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CRITERIA FOR SINGULARITIES FOR MAPPINGS FROM TWO-MANIFOLD TO THE PLANE. THE NUMBER AND SIGNS OF CUSPS

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Abstract

Let $M \subset \mathbf{R}^{n+2}$ be a two-dimensional complete intersection. We show how to check whether a mapping $f : M \to \mathbf{R}^2$ is 1-generic with only folds and cusps as singularities. In this case we give an effective method to count the number of positive and negative cusps of a polynomial f, using the signatures of some quadratic forms.

1. Introduction

In [13], Whitney investigated a smooth mapping between two surfaces. He proved that for a generic mapping the only possible types of singular points are folds and simple cusps. With smooth oriented 2-dimensional manifolds M and N, and a smooth mapping $f : M \to N$ with a simple cusp $p \in M$ one can associate a sign $\mu(p) = \pm 1$ defined as the local topological degree of the germ of f at p.

In [6], the authors studied smooth mappings from the plane to the plane, and they presented methods of checking whether a map is a generic one with only folds and simple cusps as singular points. They also gave the effective formulas to determine the number of positive and negative cusps in therms of signatures of quadratic forms.

Criteria for types of Morin singularities of mappings from $\mathbf{R}^m$ to $\mathbf{R}^n$ (in case $m \neq n$) were presented in [9, 10]. In case $m = n = 2$ Morin singularities are folds and cusps. Some results concerning the algebraic sum of cusps are contained in [2], [8], and in [3] in the complex case.

In this paper we investigate properties of mappings $f = \tilde{f}|_M : M \to \mathbf{R}^2$, where $M = h^{-1}(0)$ is a 2-dimensional complete intersection, $h : \mathbf{R}^{n+2} \to \mathbf{R}^n$, $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}^2$. We give methods for checking whether f is 1-generic (in sense of [4]) and whether a given singular point $p \in M$ of f is a fold point or a simple cusp (Theorem 3.3, Propositions 3.4, 3.5). We define $F : \mathbf{R}^{n+2} \to \mathbf{R}^2$ associated

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with $\tilde{f}$ and h such that for a simple cusp p of f the sign of it $\mu(p) = \operatorname{sgn}\det\begin{bmatrix} DF(p) \\ Dh(p) \end{bmatrix}$ (Theorem 4.2).

In the case where $\tilde{f}$ and h are polynomial mappings, we construct an ideal $S \subset \mathbf{R}[x] = \mathbf{R}[x_1, \ldots, x_{n+2}]$ such that if $S = \mathbf{R}[x]$ then f is 1-generic with only folds and simple cusps as singular points (Proposition 5.1). Then we define an ideal J such that the set of its real zeros $V(J)$ is the set of simple cusps of f. If $S = \mathbf{R}[x]$ and $\dim_{\mathbf{R}} \mathbf{R}[x]/J < \infty$ then the number of simple cusps and the algebraic sum of them can be expressed in terms of signatures of some associated quadratic forms (Proposition 5.2).

In the whole article by smooth we will mean C^∞ class.

2. Preliminaries

Let M, N be smooth manifolds such that $m = \dim M$ and $n = \dim N$. Take $p \in M$. For smooth mappings $f, g : M \to N$ such that $f(p) = g(p) = q$, we say that f has first order contact with g at p if $Df(p) = Dg(p)$, as mappings $T_pM \to T_qN$. Then $J^1(M,N)_{(p,q)}$ denotes the set of equivalence classes of mappings $f : M \to N$, where $f(p) = q$, having the same first order contact at p. Let

$$J^1(M,N) = \bigcup_{(p,q)\in M\times N} J^1(M,N)_{(p,q)}$$

denote the 1-jet bundle of smooth mappings from M to N.

With any smooth $f : M \to N$ we can associate a canonical mapping $j^1f : M \to J^1(M,N)$. Take $\sigma \in J^1(M,N)$, represented by f. Then by corank σ we denote the corank $Df(p)$. Put $S_r = \{\sigma \in J^1(M,N) \mid \operatorname{corank} \sigma = r\}$. According to [4, II, Theorem 5.4], S_r is a submanifold of $J^1(M,N)$, with $\operatorname{codim} S_r = r(|m-n| + r)$. Put $S_r(f) = \{x \in M \mid \operatorname{corank} Df(p) = r\} = (j^1f)^{-1}(S_r)$.

Definition 2.1. We say that $f : M \to N$ is 1-generic if $j^1f \pitchfork S_r$, for all r.

According to [4, II, Theorem 4.4], if $j^1f \pitchfork S_r$ then either $S_r(f) = \emptyset$ or $S_r(f)$ is a submanifold of M, with $\operatorname{codim} S_r(f) = \operatorname{codim} S_r$.

In the remaining we will need the following useful fact.

Lemma 2.2. *Let M, N and P be smooth manifolds, and let $f : M \to N$, $a : P \to M$, $b : P \to N$ be such that $b = f \circ a$. If a is a smooth surjective submersion, b is smooth, then f is also smooth. If in addition b is a submersion, then so is f.*

Let

$$h = (h_1, \ldots, h_n) : \mathbf{R}^{n+k} \to \mathbf{R}^n$$
$$f = (f_1, \ldots, f_l) : \mathbf{R}^{n+k} \to \mathbf{R}^l$$

be C^1 maps, $M := h^{-1}(0)$. Suppose that each point $p \in M$ is a regular point of h, i.e. rank $Dh(p) = n$ in each $p \in M$. Then M is an orientable C^1 k-manifold called a complete intersection. It is easy to verify that for each point $p \in M$

$$\text{rank } Df|_M(p) = \text{rank}\begin{bmatrix} Df(p) \\ Dh(p) \end{bmatrix} - n. \tag{1}$$

Assume that $N = \mathbf{R}^2$ and $M = h^{-1}(0)$, where $h : \mathbf{R}^{n+2} \to \mathbf{R}^n$ is a smooth mapping such that rank $Dh(x) = n$ for all $x \in M$. In that case M is a smooth 2-manifold.

