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LINEAR WEINGARTEN SUBMANIFOLDS IMMERSED IN A SPACE FORM

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Abstract

In this paper, we deal with complete linear Weingarten submanifolds M^n immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}_c^{n+p}$ of constant sectional curvature c. Under an appropriated restriction on the norm of the traceless part of the second fundamental form, we show that such a submanifold M^n must be either totally umbilical or isometric to a Clifford torus, if $c = 1$, a circular cylinder, if $c = 0$, or a hyperbolic cylinder, if $c = -1$. We point out that our results are natural generalizations of those ones obtained in [2] and [6].

1. Introduction

The problem of characterizing hypersurfaces immersed with constant mean curvature or with constant scalar curvature in a Riemannian space form $\mathbf{Q}_c^{n+1}$ of constant sectional curvature c constitutes an important thematic into the theory of isometric immersions. In the seminal work [5], Cheng and Yau introduced a new self-adjoint differential operator, the so-called square operator, acting on smooth functions defined on Riemannian manifolds and used it to classify closed hypersurfaces with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in $\mathbf{Q}_c^{n+1}$. Later on, Li [9] extended the results due to Cheng and Yau [5] in terms of the squared norm of the second fundamental form of the hypersurface M^n.

In [10], Li studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed a unit sphere with scalar curvature proportional to the mean curvature. Next, Li et al. [11] extended the result of [5] and [10] by considering closed *linear Weingarten* hypersurfaces immersed in the unit sphere $\mathbf{S}^{n+1}$, that is, closed hypersurfaces of $\mathbf{S}^{n+1}$ whose mean curvature and normalized

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scalar curvature are linearly related. In this setting, they showed that if M^n is a closed linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbf{S}^{n+1}$, then M^n is either totally umbilical or isometric to a Clifford torus.

Afterwards, Guo and Li [6] studied submanifolds in the unit sphere $\mathbf{S}^{n+p}$ with constant scalar curvature and parallel normalized mean curvature vector field (that is, the normalized mean curvature vector field is parallel as a section of the normal bundle) and, in this setting, they generalized the results of [9]. Meanwhile, the first and fourth authors jointly with Aquino [2], working with a suitable Cheng-Yau's modified operator, extended the results of [11] to the context of complete linear Weingarten hypersurfaces immersed in a Riemannian space form $\mathbf{Q}_c^{n+1}$. More precisely, under the assumption that the mean curvature attains its maximum and supposing an appropriated restriction on the norm of the traceless part of the second fundamental form, they proved that such a hypersurface must be either totally umbilical or isometric to a Clifford torus, if $c = 1$, a circular cylinder, if $c = 0$, or a hyperbolic cylinder, if $c = -1$.

Here, we deal with complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}_c^{n+p}$. In this setting, we extend the technique developed in [2] in order to characterize such submanifolds under an appropriated restriction on the norm of the traceless part of the second fundamental form, obtaining natural generalizations of the main results of [2] and [6].

This manuscript is organized in the following way: In Section 2 we recall some basic facts concerning submanifolds immersed in a Riemannian space form $\mathbf{Q}_c^{n+p}$. Next, in Section 3 we develop a Simon's type formula for submanifolds with parallel normalized mean curvature vector field in $\mathbf{Q}_c^{n+p}$ (cf. Proposition 3.1). Afterwards, in Section 4 we present some auxiliary lemmas. Finally, in Section 5 we establish our characterization theorems concerning complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector field in $\mathbf{Q}_c^{n+p}$ (cf. Theorems 5.1 and 5.2).

2. Preliminaries

Let M^n be an n-dimensional connected submanifold immersed in a Riemannian space form $\mathbf{Q}_c^{n+p}$, with constant sectional curvature c. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $\mathbf{Q}_c^{n+p}$. We choose a local field of orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbf{Q}_c^{n+p}$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n. We will use the following convection for indices

$$1 \le A, B, C, \dots \le n+p, \quad 1 \le i, j, k, \dots \le n \quad \text{and} \quad n+1 \le \alpha, \beta, \gamma, \dots n+p.$$

With restricting on M^n, the second fundamental form, the curvature tensor of M^n and the normal curvature tensor of M^n are given by

$$\omega_{i\alpha} = \sum_j h^\alpha_{ij}\omega_j, \quad A = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i \otimes \omega_j e_\alpha,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{k,l} R^\perp_{\alpha\beta kl}\omega_k \wedge \omega_l.$$

