof of Theorem 2.3, we obtain the following result.

Corollary 2.4. *Suppose that M^n is a complete noncompact Riemannian manifold satisfying a weighted Poincaré inequality (P_ρ) and the Ricci curvature is bounded by*

$$\mathrm{Ric}_M(x) \geq -a\rho(x)$$

for all $x \in M$. *Let* u *be a strongly* p*-harmonic function on* M *with finite* L^q *energy. If* $q \geq 2$, $q - 1 + \kappa + b > 0$ *and*

$$a < \frac{4(q - 1 + \kappa + b)}{q^2}$$

then

1. *If* $p \geq 2$ *then* u *is constant.*
2. *If* $1 < p < 2$ *then* u *does not exist.*

Remark 2.1. It is worth to notice that the conclusion of Corollary covers a part of Theorem 1.2 in [3]. In fact, to obtain the conclusion as in Corollary, Chang-Chen-Wei had to require an extra assumption on the value of q.

Now, we show that the above vanishing properties still hold true for p-harmonic functions with finite L^p energy without the condition on smoothness of p-harmonic functions. To begin with, let us recall some facts on regularity of p-harmonic functions. It is known that the regularity of (weakly) p-harmonic function u is not better than $\mathscr{C}^{1,\alpha}_{loc}$ (see [11, 18, 20] and the references therein). Moreover it is also known that $u \in W^{2,2}_{loc}$ if $p \geq 2$; $u \in W^{2,p}_{loc}$ if $1 < p < 2$ by Tolksdorf [18]. In fact, any nontrivial (weakly) p-harmonic function u on M is smooth away from the set $\mathscr{S} = \{\nabla u = 0\}$ (see [12, 20] for example). Hence, the Kato inequality (2.3) is valid outside the set $\mathscr{S}$ for $\omega := du$.

Theorem 2.5. *Suppose that* M^n *is a complete noncompact Riemannian manifold satisfying a weighted Poincaré inequality* (P_ρ) *and the Ricci curvature is bounded by*

$$\mathrm{Ric}_M(x) \geq -a\rho(x)$$

for all $x \in M$. *If* $p \geq 2$ *and*

$$a < \frac{4(p - 1 + \kappa)}{p^2}$$

then there is no non trivial weakly p*-harmonic function on* M *with finite* L^p *energy.*

Proof. Let u be a p-harmonic function on M and let $\omega = du$ then the regularity of p-harmonic function implies that u is smooth outside the singular set $\mathscr{S} := \{du = 0\}$. Therefore, ω is a smooth p-harmonic form on $M_+ := M \backslash \mathscr{S}$. Choose a nonnegative smooth function η with compact support on M_+. By (2.7), for $q = p$, there exists a constant $C > 0$ such that

$$\int_{M_+} \eta^2 |\omega|^{p-2} |\nabla|\omega||^2 \leq C \int_{M_+} |\omega|^p |\nabla \eta|^2 \tag{2.11}$$

provided that

$$a < \frac{4(p - 1 + \kappa)}{p^2}.$$

Following the proof of Theorem 2.3, we want to show that (2.11) holds true for every $\psi \in C_0^\infty(M)$ by using a version of the Duzaar-Fuchs cut-off technique (see also [6, 13, 19]). We define

$$\varphi_\varepsilon = \min\left\{\frac{|du|}{\varepsilon}, 1\right\}$$

for $\varepsilon > 0$, then set $\eta_\varepsilon = \psi^2\varphi_\varepsilon$. It is easy to see that η_ε is a compactly supported continuous function and $\eta_\varepsilon = 0$ on $M\backslash M_+$. By regularity of p-harmonic function, we know that $\eta_\varepsilon \in W_0^{1,2}(M_+)$. As $\varepsilon \to 0$, $\varphi_\varepsilon \to 1$ pointwisely in M_+. Using an argument of Veronelli (see [19], page 20), we can replace η by η_ε in (2.11) and get

$$\begin{aligned}(2.12)\qquad &\int_{M_+} \psi^4(\varphi_\varepsilon)^2|du|^{p-2}|\nabla|du||^2 \\ &\leq 6C\int_{M_+} |du|^p|\nabla\psi|^2\psi^2(\varphi_\varepsilon)^2 + 3C\int_{M_+} |du|^p|\nabla\varphi_\varepsilon|^2\psi^4.\end{aligned}$$