We have $J^1(\mathbf{R}^{n+2}, \mathbf{R}^2) \simeq \mathbf{R}^{n+2} \times \mathbf{R}^2 \times M(2, n+2)$, where $M(2, n+2)$ is the space of real $2 \times (n+2)$-matrices.

Let us define

$$G = \{\sigma = (x, y, A) \in J^1(\mathbf{R}^{n+2}, \mathbf{R}^2) \,|\, x \in M\} = \bigcup_{(p,q) \in M \times \mathbf{R}^2} J^1(\mathbf{R}^{n+2}, \mathbf{R}^2)_{(p,q)}.$$

Then G is a submanifold of $J^1(\mathbf{R}^{n+2}, \mathbf{R}^2)$, and $\dim G = 2n + 8$.

We define a relation $\sim$ in $G : (x_1, y_1, A_1) = \sigma_1 \sim \sigma_2 = (x_2, y_2, A_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$, and $A_1|_{T_{x_1}M} = A_2|_{T_{x_1}M}$ considered as linear mappings on $T_{x_1}M \subset T_{x_1}\mathbf{R}^{n+2}$.

Proposition 2.3. *$G/_\sim$ is a smooth manifold diffeomorphic to $J^1(M, \mathbf{R}^2)$ such that the projection $pr : G \to G/_\sim$ is a submersion.*

Proof. Using [11, Part II, Chap. III, Sec. 12, Th. 1 and Th. 2], to verify that $G/_\sim$ is a smooth manifold such that the projection $pr : G \to G/_\sim$ is a submersion, it is enough to show that

a) the set $R = \{(\sigma_1, \sigma_2) \in G \times G \,|\, \sigma_1 \sim \sigma_2\}$ is a submanifold of $G \times G$,

b) the projection $\pi : R \to G$ is a submersion.

Take $x \in M$, then in a neighbourhood of x in $\mathbf{R}^{n+2}$ there exists a smooth non-vanishing vector field $(v_1, v_2) \in \mathbf{R}^{n+2} \times \mathbf{R}^{n+2}$ such that

$$\text{Span}\{v_1, v_2\} = (\text{Span}\{\nabla h_1, \ldots, \nabla h_n\})^\perp$$

at every point of this neighbourhood. Then at points of M vectors v_1, v_2 span the tangent space to M.

Let us define $\gamma : J^1(\mathbf{R}^{n+2}, \mathbf{R}^2) \times J^1(\mathbf{R}^{n+2}, \mathbf{R}^2) \to \mathbf{R}^{2n+8}$ by

$$\begin{aligned} \gamma(\sigma_1, \sigma_2) &= \gamma((x_1, y_1, A_1), (x_2, y_2, A_2)) \\ &= (x_1 - x_2, y_1 - y_2, A_1 v_1(x_1) - A_2 v_1(x_1), A_1 v_2(x_1) - A_2 v_2(x_1), h(x_1)). \end{aligned}$$

Hence $\gamma(\sigma_1, \sigma_2) = 0$ if and only if $(\sigma_1, \sigma_2) \in R$. Then locally $\gamma^{-1}(0) = R$. Moreover γ is a submersion at points from R, so R is a submanifold of $G \times G$, and a) is proven.

Using equation (1) it is easy to see that rank $D\pi = 2n + 8 = \dim G$, so π is a submersion and we have b).

Now we will prove that $G/_\sim$ is diffeomorphic to $J^1(M, \mathbf{R}^2)$. Since M is a submanifold of $\mathbf{R}^{n+2}$, there exists a tubular neighbourhood U of M in $\mathbf{R}^{n+2}$ with a smooth retraction $r : U \to M$, which is also a submersion.

Let us define $\Psi : J^1(M, \mathbf{R}^2) \to G/_\sim$ by

$$\Psi(\sigma) = \Psi([g]) = [g \circ r] \in G/_\sim.$$

Note that Ψ is a well-defined bijection and Ψ^{-1} is given by $G/_\sim \ni [g] \mapsto [g|_M] \in J^1(M, \mathbf{R}^2)$. The mapping $\Psi^{-1} \circ pr : G \to J^1(M, \mathbf{R}^2)$ can be given by $G \ni [g] \mapsto [g|_M] \in J^1(M, \mathbf{R}^2)$ and we see that it is a smooth submersion. So according to Lemma 2.2, Ψ^{-1} is also a smooth submersion. Since Ψ^{-1} is bijective, it is a diffeomorphism. □

3. Checking 1-genericity and recognizing folds and cusps

Let $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}^2$ be smooth and put $f = \tilde{f}|_M : M \to \mathbf{R}^2$, where $M = h^{-1}(0)$ is a 2-dimensional complete intersection. Using mappings h and $\tilde{f}$ defined on $\mathbf{R}^{n+2}$, we will present an effective method to check whether f is 1-generic.