The components h^α_{ijk} of the covariant derivative ∇A satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ki}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.1}$$

The Gauss equation is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.2}$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk} \tag{2.3}$$

and

$$R = \frac{1}{(n-1)}\sum_i R_{ii}. \tag{2.4}$$

From (2.3) and (2.4), we get the following relation

$$n(n-1)R = n(n-1)c + n^2H^2 - |A|^2, \tag{2.5}$$

where $|A|^2 = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the norm square of the second fundamental form and, being $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n}\sum_\alpha\left(\sum_k h^\alpha_{kk}\right)e_\alpha$ the mean curvature vector field, $H = |\mathbf{H}|$ is the mean curvature function of M^n.

By exterior differentiation of (2.1), we have the following Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R^\perp_{\beta\alpha kl}. \tag{2.6}$$

The Codazzi equation and the Ricci equation are given by

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik} \tag{2.7}$$

and

$$R^{\perp}_{\alpha\beta ij} = \sum_k (h^{\alpha}_{ik} h^{\beta}_{kj} - h^{\alpha}_{jk} h^{\beta}_{ki}). \tag{2.8}$$

3. A Simon's type formula

From now on, we will deal with submanifolds M^n of $\mathbf{Q}^{n+p}_c$ having *parallel normalized mean curvature vector field*, which means that the mean curvature function H is positive and that the corresponding normalized mean curvature vector field $\dfrac{\mathbf{H}}{H}$ is parallel as a section of the normal bundle.

In this context, we can choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ such that $e_{n+1} = \dfrac{\mathbf{H}}{H}$. Thus,

$$H^{n+1} = \frac{1}{n}\,\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\,\mathrm{tr}(h^{\alpha}) = 0, \quad \alpha \geq n+2. \tag{3.1}$$

We will also consider the following symmetric tensor

$$\Phi = \sum_{\alpha, i, j} \Phi^{\alpha}_{ij} \omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \quad n+2 \leq \alpha \leq n+p. \tag{3.2}$$

Let $|\Phi|^2 = \sum_{\alpha,i,j} (\Phi^{\alpha}_{ij})^2$ be the square of the length of Φ. It is easy to check that Φ is traceless with

$$|\Phi|^2 = |A|^2 - nH^2. \tag{3.3}$$

Now, we are in position to show the following suitable Simon-type formula

PROPOSITION 3.1. *Let M^n be an n-dimensional $(n \geq 2)$ submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}^{n+p}_c$. Then, we have*

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j,\alpha} nH^{\alpha}_{ij} h^{\alpha}_{ij} + cn|\Phi|^2 + n \sum_{\beta,i,j,k} H h^{n+1}_{ij} h^{\beta}_{jk} h^{\beta}_{ki}$$
$$- \sum_{i,j,k,l} \left(\sum_{\alpha} h^{\alpha}_{ij} h^{\alpha}_{kl} \right)^2 - \sum_{i,j,\alpha,\beta} (R^{\perp}_{\alpha\beta ij})^2.$$

Proof. Taking into account that

$$\frac{1}{2}\Delta|A|^2 = \sum_{\alpha,i,j} h^{\alpha}_{ij} \Delta h^{\alpha}_{ij} + \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2, \tag{3.4}$$

where the Laplacian Δh_{ij}^{α} of h_{ij}^{α} is defined by $\Delta h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha}$, using Codazzi equation (2.7) into (3.4) we have

$$\frac{1}{2}\Delta|A|^2 = \sum_{\alpha,i,j} h_{ij}^{\alpha}\left(\sum_k h_{ijkk}^{\alpha}\right) + \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 = |\nabla A|^2 + \sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{kijk}^{\alpha}. \tag{3.5}$$

Thus, from (2.6) and (3.5) we conclude that

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 &+ \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j - \sum_{\alpha,i,j,k} h_{ijj}^{\alpha}h_{kki}^{\alpha} + \sum_{\alpha,i,j,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mj} \\ &+ \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\beta,\alpha,i,j,k} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}^{\perp}.\end{aligned} \tag{3.6}$$