Observe that

$$(2.13)\qquad \int_{M_+} |du|^p|\nabla\varphi_\varepsilon|^2\psi^4 \leq \varepsilon^{p-2}\int_{M_+} |\nabla|du||^2\psi^4\chi_{\{|du|\leq\varepsilon\}}$$

and the right hand side vanishes by dominated convergence as $\varepsilon \to 0$, because $|\nabla|du|| \in L^2_{loc}(M)$. Now, letting $\varepsilon \to 0$ and applying Fatou Lemma to the integral on the left hand side and dominated convergence to the first integral in the right hand side of (2.12), we obtain

$$(2.14)\qquad \int_{M_+} \psi^4|du|^{p-2}|\nabla|du||^2 \leq 6C\int_{M_+} |du|^p|\nabla\psi|^2\psi^2,$$

where $\psi \in \mathscr{C}_0^\infty(M)$. Choose a nonnegative smooth function ψ such that

$$\psi = \begin{cases} 1 & \text{on } B(R) \\ 0 & \text{on } M\backslash B(2R)\end{cases}$$

and $|\nabla\psi| \leq \dfrac{2}{R}$. Then the inequality (2.14) implies

$$\int_{M_+} |du|^{p-2}|\nabla|du||^2 \leq \frac{4C}{R^2}\int_{M_+} |du|^p.$$

Letting $R \to \infty$, we see that $|du|$ is constant, since $|du| \in L^p(M)$. Note that $u \in \mathscr{C}^1(M)$, $du = 0$ on ∂M_+. Therefore, if $\partial M_+ \neq \emptyset$ then we have $du = 0$ on M_+. This is a contradiction. So that $M_+ = M$, consequently, du is constant on M. Since M satisfies a weighted Poincaré inequality, M must have infinity volume. Therefore $du = 0$ because $|du| \in L^p(M)$. This implies that u is constant, which completes the proof. $\square$

Remark 2.2. 1. In [3], for $p \ge 2$, Chang et al. proved that every weakly p-harmonic function u with finite L^p-energy is constant when M satisfies a weighted Poincaré inequality (P_ρ) with $\mathrm{Ric} \ge -a\rho$, where

$$a < \frac{4(p-1+\kappa_C)}{p^2}$$

and

$$\kappa_C = \max\left\{\frac{1}{m-1}, \min\left\{\frac{(p-1)^2}{m}, 1\right\}\right\}.$$

It is easy to see that $\kappa_C \le \kappa$. Therefore, the range of values of a we obtained are better than those in [3].

2. Recently, using a different method, Seo and the second author prove in [5] that if $p, q \ge 2$ and

$$a < \frac{4(q-1+\kappa)}{q^2}$$

then there is no non trivial weakly p-harmonic function on M with finite L^q energy. This improves Chang-Chen-Wei's results in case that $p \ge 2$.

3. Applications

Let us recall the following curvature estimate given by Leung ([8]).

Lemma 3.1 ([8]). *Let M be an n-dimensional submanifold immersed in a Riemannian manifold N with sectional curvature K_N satisfying that $K_N \ge -c$, where $c \ge 0$. Then the Ricci curvature Ric_M of M satisfies*

$$\mathrm{Ric}_M \ge -(n-1)c + \frac{1}{n^2}\{2(n-1)|H|^2 - (n-2)\sqrt{n-1}|H|\sqrt{n|A|^2 - |H|^2}\} - \frac{n-1}{n}|A|^2.$$

To show a geometric application of Theorem 2.5, let us recall the following result about the existence of p-harmonic function on a Riemannian manifold.

Theorem 3.2 ([3, 16]). *Let M be a Riemannian manifold with at least two p-nonparabolic ends. Then, there exists a non-constant, bounded p-harmonic function $u \in C^{1,\alpha}(M)$ for some $\alpha > 0$ such that $|\nabla u| \in L^p(M)$.*

Now, we prove a geometric property of stable minimal hypersurface via p-harmonic function theory.

Theorem 3.3. *Suppose that M is an immersed stable minimal hypersurface in a Riemannian manifold N with sectional curvature K_N bounded from below by $-c$,* *where $c > 0$ is a fixed constant. If M has positive spectrum λ_1 then M has at most one p-non parabolic end provided that*

$$\lambda_1(M) > \frac{2(n-1)c}{\dfrac{4(p-1+\kappa)}{p^2} - \dfrac{n-1}{n}}$$

for any $2 \le p < \dfrac{2(n+\sqrt{2n})}{n-1}$.