Put $\Phi : G/_\sim \to \mathbf{R}$ as

$$\Phi([(x, y, A)]) = \det \begin{bmatrix} A \\ Dh(x) \end{bmatrix}.$$

Notice that if $[(x, y, A)] \in G/_\sim$ is represented by g defined near $x \in \mathbf{R}^{n+2}$, then $\Phi([g]) = \det \begin{bmatrix} Dg(x) \\ Dh(x) \end{bmatrix}$.

Lemma 3.1. *Φ is well-defined.*

Proof. Take (x, y, A_1) and (x, y, A_2) representing the same element in $G/_\sim$. Then $A_1 v_1 = A_2 v_1$ and $A_1 v_2 = A_2 v_2$, where $v_1, v_2 \in \mathbf{R}^{n+2}$ span $T_x M$, and so they both are orthogonal to all vectors $\nabla h_i(x)$.

Hence we have

$$\begin{aligned}
\det\left(\begin{bmatrix} A_1 \\ Dh(x) \end{bmatrix} [\, v_1 \quad v_2 \quad \nabla h_1(x) \quad \cdots \quad \nabla h_n(x) \,] \right) &= \det \begin{bmatrix} A_1 v_1 \quad A_1 v_2 & * \\ \mathbf{0} & Dh(x) Dh(x)^T \end{bmatrix} \\
&= \det \begin{bmatrix} A_2 v_1 \quad A_2 v_2 & ** \\ \mathbf{0} & Dh(x) Dh(x)^T \end{bmatrix} \\
&= \det\left(\begin{bmatrix} A_2 \\ Dh(x) \end{bmatrix} [\, v_1 \quad v_2 \quad \nabla h_1(x) \quad \cdots \quad \nabla h_n(x) \,] \right).
\end{aligned}$$

Since $\det[v_1 \quad v_2 \quad \nabla h_1(x) \quad \cdots \quad \nabla h_n(x)] \neq 0$, we obtain

$$\det \begin{bmatrix} A_1 \\ Dh(x) \end{bmatrix} = \det \begin{bmatrix} A_2 \\ Dh(x) \end{bmatrix}.$$ □

LEMMA 3.2. *Φ is a submersion at every $[(x, y, A)] \in G/_{\sim}$ such that* $\operatorname{rank}\begin{bmatrix} A \\ Dh(x) \end{bmatrix} \geqslant n+1$.

Proof. Put $\tilde{\Phi} : G \to \mathbf{R}$ as $\tilde{\Phi}(x, y, A) = \det\begin{bmatrix} A \\ Dh(x) \end{bmatrix}$. Then $\tilde{\Phi}(x, y, A)$ can be expressed as a linear combination of elements of one of rows of the matrix A, whose coefficients are appropriates $(n+1)$-minors of the matrix $\begin{bmatrix} A \\ Dh(x) \end{bmatrix}$. Since at least one of these minors is not 0, $\tilde{\Phi}$ is a submersion at (x, y, A). Notice that $\tilde{\Phi} = \Phi \circ pr$, so by Lemma 2.2, Φ is a submersion at $[(x, y, A)]$. □

For a smooth mapping $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}^2$ we define $d : \mathbf{R}^{n+2} \to \mathbf{R}$ as

$$d(x) = \det\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix}.$$

According to (1) for $f = \tilde{f}|_M : M \to \mathbf{R}^2$ we have $x \in S_i(f)$ if and only if $\operatorname{rank}\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix} = n+2-i$, for $i = 1, 2$, and so $S_1(f) \cup S_2(f) = d^{-1}(0) \cap M$.

THEOREM 3.3. *A mapping $f = \tilde{f}|_M : M \to \mathbf{R}^2$ is 1-generic if and only if $d|_M$ is a submersion at points from $d^{-1}(0) \cap M$, i.e.* $\operatorname{rank}\begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix} = n+1$, *for $x \in d^{-1}(0) \cap M$. If that is the case, then $S_1(f) = d^{-1}(0) \cap M$.*

Proof. Let $x \in S_1(f)$. According to Lemma 3.2, Φ is a submersion at $\Psi(j^1 f(x))$. Notice that there exists a small enough neighbourhood U of $\Psi(j^1 f(x))$ such that $\Phi|_U$ is a submersion and

$$U \cap \Psi(S_1) = \Phi|_U^{-1}(0).$$

We have $j^1 f \pitchfork S^1$ at x if and only if $\Psi(j^1 f) \pitchfork \Psi(S^1)$ at x. According to [4, II, Lemma 4.3], $\Psi(j^1 f) \pitchfork \Psi(S^1)$ at x if and only if $\Phi|_U \circ \Psi \circ j^1 f$ is a submersion at x.

Let us see that $\Phi|_U \circ \Psi \circ j^1 f(x) = d(x)$ for $x \in M$. We get that for $x \in S_1(f)$, $j^1 f \pitchfork S^1$ at x if and only if $d|_M : M \to \mathbf{R}$ is a submersion at x, i.e. $\operatorname{rank}\begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix} = n+1$.

Note that since $\operatorname{codim} S_2 = 4$, $j^1 f \pitchfork S_2$ if and only if $S_2(f) = \emptyset$. On the other hand, if $x \in S_2(f)$, then

$$\operatorname{rank}\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix} = n,$$

the elements of $Dd(x) = D\left(\det\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix}\right)$ are linear combinations of $(n+1)$-minors of this matrix, and so $Dd(x) = (0, \dots, 0)$. We get that if $d|_M$ is a submersion at points from $d^{-1}(0) \cap M$, then $S_2(f) = \emptyset$. $\square$

From now on we assume that $f = \tilde{f}|_M : M \to \mathbf{R}^2$ is 1-generic. Then by Theorem 3.3, for x near $S_1(f)$, the vectors $\nabla h_1(x), \dots, \nabla h_n(x), \nabla d(x)$ are linearly independent and $S_1(f)$ is 1-dimensional submanifold of M.