Hence, observing that $\sum_{\alpha,i,j,k} h_{ijj}^{\alpha}h_{kki}^{\alpha} = n^2|\nabla^{\perp}\mathbf{H}|^2$, where $|\nabla^{\perp}\mathbf{H}|^2 = \sum_{\alpha,i}(H_i^{\alpha})^2$, from (3.6) we get

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - n^2|\nabla^{\perp}\mathbf{H}|^2 &+ \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk} \\ &+ \sum_{\alpha,i,j,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mj} + \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}^{\perp}.\end{aligned} \tag{3.7}$$

But, using once more Codazzi equation (2.7) we obtain that

$$-n^2|\nabla^{\perp}\mathbf{H}|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j = \sum_{i,j,\alpha} nH_{ij}^{\alpha}h_{ij}^{\alpha}. \tag{3.8}$$

From (2.2) and (2.3) we also conclude that

$$\begin{aligned}&\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mj} + \sum_{\beta,\alpha,i,j,k} h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}^{\perp} \\ &\quad = c|\Phi|^2 - \sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha}h_{ij}^{\beta}h_{mk}^{\alpha}h_{mk}^{\beta} + n\sum_{\alpha,\beta,i,j,m} H^{\beta}h_{mj}^{\beta}h_{ij}^{\alpha}h_{im}^{\alpha} \\ &\quad - \sum_{\alpha,\beta,i,j,m,l} h_{ij}^{\alpha}h_{im}^{\alpha}h_{ml}^{\beta}h_{ij}^{\beta} + \sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}h_{jm}^{\beta}h_{ik}^{\beta} + \sum_{\alpha,\beta,i,j,k} h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}^{\perp}.\end{aligned} \tag{3.9}$$

On the other hand, from (3.1) we get

$$\sum_{\alpha,\beta,i,j,m} H^{\beta}h_{mj}^{\beta}h_{ij}^{\alpha}h_{im}^{\alpha} = \sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}, \tag{3.10}$$

and

$$\sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha}h_{ij}^{\beta}h_{mk}^{\alpha}h_{mk}^{\beta} = \sum_{i,j,k,l}\left(\sum_{\alpha,\beta} h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}\right) = \sum_{i,j,k,l}\left(\sum_{\alpha} h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^2. \tag{3.11}$$

Furthermore, using (2.8) we have

$$
\begin{aligned}
\sum_{\alpha,\beta,j,k}(R^{\perp}_{\alpha\beta jk})^2 &= \sum_{\alpha,\beta,j,k}\left[\sum_i (H^{\beta}_{ji}h^{\alpha}_{ik} - h^{\alpha}_{ji}h^{\beta}_{ik})\right]R^{\perp}_{\beta\alpha jk} \\
&= \sum_{\alpha,\beta,i,j,k} h^{\beta}_{ji}h^{\alpha}_{ik}R^{\perp}_{\beta\alpha jk} - \sum_{\alpha,\beta,i,j,k} h^{\alpha}_{ji}h^{\beta}_{ik}R^{\perp}_{\beta\alpha jk} \\
&= \sum_{\alpha,\beta,i,j,m,l} h^{\alpha}_{ij}h^{\alpha}_{im}h^{\beta}_{ml}h^{\beta}_{lj} - \sum_{\alpha,\beta,i,j,k,m} h^{\alpha}_{ij}h^{\alpha}_{km}h^{\beta}_{jm}h^{\beta}_{ik} \\
&\quad - \sum_{\alpha,\beta,i,j,k} h^{\alpha}_{ji}h^{\beta}_{ik}R^{\perp}_{\beta\alpha jk}.
\end{aligned} \tag{3.12}
$$

Therefore, considering (3.8), (3.9), (3.10), (3.11) and (3.12) in (3.7), we conclude the proof. □

4. Key lemmas

In order to establish our main results, we devote this section to present some auxiliary lemmas. The first one can be proven reasoning in a similar way of that in Proposition 2.2 of [14] (see also Lemma 3.2 of [2]).

Lemma 4.1. *Let M^n be a linear Weingarten submanifold immersed in a Riemannian space form $\mathbf{Q}^{n+p}_c$, with $R = aH + b$ for some $a, b \in \mathbf{R}$. Suppose that*

$$(n-1)a^2 + 4n(b-c) \geq 0. \tag{4.1}$$

Then

$$|\nabla A|^2 \geq n^2|\nabla H|^2. \tag{4.2}$$

Moreover, the equality holds in (4.2) *on M^n if, and only if, M^n is an isoparametric submanifold of $\mathbf{Q}^{n+p}_c$.*

We will also need the following algebraic lemma, whose proof can be found in [12].

