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GLOBAL ATTRACTORS FOR A KIRCHHOFF TYPE PLATE EQUATION WITH MEMORY

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Abstract

A Kirchhoff type plate equation with memory is investigated. Under the suitable assumptions, we establish the existence of a global attractor by using the contraction function method.

1. Introduction

In this paper, we are concerned with the following Kirchhoff type plate equation with memory:

$$u_{tt} + \alpha u_t + \Delta^2 u - \int_0^\infty \mu(s)\Delta^2 u(t-s)\,ds + \lambda u + (p - \beta\|\nabla u\|_2^2)\Delta u + f(u) = g(x) \quad \text{in } \Omega \times \mathbf{R}^+, \tag{1.1}$$

where $\Omega \subseteq \mathbf{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$. Here α, β and λ are given positive constants, $p \in \mathbf{R}$, μ is the memory kernel, and $g \in L^2(\Omega)$ is a forcing term.

Equation (1.1) with the memory term $\int_0^\infty \mu(s)\Delta^2 u(t-s)\,ds$, where the function μ is called kernel, can be regarded as a fourth order viscoelastic plate equation with a lower order perturbation, and it can be also regarded as an elastoplastic flow equation with some kind of memory effect ([3], [14], [15]).

In this paper, we consider (1.1) with boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \times \mathbf{R}, \tag{1.2}$$

and initial conditions

$$u(x,\tau) = u_0(x,\tau), \quad u_t(x,\tau) = \partial_t u_0(x,\tau), \quad (x,\tau) \in \Gamma \times (-\infty, 0], \tag{1.3}$$

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where ν is the unit outer normal on Γ, $u_0 : \Omega \times (-\infty, 0] \to \mathbf{R}$ is the prescribed past history of u.

Fourth order equations with lower order perturbation are related to models of elastoplastic microstructure flows. Woinowsky-Krieger [18] first proposed the one-dimensional nonlinear equation of vibration of beams, which is given by

$$u_{tt} + \alpha u_{xxxx} - \left(\beta + \gamma \int_0^L |u_x|^2 \, dx\right) u_{xx} = 0, \tag{1.4}$$

where L is the length of the beam, α, γ are positive physical constants and $\beta \in \mathbf{R}$. The nonlinear part of (1.4) represents for the extensible effect for the beam whose ends are restrained to remain in a fixed distance apart in its transverse vibrations. The global existence and the global attractors of solutions for the plate equation were studied in [1, 15, 16, 19, 20] and references therein. Yang [20] proved the existence of a uniform attractor for non-autonomous plate equations with a localized damping and critical nonlinearity when $\mu(s) \equiv 0$ and $p = \beta \equiv 0$. Wu [19] proved the existence and uniqueness of global solutions as well as the existence of a global attractor for a nonlinear plate equation with thermal memory. Barbosa and Ma [1] obtained the existence of a finite-dimensional global attractor as well as the existence of exponential attractors for an extensible plate equation with thermal memory under the case that $p = 0$ and $M(\|\nabla u\|^2)$ satisfies the proper conditions. Silva and Ma [15, 16] proved the exponential stability and global attractor of plate equations with memory and perturbation of p-Laplacian type. Kang [12] showed the existence of global attractors to the following suspension bridge equation with memory

$$\begin{cases} u_{tt} + \alpha\Delta^2 u - \int_0^\infty \mu(s)\Delta^2 u(t-s)\, ds + ku^+ + f(u) = g(x) \quad \text{in } \Omega \times \mathbf{R}^+, \\ u = 0,\ \Delta u = 0 \quad \text{on } \Gamma \times \mathbf{R}, \\ u(x,\tau) = u_0(x,\tau),\ u_t(x,\tau) = \partial_t u_0(x,\tau), \quad (x,\tau) \in \Gamma \times (-\infty, 0]. \end{cases}$$

Motivated by above results, A natural question is to determine whether or not the system (1.1) admits a global attractor. Our objective in the present paper is to show (1.1) does have a global attractor when the growth of $f(u)$ are up to the critical range and the memory kernel $\mu \geq 0$ decays exponentially. The main result is Theorem 2.2.