For $x \in \mathbf{R}^{n+2}$ and the matrix $\begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix}$, by $w_i(x)$ we will denote its $(n+1)$-minors obtained by removing i-th column. We define a vector field $v : \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ as

$$v(x) = (-w_1(x), w_2(x), \dots, (-1)^{n+2} w_{n+2}(x)).$$

Then for $x \in S_1(f)$ the vector $v(x)$ is a generator of

$$T_x S_1(f) = (\mathrm{Span}\{\nabla h_1(x), \dots, \nabla h_n(x), \nabla d(x)\})^{\perp}.$$

Put $F = (F_1, F_2) : \mathbf{R}^{n+2} \to \mathbf{R}^2$ as

$$F(x) = D\tilde{f}(x)(v(x)).$$

We will call $p \in S_1(f)$ a **fold point** if it is a regular point of $f|_{S_1(f)}$.

Proposition 3.4. *For a* 1-*generic* f *and a point* $p \in S_1(f)$ *the following are equivalent*:

(a) p *is a fold point*;

(b) $\mathrm{rank}\begin{bmatrix} D\tilde{f}(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$;

(c) $F(p) \neq 0$.

Proof. Since f is 1-generic, $S_1(f) = (h, d)^{-1}(0)$ is a complete intersection, and so the equivalence of the first two conditions is a simple consequence of the equation (1).

We see that $F(p) \neq 0$ iff $\langle \nabla\tilde{f}_1(p), v(p)\rangle \neq 0$ or $\langle \nabla\tilde{f}_2(p), v(p)\rangle \neq 0$ iff at least one of $\nabla\tilde{f}_1(p)$, $\nabla\tilde{f}_2(p)$ does not belong to $\mathrm{Span}\{\nabla h_1(x), \dots, \nabla h_n(x), \nabla d(x)\}$ iff $\mathrm{rank}\begin{bmatrix} D\tilde{f}(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$. So we get (b) $\Leftrightarrow$ (c). $\square$

If $f = (f_1, f_2) : M \to \mathbf{R}^2$ is 1-generic, then for $p \in S_1(f)$ one of the following two conditions can occur.

$$T_p S_1(f) + \ker Df(p) = \mathbf{R}^2, \tag{2}$$

$$T_p S_1(f) = \ker Df(p). \tag{3}$$

It is easy to see that $p \in S_1(f)$ satisfies (2) if and only if $F(p) \neq 0$, and then p is a fold point.

Assume that condition (3) holds at $p \in S_1(f)$. By the previous Proposition this is equivalent to the condition $F(p) = 0$.

Take a smooth function k on M such that $k \equiv 0$ on $S_1(f)$ and $Dk(p) \neq 0$ (our mapping $d|_M$ satisfies both these conditions). Let ξ be a non-vanishing vector field along $S_1(f)$ such that ξ is in the kernel of Df at each point of $S_1(f)$ near p. Then $Dk(\xi)$ is a function on $S_1(f)$ having a zero at p. The order of this zero does not depend on the choice of ξ or k (see [4, p. 146]), so in our case it equals the order of $Dd|_M(\xi)$ at p. Following [4] we will say that p is a **simple cusp** (or **cusp** for short) if p is a simple zero of $Dd|_M(\xi)$. If this is the case, then locally near p the mapping f has a form $(x_1, x_2) \mapsto (x_1, x_2^3 + x_1 x_2)$ (see [13], [4]).

Proposition 3.5. *Assume that f is 1-generic and $p \in S_1(f)$. Then p is a simple cusp if and only if $F(p) = 0$ and* $\operatorname{rank}\begin{bmatrix} DF(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n + 2.$

Proof. Take $p \in S_1(f)$. Note that $F(p) = 0$ is equivalent to the condition $T_p S_1(f) = \ker Df(p)$. So we assume that $F(p) = 0$.

Let us take a small neighbourhood $U \subset \mathbf{R}^{n+2}$ of p and a smooth vector field $w : U \to \mathbf{R}^{n+2}$ such that

$$\operatorname{Span}\{w(x)\} = (\operatorname{Span}\{\nabla h_1(x), \dots, \nabla h_n(x), v(x)\})^{\perp} \quad \text{and} \quad \langle \nabla d(x), w(x) \rangle \neq 0,$$

for $x \in U$. We define a smooth vector field $\xi_i : S_1(f) \cap U \to \mathbf{R}^{n+2}$ for $i = 1, 2$ by

$$\xi_i(x) = \frac{F_i(x)}{\langle \nabla d(x), w(x) \rangle} w(x) - \frac{\langle \nabla \tilde{f}_i(x), w(x) \rangle}{\langle \nabla d(x), w(x) \rangle} v(x).$$

By our assumptions

$$\operatorname{rank}\begin{bmatrix} D\tilde{f}(p) \\ Dh(p) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Dd(p) \\ Dh(p) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} D\tilde{f}(p) \\ Dd(p) \\ Dh(p) \end{bmatrix} = n + 1,$$

and then there exist $\alpha, \beta \in \mathbf{R}$ such that $\alpha^2 + \beta^2 \neq 0$, $\nabla d(p) = \alpha \nabla \tilde{f}_1(p) + \beta \nabla \tilde{f}_2(p) +$ some linear combination of $\nabla h_i(p)$. So

$$0 \neq \langle \nabla d(p), w(p) \rangle = \alpha \langle \nabla \tilde{f}_1(p), w(p) \rangle + \beta \langle \nabla \tilde{f}_2(p), w(p) \rangle,$$

and then $\langle \nabla \tilde{f}_1(p), w(p) \rangle \neq 0$ or $\langle \nabla \tilde{f}_2(p), w(p) \rangle \neq 0$. Hence at least one of $\xi_i(p) = -\dfrac{\langle \nabla \tilde{f}_i(p), w(p) \rangle}{\langle \nabla d(p), w(p) \rangle} v(p)$ is different from 0. Of course $\xi_i(p) \in T_p S_1(f) = \operatorname{Span}\{v(p)\}$.