Lemma 4.2. *Let $B, C : \mathbf{R}^n \to \mathbf{R}^n$ be symmetric linear maps that $BC - CB = 0$ and* $\operatorname{tr} B = \operatorname{tr} C = 0$, *then*

$$-\frac{n-2}{\sqrt{n(n-1)}}|B|^2|C| \leq \operatorname{tr}(B^2C) \leq \frac{n-2}{\sqrt{n(n-1)}}|B|^2|C|. \tag{4.3}$$

Consider the following algebraic lemma, whose proof can be found in [8].

Lemma 4.3. *Let $B^1, B^2, \ldots, B^n$ be symmetric $(n \times n)$-matrices. Set $S_{\alpha\beta} =$* $\operatorname{tr}(B^{\alpha}B^{\beta})$, *$S_{\alpha} = S_{\alpha\alpha}$, $S = \sum_{\alpha} S_{\alpha}$, then*

$$\sum_{\alpha,\beta}|B^{\alpha}B^{\beta} - B^{\beta}B^{\alpha}|^2 + \sum_{\alpha,\beta} S^2_{\alpha\beta} \leq \frac{3}{2}\left(\sum_{\alpha} S_{\alpha}\right)^2. \tag{4.4}$$

Now, let $\phi = \sum_{i,j} \phi_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}^{n+1}.$$

Following Cheng-Yau [5], we introduce a operator $\square$ associated to ϕ acting on any smooth function f by

$$\square f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij}. \tag{4.5}$$

Here, in order to study linear Weingarten submanifolds, we will consider, for each $a \in \mathbf{R}$, an appropriated Cheng-Yau's modified operator, which is given by

$$L = \square - \frac{n-1}{2} a\Delta. \tag{4.6}$$

By taking a local orthonormal frame field $e_1, \dots, e_n$ at $q \in \Sigma^n$ such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$, we can show the following sufficient criterion of ellipticity in a similar way of Lemma 3.3 in [2].

Lemma 4.4. *Let M^n be a linear Weingarten submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}_c^{n+p}$, such that $R = aH + b$ with $b > c$. Then, L is elliptic.*

We will also use the following result obtained by Caminha [3], which extends a previous one due to Yau in [16] (cf. Proposition 2.1 of [3]). In what follows, let $\mathscr{L}^1(M^n)$ denote the space of Lebesgue integrable functions on M^n.

Lemma 4.5. *Let X be a smooth vector field on an n-dimensional complete oriented Riemannian manifold M^n such that* div X *does not change sign on M^n. If $|X| \in \mathscr{L}^1(M^n)$, then* div $X = 0$.

5. Main results

In this section we present our characterization results concerning complete linear Weingarten submanifolds immersed in a space form. The first one will be obtained applying the Hopf's strong maximum principle and it is a natural extension of Theorem 1.1 of [2].

Theorem 5.1. *Let M^n be a complete linear Weingarten submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}_c^{n+p}$ ($c = 1, 0, -1$ and $n \geq 4$), such that $R = aH + b$ with $a \geq 0$ and $b > c$. In the case that $c = -1$, assume in addition that $R > 0$. If H attains its maximum on M^n and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR - (n-2)c)}, \tag{5.1}$$

then

i. *either* $|\Phi| \equiv 0$ *and* M^n *is totally umbilical,*

ii. *or* $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$ *and* M^n *is isometric to a*

   (a) *Clifford torus* $\mathbf{S}^1(\sqrt{1-r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{S}^{n+1} \hookrightarrow \mathbf{S}^{n+p}$, *when* $c = 1$,

   (b) *circular cylinder* $\mathbf{R} \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{R}^{n+1} \hookrightarrow \mathbf{R}^{n+p}$, *when* $c = 0$,

   (c) *hyperbolic cylinder* $\mathbf{H}^1(-\sqrt{1+r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{H}^{n+1} \hookrightarrow \mathbf{H}^{n+p}$, *when* $c = -1$,

   where r *is constant and equal to* $\sqrt{\dfrac{n-2}{nR}}$.