Proof. Suppose that M has at least two p-nonparabolic end then there exists a nontrivial p-harmonic function u on M with finite L^p energy. Assume that u is smooth and let $\omega = du$, so ω is a smooth p-harmonic 1-form. Since M is minimal, Lemma 3.1 implies

$$\mathrm{Ric}_M \ge -(n-1)c - \frac{n-1}{n}|A|^2.$$

Therefore, by Bochner formula, we have

$$\begin{aligned}\frac{1}{2}\Delta(|\omega|^2) &= \langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + \mathrm{Ric}_M(\omega,\omega)\\ &\ge \langle \Delta\omega, \omega\rangle + (1+\kappa)|\nabla|\omega||^2 - \left((n-1)c + \frac{n-1}{n}|A|^2\right)|\omega|^2.\end{aligned}$$

For any $\varphi \in \mathscr{C}_0^\infty(M)$, the stability of M means

$$\int_M (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\varphi \le \int_M |\nabla\varphi|^2$$

where ν donetes the unit normal vector of M, and $\overline{\mathrm{Ric}}(\nu,\nu)$ is the Ricci curvature of N in the direction ν. Hence,

$$\int_M (|A|^2 - nc)\varphi^2 \le \int_M |\nabla\varphi|^2.$$

On the other hand, the variational principle of $\lambda_1(M)$ gives

$$\int_M \varphi^2 \le \frac{1}{\lambda_1(M)}\int_M |\nabla\varphi|^2.$$

This implies that M satisfies a weighted Poincaré inequality (P_ρ) with the weighted function

$$\rho = \frac{\dfrac{n-1}{n}(|A|^2 - nc) + 2(n-1)c}{\dfrac{n-1}{n} + \dfrac{2(n-1)c}{\lambda_1(M)}} = \frac{\dfrac{n-1}{n}|A|^2 + (n-1)c}{\dfrac{n-1}{n} + \dfrac{2(n-1)c}{\lambda_1(M)}}$$

and

$$\begin{aligned}\mathrm{Ric}_M &\geq -(n-1)c - \frac{n-1}{n}|A|^2 \\ &= -a\rho\end{aligned}$$

where

$$a = \frac{n-1}{n} + \frac{2(n-1)c}{\lambda_1(M)}.$$

Now, we require that

$$a < \frac{4\left(p - \dfrac{n-2}{n-1}\right)}{p^2} \tag{3.1}$$

then

$$a < \frac{4(p-1+\kappa)}{p^2}.$$

Consequently, every smooth p-harmonic function u with finite L^p energy must be constant provided that

$$\frac{2(n-1)c}{\lambda_1(M)} < \frac{4(p-1+\kappa)}{p^2} - \frac{n-1}{n}$$

or equivalently,

$$\lambda_1(M) > \frac{2(n-1)c}{\dfrac{4(p-1+\kappa)}{p^2} - \dfrac{n-1}{n}}.$$

Note that the condition $2 \leq p < \dfrac{2(n+\sqrt{2n})}{n-1}$ implies

$$\frac{4(p-1+\kappa)}{p^2} - \frac{n-1}{n} \geq \frac{4\left(p - \dfrac{n-2}{n-1}\right)}{p^2} - \frac{n-1}{n} > 0.$$

In conclusion, we have shown that u is trivial. This gives a contradiction.

Now if u is not smooth, by using a version of the Duzaar-Fuchs cut-off trick and repeating the proof of Lemma 2.3, we can show that u is constant if

$$\lambda_1(M) > \frac{2(n-1)c}{\dfrac{4(p-1+\kappa)}{p^2} - \dfrac{n-1}{n}}.$$

The proof is complete. □

If $c = 0$, we can remove the condition on the spectrum $\lambda_1(M)$ to obtain the following theorem.

THEOREM 3.4. *Suppose that M is an immersed stable minimal hypersurface in a Riemannian manifold N with non negative sectional curvature K_N. If*

$$2 \leq p < \frac{2(n + \sqrt{2n})}{n - 1}$$

then M has at most one p-non parabolic end.

Since the proof of Theorem 3.4 is similar to the proof of Theorem 3.3, we omit the details.

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