Since for $x \in S_1(f) \cap U$ we have $\xi_i(x) \in (\mathrm{Span}\{\nabla h_1(x), \dots, \nabla h_n(x)\})^{\perp}$, $\langle \nabla \tilde{f}_i(x), \xi_i(x) \rangle = 0$, and $\operatorname{rank}\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix} = n+1$. It is easy to see that

$$\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix} \xi_i(x) = 0,$$

and so $\xi_i(x) \in \ker(Df(x))$ for $i = 1, 2$.

Notice that $Dd|_M(x)\xi_i(x) = \langle \nabla d(x), \xi_i(x) \rangle = F_i(x)$ for $x \in S_1(f) \cap U$. Take i such that $\xi_i(p) \neq 0$. We get that p is a simple cusp if and only if p is a simple zero of $F_i|_{S_1(f)}$, then $\operatorname{rank}\begin{bmatrix} DF(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$.

On the other hand, if for $j = 1, 2$, $\operatorname{rank}\begin{bmatrix} DF_j(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$, then p is a simple zero of $F_i|_{S_1(f)}$. So let us assume, that for example $\operatorname{rank}\begin{bmatrix} DF_2(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+1$ and $\operatorname{rank}\begin{bmatrix} DF_1(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$. Since for $x \in S_1(f) \cap U$, $\operatorname{rank}\begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix} = n+1$, there exist smooth α, β such that $\alpha^2(x) + \beta^2(x) \neq 0$ and $\alpha(x)F_1(x) + \beta(x)F_2(x) = 0$ for $x \in S_1(f) \cap U$. Then differentiating the above equality in $S_1(f) \cap U$ we get $\beta(p) \neq 0$ and we obtain $\langle \nabla \tilde{f}_2(p), w(p) \rangle = 0$. So $\xi_2(p) = 0$, that means i must be 1, and $\operatorname{rank}\begin{bmatrix} DF_i(p) \\ Dh(p) \\ Dd(p) \end{bmatrix} = n+2$ implies that p is a simple zero of $F_i|_{S_1(f)}$. $\square$

4. Signs of cusps

Let $f : M \to \mathbf{R}^2$ be a smooth map on a smooth oriented 2-dimensional manifold. For a simple cusp p of f we denote by $\mu(p)$ the local topological degree $\deg_p f$ of the germ $f : (M, p) \to (\mathbf{R}^2, f(p))$. From the local form of f near p it is easy to see that $\mu(p) = \pm 1$. We will call it the **sign of the cusp** p.

In [6], the authors investigated the algebraic sum of cusps of a 1-generic mapping $g = (g_1, g_2) : \mathbf{R}^2 \to \mathbf{R}^2$. They defined $G : \mathbf{R}^2 \to \mathbf{R}^2$ as $G(x) = Dg(x)\zeta(x)$, where $\zeta(x) = (\zeta_1(x), \zeta_2(x)) = \left(-\frac{\partial}{\partial x_2}\det Dg(x), \frac{\partial}{\partial x_1}\det Dg(x)\right)$ is tangent to $S_1(g)$ for $x \in S_1(g)$.

According to [6, Proposition 1], for a simple cusp $q \in \mathbf{R}^2$ of g, we have $\det DG(q) \neq 0$ and $\mu(q) = \operatorname{sgn}\det DG(q)$.

Using the facts and proofs from [6, Section 3.] it is easy to show the following.

LEMMA 4.1. *Let $\eta = (\eta_1, \eta_2)$ be a non-zero vector field on $\mathbf{R}^2$. Assume that in some neighbourhood of the simple cusp q of g there exists a smooth non-vanishing function s such that on $S_1(g)$ we have $s(x)\eta(x) = \zeta(x)$. Then for $\tilde{G}(x) = Dg(x)\eta(x)$*

$$\operatorname{sgn}\det DG(q) = \operatorname{sgn}\det D\tilde{G}(q).$$

Proof. Following [6, Section 3.] we can assume that $q = 0$ and there exist $\alpha, \beta \neq 0$ such that

$$Dg(0) = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad \zeta(0) = (\beta, 0), \quad \frac{\partial^2 g_2}{\partial x_1^2}(0) = 0.$$

We can take a smooth $\varphi : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that locally $S_1(g) = \{(t, \varphi(t))\}$. Then $\varphi'(0) = 0$ and

$$\frac{d}{dt} s(t, \varphi(t))\eta_2(t, \varphi(t)) = \frac{d}{dt}\zeta_2(t, \varphi(t)),$$

hence $s(0)\dfrac{\partial \eta_2}{\partial x_1}(0) = \dfrac{\partial \zeta_2}{\partial x_1}(0)$. Easy computations show that $\det DG(0) = s^2(0)\det D\tilde{G}(0)$. □

Let us recall that $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}^2$ is smooth and $f = \tilde{f}|_M : M \to \mathbf{R}^2$ is 1-generic, $M = h^{-1}(0)$ is a complete intersection. In the previous section we have defined a vector field $v : \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ such that for $x \in S_1(f)$ the vector $v(x)$ spans $T_x S_1(f)$, and the mapping $F(x) = D\tilde{f}(x)v(x)$.