Proof. From (2.5), (4.5), (4.6) and Proposition 3.1 that

$$
\begin{aligned}
(5.2)\qquad L(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} - \frac{n-1}{2}a\Delta(nH)\\
&= \frac{1}{2}n^2\Delta H^2 - n^2|\nabla H|^2 - \frac{n-1}{2}a\Delta(nH) - n\sum_{i,j}h_{ij}^{n+1}H_{ij}\\
&= (|\nabla A|^2 - n^2|\nabla H|^2) + cn|\Phi|^2 + n\sum_{\beta,i,j,k}Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}\\
&\quad - \sum_{i,j,k,l}\left(\sum_{\alpha}h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^{\perp})^2.
\end{aligned}
$$

From (3.1) and (3.2) we have

$$
\begin{aligned}
(5.3)\qquad &\sum_{i,j,k,\beta}Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}\\
&= \sum_{i,j,k}Hh_{ij}^{n+1}h_{jk}^{n+1}h_{ki}^{n+1} + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k}Hh_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta}\\
&= H\,\mathrm{tr}(\Phi^{n+1}+HI)^3 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k}H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta} + \sum_{\beta=n+2}^{n+p}H^2|\Phi^{\beta}|^2\\
&= H\,\mathrm{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p}H^2|\Phi^{\beta}|^2\\
&\quad + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k}H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta}.
\end{aligned}
$$

Noticing that $\mathrm{tr}\,\Phi^{\alpha} = 0$ and $\Phi^{n+1}\Phi^{\beta} - \Phi^{\beta}\Phi^{n+1} = 0$, $n+2 \le \beta \le n+p$, from Lemma 4.2 we obtain

$$(5.4)\quad H\ \mathrm{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^\beta|^2 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi_{ij}^{n+1}\Phi_{jk}^\beta\Phi_{ki}^\beta$$

$$\geq -\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|^3 + 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4$$

$$-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\beta=n+2}^{n+p} H|\Phi^{n+1}|\,|\Phi^\beta|^2$$

$$= 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|\,|\Phi|^2.$$

Hence, from (5.3) and (5.4) we have

$$(5.5)\quad \sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^\beta h_{ki}^\beta \geq 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|\,|\Phi|^2.$$

From Ricci equation (2.3) we get

$$(5.6)\quad \sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 + \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2$$

$$= \sum_{\alpha,\beta}(\mathrm{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha\neq n+1,\beta\neq n+1,i,j}(R_{\alpha\beta ij}^\perp)^2$$

$$= [\mathrm{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(A^{n+1}A^\beta)]^2$$

$$+ \sum_{\alpha\neq n+1,\beta\neq n+1}(\mathrm{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha\neq n+1,\beta\neq n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2.$$

But, using (3.2) and Lemma 4.3 we obtain

$$(5.7)\quad \sum_{\alpha\neq n+1,\beta\neq n+1}[\mathrm{tr}(A^\alpha A^\beta)]^2 + \sum_{\alpha\neq n+1,\beta\neq n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2$$

$$\leq \frac{3}{2}\left(\sum_{\beta\neq n+1}\mathrm{tr}(A^\beta A^\alpha)\right)^2 \leq \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^\beta|\right)^2.$$

Hence, from (5.6) and (5.7) we have

$$(5.8)\quad \sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 + \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2$$

$$\leq [\mathrm{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(A^{n+1}A^\beta)]^2 + \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^\beta|^2\right)^2$$

$$= |\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(\Phi^{n+1}\Phi^{\beta})]^2$$
$$+\frac{3}{2}(|\Phi|^2 - |\Phi^{n+1}|^2)^2$$
$$\leq \frac{5}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2|\Phi^{n+1}|^2(|\Phi|^2 - |\Phi^{n+1}|^2)$$
$$+\frac{3}{2}|\Phi|^4 - 3|\Phi|^2|\Phi^{n+1}|^2$$
$$=\frac{1}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 - |\Phi|^2|\Phi^{n+1}|^2 + \frac{3}{2}|\Phi|^4.$$

Therefore, from (5.2), (5.5) and (5.8) we get

$$L(nH) \geq cn|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi^{n+1}|\,|\Phi|^2 + nH^2|\Phi|^2 \tag{5.9}$$
$$-\frac{1}{2}|\Phi^{n+1}|^4 + |\Phi|^2|\Phi^{n+1}|^2 - \frac{3}{2}|\Phi|^4$$
$$= |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(H^2+c)\right)$$
$$+ (|\Phi| - |\Phi^{n+1}|)$$
$$\times\left(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2\right).$$