THEOREM 4.2. *Let us assume that p is a simple cusp of a 1-generic map $f : M \to \mathbf{R}^2$, where $f = \tilde{f}|_M$ and $M = h^{-1}(0)$ is a complete intersection. Then $\mu(p) = \operatorname{sgn}\det \begin{bmatrix} DF(p) \\ Dh(p) \end{bmatrix}$.*

Proof. We can choose a chart ϕ of $\mathbf{R}^{n+2}$ defined in some neighbourhood of p such that both ϕ and the corresponding chart ϕ_M of M, i.e. $\phi|_M = (\phi_M, 0) : M \to \mathbf{R}^2 \times \{0\}$, preserve the orientations. Put $q = \phi_M(p)$ and take G as above for the mapping $g = f \circ \phi_M^{-1} : (\mathbf{R}^2, q) \to \mathbf{R}^2$.

For $x \in M$ we define $\eta = (\eta_1, \eta_2)$ as $D\phi(x)v(x) = (\eta_1(x), \eta_2(x), 0, \dots, 0)$. Let $y \in \mathbf{R}^2$ be such that $\phi(x) = (y, 0, \dots, 0)$, i.e. $\phi_M(x) = y$. Since $\eta(x) = \eta(\phi_M^{-1}(y))$ is a non-zero vector in the tangent space at y of $\phi_M(S_1(f)) = S_1(g) \subset \mathbf{R}^2$, as well as $\zeta(y)$, there exists a smooth non-vanishing mapping $s : (\mathbf{R}^2, q) \to \mathbf{R}$ such that $\zeta(y) = s(y)\eta(\phi_M^{-1}(y))$ for $y \in \phi_M(S_1(f))$.

According to [6, Proposition 1.],

$$\mu(p) = \deg_p f = \deg_q g = \operatorname{sgn}\det DG(q) \neq 0.$$

Define $\tilde{G}(y) = Dg(y)\eta(\phi_M^{-1}(y))$. Then from Lemma 4.1

$$\operatorname{sgn}\det DG(q) = \operatorname{sgn}\det D\tilde{G}(q).$$

Notice that

$$F(\phi_M^{-1}(y)) = D\tilde{f}(\phi^{-1}(y,0))D\phi^{-1}(y,0)D\phi(\phi^{-1}(y,0))v(\phi^{-1}(y,0))$$
$$= D(\tilde{f} \circ \phi^{-1})(y,0)(\eta(\phi^{-1}(y,0)),0) = Dg(y)(\eta(\phi_M^{-1}(y))) = \tilde{G}(y).$$

According to [12, Lemma 3.1.]

$$\operatorname{sgn}\det D\tilde{G}(q) = \operatorname{sgn}\det D(F \circ \phi_M^{-1})(q) = \operatorname{sgn}\det \begin{bmatrix} DF(p) \\ Dh(p) \end{bmatrix}. \qquad \square$$

5. Algebraic sum of cusps of a polynomial mapping

Now we recall a well-known fact. Take an ideal $J \subset \mathbf{R}[x] = \mathbf{R}[x_1, \dots, x_m]$ such that the $\mathbf{R}$-algebra $\mathscr{A} = \mathbf{R}[x]/J$ is finitely generated over $\mathbf{R}$, i.e. $\dim_{\mathbf{R}} \mathscr{A} < \infty$. Denote by $V(J)$ the set of real zeros of the ideal J.

For $h \in \mathscr{A}$, we denote by $T(h)$ the trace of the $\mathbf{R}$-linear endomorphism $\mathscr{A} \ni a \mapsto h \cdot a \in \mathscr{A}$. Then $T : \mathscr{A} \to \mathbf{R}$ is a linear functional. Take $\delta \in \mathbf{R}[x]$. Let $\Theta : \mathscr{A} \to \mathbf{R}$ be the quadratic form given by $\Theta(a) = T(\delta \cdot a^2)$.

According to [1], [7], the signature $\sigma(\Theta)$ of Θ equals

$$\sigma(\Theta) = \sum_{p \in V(J)} \operatorname{sgn} \delta(p), \tag{4}$$

and if Θ is non-degenerate then $\delta(p) \neq 0$ for each $p \in V(J)$.

In this Section we will present that the results from Sections 3, 4 can be applied to compute the number and the algebraic sum of cusps in the polynomial case. So take polynomial mappings $\tilde{f} : \mathbf{R}^{n+2} \to \mathbf{R}^2$ and $h = (h_1, \dots, h_n) : \mathbf{R}^{n+2} \to \mathbf{R}^n$ such that $M = h^{-1}(0)$ is a complete intersection. Put $f = \tilde{f}|_M : M \to \mathbf{R}^2$. Let us recall that $d(x) = \det \begin{bmatrix} D\tilde{f}(x) \\ Dh(x) \end{bmatrix}$, $v(x) = (-w_1(x), w_2(x), \dots, (-1)^{n+2} w_{n+2}(x))$, where $w_i(x)$ are $(n+1)$-minors obtained by removing i-th column from the matrix $\begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix}$, and $F(x) = D\tilde{f}(x)v(x)$.

Let us define ideals $I, S \subset \mathbf{R}[x] = \mathbf{R}[x_1, \dots, x_{n+2}]$ as

$$I = \langle h_1, \dots, h_n, d, w_1, \dots, w_{n+2} \rangle,$$

$$S = \left\langle h_1, \dots, h_n, d, F_1, F_2, \det \begin{bmatrix} DF_1 \\ Dd \\ Dh \end{bmatrix}, \det \begin{bmatrix} DF_2 \\ Dd \\ Dh \end{bmatrix} \right\rangle.$$

One may check that $S \subset I$.

Proposition 5.1. (a) *If $I = \mathbf{R}[x]$ then f is 1-generic.*

(b) *If $S = \mathbf{R}[x]$ then f is 1-generic, and has only folds and simple cusps as singular points. If that is the case, then the set of simple cusps $\{x \in \mathbf{R}^{n+2} \mid h_1(x) = \cdots = h_n(x) = d(x) = F_1(x) = F_2(x) = 0\}$ is an algebraic set of isolated points, so it is finite.*

Proof. If $I = \mathbf{R}[x]$ then the set $V(I)$ of real zeros of I is empty. We have $V(I) = \{x \in M \mid d(x) = w_1(x) = \cdots = w_{n+2}(x) = 0\}$. Since the dimension of the matrix $\begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix}$ is $(n+1) \times (n+2)$, we obtain

$$\emptyset = V(I) = \{x \in M \mid d(x) = 0, \operatorname{rank} \begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix} < n+1\}.$$

So we get that for all $x \in d^{-1}(0) \cap M$, $\operatorname{rank} \begin{bmatrix} Dd(x) \\ Dh(x) \end{bmatrix} = n+1$. According to Theorem 3.3, f is 1-generic, so we get (a).

Since $S \subset I$, if $S = \mathbf{R}[x]$, then $I = \mathbf{R}[x]$, and so by (a) f is 1-generic. By Theorem 3.3, $S_1(f) = d^{-1}(0) \cap M = d^{-1}(0) \cap h^{-1}(0)$. Moreover if $S = \mathbf{R}[x]$, then $V(S) = \emptyset$, and we obtain

$$\emptyset = V(S) \supset \left\{ x \in S_1(f) \mid F_1(x) = F_2(x) = 0, \operatorname{rank} \begin{bmatrix} DF(x) \\ Dd(x) \\ Dh(x) \end{bmatrix} < n+2 \right\}.$$

Hence for $x \in S_1(f)$ we have either

$$F(x) \neq 0$$

or

$$F(x) = 0 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} DF(x) \\ Dd(x) \\ Dh(x) \end{bmatrix} = n+2.$$

According to Propositions 3.4, 3.5, f has only folds and simple cusps as singular points. If that is the case, for $x \in S_1(f)$, the point x is a simple cusp if and only if $F(x) = 0$, so we get (b). □

Let us assume that $S = \mathbf{R}[x]$. Put $J = \langle h_1, \ldots, h_n, d, F_1, F_2 \rangle$, and $\mathscr{A} = \mathbf{R}[x]/J$, and assume that $\dim_{\mathbf{R}} \mathscr{A} < \infty$. Then according to the previous Proposition, f is 1-generic, and has only folds and simple cusps as singular points. Moreover $V(J)$ is the set of simple cusps of f, it is finite, and so we can count the **algebraic sum of cusps**, i.e. $\sum_{p \in V(J)} \mu(p)$. Let us define quadratic forms $\Theta_1, \Theta_2 : \mathscr{A} \to \mathbf{R}$ by $\Theta_1(a) = T(1 \cdot a^2)$, $\Theta_2(a) = T(\delta \cdot a^2)$, where $\delta(x) = \det \begin{bmatrix} DF(x) \\ Dh(x) \end{bmatrix}$.

PROPOSITION 5.2. *Assume that* $S = \mathbf{R}[x]$ *and* $\dim_{\mathbf{R}} \mathscr{A} < \infty$. *Then for the mapping* f

(a) *the number of cusps* $\#V(J) = \sigma(\Theta_1)$,

(b) *the algebraic sum of cusps* $\sum_{p \in V(J)} \mu(p) = \sigma(\Theta_2)$.

Proof. Since $S = \mathbf{R}[x]$, according to Proposition 5.1, f is 1-generic, has only folds and simple cusps as singular points, and the set $V(J)$ of simple cusps of f is finite.

By the formula (4) we get

$$\sigma(\Theta_1) = \sum_{p \in V(J)} \operatorname{sgn}(1) = \# V(J).$$

Let us notice that by Theorem 4.2 for a simple cusp p of f, $\operatorname{sgn} \delta(p) = \mu(p)$. Then using once again the formula (4) we obtain

$$\sigma(\Theta_2) = \sum_{p \in V(J)} \operatorname{sgn} \delta(p) = \sum_{p \in V(J)} \mu(p). \qquad \square$$

Using Propositions 5.1, 5.2, and Singular ([5]) we computed the following examples.

The first example we will present in details.

Example 5.3. Put $\tilde{f} = (x^2 - 2xy + x, 2z) : \mathbf{R}^3 \to \mathbf{R}^2$ and $h = x^2 + y^2 + z^2 - 1 : \mathbf{R}^3 \to \mathbf{R}$. Then $h^{-1}(0)$ is a 2-dimensional sphere.