On the other hand, from (2.5) and (3.3) we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R-c). \tag{5.10}$$

Thus, from (5.9) and (5.10) we get

$$L(nH) \geq (|\Phi| - |\Phi^{n+1}|)\left[\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2\right. \tag{5.11}$$
$$\left.-\frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2\right]$$
$$+\frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|),$$

where $Q_R(x)$ is the function introduced by Alías, García-Martínez and Rigoli in [1] and which is given by

$$Q_R(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(R-c)} + n(n-1)R.$$

On the other hand, we note that holds the following algebraic inequality (3.5) of [6]

$$(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \leq \frac{32}{27}|\Phi|^3. \tag{5.12}$$

Moreover, since that $a \geq 0$ and $b > c$ and using (2.5), we also have

$$n^2H^2 \geq n^2H^2 - n(n-1)aH = |A|^2 + n(n-1)(b-c) \geq |A|^2 = |\Phi|^2 + nH^2,$$

which give us

$$H \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|. \tag{5.13}$$

Thus, from (5.12) and (5.13) we conclude that

$$\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \geq \left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3. \tag{5.14}$$

But, taking into account our assumption that $n \geq 4$, we have that

$$\frac{n-2}{n-1} - \frac{16}{27} > 0. \tag{5.15}$$

Consequently, from (5.11), (5.14) and (5.15) we get that

$$\begin{aligned} L(nH) &\geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) + (|\Phi| - |\Phi^{n+1}|)\left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3 \\ &\geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) \end{aligned} \tag{5.16}$$

We will observe that, from ours hypothesis on a, b and M^n be linear Weingarten, we get

$$R = aH + b \geq b > c. \tag{5.17}$$

Hence, when $c = 0, 1$, from (5.17) we must have $R > 0$. Thus, $Q_R(0) = n(n-1)R > 0$ and the function $Q_R(x)$ is strictly decreasing for $x \geq 0$, with $Q_R(x^*) = 0$ at

$$x^* = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}} > 0,$$

once

$$nR - (n-2)c = naH + n(b-c) + 2c > 2c$$

and, consequently, if $c = 0, 1$, we get $nR - (n-2)c > 0$. Moreover, since we are assuming that $R > 0$ when $c = -1$, we also have that $nR + (n-2) > 0$.

Thus, from our restriction (5.1) on the norm of Φ, we obtain that $Q_R(|\Phi|) \geq 0$. Hence, from (5.16)

$$L(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) \geq 0.$$

Since Lemma 4.4 guarantees that L is elliptic and as we are supposing that H attains its maximum on M^n, from (5.16) we conclude that H is constant on M^n. Thus, returning to (5.2), we get that equality holds in (4.2). So, Lemma 4.1 guarantees that M^n is a isoparametric submanifold of $\mathbf{Q}_c^{n+p}$ and, in particular, $|\Phi|$ is constant. Now, suppose that M^n is not totally umbilical, which means that $|\Phi|$ a positive constant. In this case, taking into account (5.15), from (5.16) we conclude that $|\Phi| = |\Phi^{n+1}|$ and, consequently, $\Phi^\alpha = 0$, for all $n+2 \leq \alpha \leq n+p$. Thus, since e_{n+1} is parallel in the normal bundle of M^n, we are in position to apply Theorem 1 of [15] to conclude that M^n is, in fact, isometrically immersed in a $(n+1)$-dimensional totally geodesic submanifold $\mathbf{Q}_c^{n+1}$ of $\mathbf{Q}_c^{n+p}$. Hence, by the classical results on isoparametric hypersurfaces of real space forms [4, 7, 13] and taking into account that $R > 0$, we conclude that either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$ and M^n is isometric to a

(a) Clifford torus $\mathbf{S}^1(\sqrt{1-r^2}) \times \mathbf{S}^{n-1}(r)$, whit $0 < r < 1$, if $c = 1$,

(b) circular cylinder $\mathbf{R} \times \mathbf{S}^{n-1}(r)$, whit $r > 0$, if $c = 0$,

(c) hyperbolic cylinder $\mathbf{H}^1(-\sqrt{1+r^2}) \times \mathbf{S}^{n-1}(r)$, whit $r > 0$, if $c = -1$.

When $c = 1$, for a given radius $0 < r < 1$, is a standard fact that the product embedding $\mathbf{S}^1(\sqrt{1-r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{S}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \frac{r}{\sqrt{1-r^2}}, \quad \lambda_2 = \cdots = \lambda_n = -\frac{\sqrt{1-r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2 - (n-1)}{nr\sqrt{1-r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1-r^2)}.$$

When $c = 0$, for a given radius $r > 0$, $\mathbf{R} \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{R}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = 0, \quad \lambda_2 = \cdots = \lambda_n = \frac{1}{r}.$$

In this case,

$$H = \frac{n-1}{nr} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2}.$$

Finally, when $c = -1$, for a given radius $r > 0$, $\mathbf{H}^1(-\sqrt{1+r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{H}^{n+1}$ has constant principal curvatures given by

$$\lambda_1 = \frac{r}{\sqrt{1+r^2}}, \quad \lambda_2 = \cdots = \lambda_n = \frac{\sqrt{1+r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2 + (n-1)}{nr\sqrt{1+r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1+r^2)}.$$

Therefore, in order to finish our proof, from equations (2.5) and (3.3) and with algebraic computations it is not difficult to verify that in all these previously described situations we must have $r = \sqrt{\frac{n-2}{nR}}$. □

We will close our paper applying Lemma 4.5 in order to get the following

THEOREM 5.2. *Let M^n be a complete linear Weingarten submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbf{Q}_c^{n+p}$ ($c = 1, 0, -1$ and $n \geq 4$), such that $R = aH + b$ with $a \geq 0$ and $b \geq c$. In the case that either $b = c = 0$ or $c = -1$, assume in addition that $R > 0$. If H is bounded on M^n, $|\nabla H| \in \mathscr{L}^1(M^n)$ and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)},$$

then

i. *either $|\Phi| \equiv 0$ and M^n is totally umbilical,*
ii. *or $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$ and M^n is isometric to a*
   (a) *Clifford torus $\mathbf{S}^1(\sqrt{1-r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{S}^{n+1} \hookrightarrow \mathbf{S}^{n+p}$, when $c = 1$,*
   (b) *circular cylinder $\mathbf{R} \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{R}^{n+1} \hookrightarrow \mathbf{R}^{n+p}$, when $c = 0$,*
   (c) *hyperbolic cylinder $\mathbf{H}^1(-\sqrt{1+r^2}) \times \mathbf{S}^{n-1}(r) \hookrightarrow \mathbf{H}^{n+1} \hookrightarrow \mathbf{H}^{n+p}$, when $c = -1$,*

   where r is constant and equal to $\sqrt{\frac{n-2}{nR}}$.

Proof. We observe that from (4.5) and (4.6) it is not difficult to verify that

$$L(nH) = \operatorname{div}(P(\nabla H)), \tag{5.18}$$

where

$$P = \left(n^2 H - \frac{n(n-1)}{2}a\right)I - nh^{n+1}, \tag{5.19}$$

$h^{n+1} = (h_{ij}^{n+1})$ stands for the second fundamental form of M^n in direction e_{n+1} and I the identity in the algebra of smooth vector fields on M^n.

On the other hand, since $R = aH + b$ and H is bounded on M^n, from equation (2.5) we have that A is bounded on M^n. Consequently, from (5.19) we conclude that the operator P is bounded, that is, there exists a positive constant C such that $|P| \le C$. Since we are also assuming that $|\nabla H| \in \mathscr{L}^1(M^n)$, we obtain that

$$|P(\nabla H)| \le |P|\,|\nabla H| \le C|\nabla H| \in \mathscr{L}^1(M^n).$$

Hence, we can apply Lemma 4.5 to obtain that $L(nH) = 0$ on M^n. Thus, since still holds (5.16), we can use once more Lemma 4.1 to get that M^n is an isoparametric submanifold of $\mathbf{Q}_c^{n+p}$. Therefore, at this point we can reason as in the proof of Theorem 5.1 to conclude the result. $\square$

Remark 5.3. Considering in Theorem 5.2 that case that M^n is compact, $a = 0$ and $c = 1$, we just reobtain Theorem 1.3 of [6].

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