In this case the ideal S is generated by

$$h = x^2 + y^2 + z^2 - 1,$$

$$d = -8x^2 - 8xy + 8y^2 - 4y,$$

$$F_1 = 96x^2z - 64xyz + 64y^2z + 32xz - 48yz + 8z,$$

$$F_2 = -32x^2 + 128xy + 32y^2 - 16x,$$

$$\det \begin{bmatrix} DF_1 \\ Dd \\ Dh \end{bmatrix} = 1536x^4 - 7168x^3y + 3584x^2y^2 - 3072xy^3 - 1024y^4 - 5120x^2z^2$$
$$+ 10240xyz^2 + 1280x^3 - 3328x^2y + 3072xy^2 + 768y^3 - 3584xz^2$$
$$+ 768yz^2 + 384x^2 - 896xy - 128y^2 - 256z^2 + 64x,$$

$$\det \begin{bmatrix} DF_2 \\ Dd \\ Dh \end{bmatrix} = 5120x^2z + 5120y^2z + 768xz - 1536yz + 128z.$$

Using Singular we compute the standard basis of S, which is $\{1\}$, i.e. S is the whole $\mathbf{R}[x]$.

From Proposition 5.1, $f = \tilde{f}|_{h^{-1}(0)}$ is 1-generic, and has only folds and simple cusps as singular points.

The ideal $J = \langle x^2 + y^2 + z^2 - 1, -8x^2 - 8xy + 8y^2 - 4y, 96x^2z - 64xyz + 64y^2z + 32xz - 48yz + 8z, -32x^2 + 128xy + 32y^2 - 16x\rangle$.

Singular computations shows that: the algebra $\mathscr{A} = \mathbf{R}[x]/J$ has dimension 6, its basis has a form $e_1 = xz$, $e_2 = yz$, $e_3 = x$, $e_4 = y$, $e_5 = z$, $e_6 = 1$, the matrices of the forms Θ_1, Θ_2 are

$$\begin{bmatrix} -33/500 & -81/500 & 0 & 0 & -57/100 & 0 & \\ -81/500 & 297/1000 & 0 & 0 & 21/20 & 0 & \\ 0 & 0 & -3/50 & -9/50 & 0 & -3/5 & \\ 0 & 0 & -9/50 & 9/25 & 0 & 6/5 & \\ -57/100 & & 21/20 & 0 & 0 & 57/10 & 0 \\ 0 & 0 & -3/5 & 6/5 & 0 & 6 & \end{bmatrix},$$

$$\begin{bmatrix} 339408/3125 & 709344/3125 & 0 & 0 & 527616/625 & 0 \\ 709344/3125 & -1178928/3125 & 0 & 0 & -891072/625 & 0 \\ 0 & 0 & 12672/125 & 31104/125 & 0 & 21888/25 \\ 0 & 0 & 31104/125 & -57024/125 & 0 & -8064/5 \\ 527616/625 & -891072/625 & 0 & 0 & -1050048/125 & 0 \\ 0 & 0 & 21888/25 & -8064/5 & 0 & -43776/5 \end{bmatrix},$$

and their signatures are 2 and -2 respectively. According to Proposition 5.2 it means that the mapping f has 2 simple cusps, both of them are negative.

The other examples are computed similarly and we present just the final results.

Example 5.4. Put $\tilde{f} = (xz^2 - z^2 - 2z, 2x^3z - y^3 + z^3 + 3yz - z^2 - y) : \mathbf{R}^3 \to \mathbf{R}^2$ and $h = x^2 + y^2 + z^2 - 1 : \mathbf{R}^3 \to \mathbf{R}$. Then $h^{-1}(0)$ is a 2-dimensional sphere, and $\dim_{\mathbf{R}} \mathscr{A} = 68$. The mapping $f = \tilde{f}|_{h^{-1}(0)}$ is 1-generic, has 6 simple cusps, 3 of them are negative.

Example 5.5. Put $\tilde{f} = (2xz^2 - y^2 + 2xz, -z^3 + 2xy - y^2 - x) : \mathbf{R}^3 \to \mathbf{R}^2$ and $h = x^2 + y^2 + z^2 - 1 : \mathbf{R}^3 \to \mathbf{R}$. In this case $\dim_{\mathbf{R}} \mathscr{A} = 44$, and the mapping $f = \tilde{f}|_{h^{-1}(0)}$ is 1-generic, has 8 simple cusps, 6 of them are negative.

Example 5.6. Put $\tilde{f} = (zw - 2w^2 - 2x, 3x^3 - 2yz^2 - yw + 2zw - x) : \mathbf{R}^4 \to \mathbf{R}^2$ and $h = (x^2 + y^2 - 1, z^2 + w^2 - 1) : \mathbf{R}^4 \to \mathbf{R}^2$. Then $h^{-1}(0)$ is a 2-dimensional torus, and $\dim_{\mathbf{R}} \mathscr{A} = 52$. The mapping $f = \tilde{f}|_{h^{-1}(0)}$ is 1-generic, has 16 simple cusps, 8 of them are negative.

Example 5.7. Put $\tilde{f} = (3z^3 + x^2 - xy, 2y^2z - 2z^3 + xy - 2y^2 - x) : \mathbf{R}^3 \to \mathbf{R}^2$ and $h = x^2 + y^2 - z : \mathbf{R}^3 \to \mathbf{R}$. Then $h^{-1}(0)$ is a 2-dimensional paraboloid, and $\dim_{\mathbf{R}} \mathscr{A} = 47$. The mapping $f = \tilde{f}|_{h^{-1}(0)}$ is 1-generic, has 3 simple cusps, all of them are negative.

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