We end this section by introducing the relative displacement memory that will transform (1.1) into an autonomous system. Following the arguments from Dafermos [5] and Giorgi et al. [6, 8, 10], we add a new variable η to the system corresponding to the relative displacement memory, namely

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in \Omega \times \mathbf{R}^+,\ t \geq 0. \tag{1.5}$$

By formal differentiation in (1.5) we have

$$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t), \quad (x,s) \in \Omega \times \mathbf{R}^+,\ t \geq 0,$$

and we take as initial condition $(t = 0)$

$$\eta^0(x,s) = u_0(x,0) - u_0(x,-s), \quad (x,s) \in \Omega \times \mathbf{R}^+.$$

Further, by assuming that $\mu \in L^1(\mathbf{R}^+)$, the original memory term can be rewritten as

$$\int_0^\infty \mu(s)\Delta^2 u(t-s)\,ds = \left(\int_0^\infty \mu(s)\,ds\right)\Delta^2 u - \int_0^\infty \mu(s)\Delta^2 \eta^t(s)\,ds.$$

Thus, taking for simplicity $\varrho = 1 - \int_0^\infty \mu(s)\,ds$, the original problem (1.1)–(1.3) can be transformed into the equivalent autonomous system

$$u_{tt} + \alpha u_t + \varrho\Delta^2 u + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,ds + \lambda u \tag{1.6}$$
$$+ (p - \beta\|\nabla u\|_2^2)\Delta u + f(u) = g(x) \quad \text{in } \boldsymbol{\Omega} \times \mathbf{R}^+,$$

$$\eta_t = -\eta_s + u_t, \quad \text{in } \boldsymbol{\Omega} \times \mathbf{R}^+ \times \mathbf{R}^+, \tag{1.7}$$

with boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \times \mathbf{R}^+, \quad \eta = \frac{\partial \eta}{\partial \nu} = 0, \quad \text{on } \Gamma \times \mathbf{R}^+ \times \mathbf{R}^+, \tag{1.8}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \eta^t(x,0) = 0, \quad \eta^0(x,s) = \eta_0(x,s), \tag{1.9}$$

where

$$\begin{cases} u_0(x) = u_0(x,0), & x \in \boldsymbol{\Omega}, \\ u_1(x) = \partial_t u_0(x,t)|_{t=0}, & x \in \boldsymbol{\Omega}, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), & (x,s) \in \boldsymbol{\Omega} \times \mathbf{R}^+. \end{cases}$$

Our analysis is made upon the system (1.6)–(1.9).

The remaining of the paper is organized as follows. In Sec. 2 we fix some notations and present our assumptions and main results. We also discuss briefly the methods used to prove the results. Section 3 contains a short account of the abstract results in the theory of infinite dimensional dynamical systems that will be used. Section 4 is dedicated to the proof of the results.

2. Assumptions and the main result

We begin with precise hypotheses on the problem (1.6)–(1.9). Concerning the nonlinear term $f : \mathbf{R} \to \mathbf{R}$, we assume that

$$f(0) = 0, \quad |f(u) - f(v)| \le k_0(1 + |u|^\rho + |v|^\rho)|u - v|, \quad \forall u, v \in \mathbf{R}, \tag{2.1}$$

where $k_0 > 0$ and

$$0 < \rho \le \frac{4}{N-4} \quad \text{if } N \ge 5 \quad \text{and} \quad \rho > 0 \quad \text{if } 1 \le N \le 4. \tag{2.2}$$

Condition (2.2) implies that $H_0^2(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$. Also, we say that $\rho = \dfrac{4}{N-4}$ is a critical exponent for the growth of $f(u)$ when $N \geq 5$. In addition, we assume that

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{2.3}$$

where λ_1 is the best constant in the Poincaré type inequality

$$\int_\Omega |\Delta u|^2 \, dx \geq \lambda_1 \int_\Omega |u|^2 \, dx.$$

Further, with respect to the memory component, we assume that

$$\mu \in C^1(\mathbf{R}^+) \cap L^1(\mathbf{R}^+), \quad \int_0^\infty \mu(s) \, ds = \mu_0 > 0, \quad \varrho = 1 - \mu_0 > 0, \tag{2.4}$$

$$\mu'(s) \leq 0 \leq \mu(s), \quad \forall s \in \mathbf{R}^+, \tag{2.5}$$

and that there exists a constant $k_1 > 0$ such that

$$\mu'(s) + k_1\mu(s) \leq 0, \quad \forall s \in \mathbf{R}^+. \tag{2.6}$$

In the sequel we fix some notations on the function spaces that will be used throughout remainder of this paper. Let

$$H = V_0 = L^2(\Omega), \quad V = V_1 = H_0^2(\Omega),$$

equipped with respective inner products and norms,

$$(u, v)_V = (\Delta u, \Delta v) \quad \text{and} \quad \|u\|_V = \|\Delta u\|_2.$$

As usual $(\cdot, \cdot)$ denotes L^2-inner product and $\|\cdot\|_p$ denotes L^p-norms.

In order to consider the relative displacement η as a new variable, we introduce the weighted L^2-space

$$L^2_\mu(\mathbf{R}^+; V) = \left\{ \eta : \mathbf{R}^+ \to V \,\middle|\, \int_0^\infty \mu(s)\|\eta(s)\|_V^2 \, ds < \infty \right\},$$

which is nonempty due to the assumptions (2.4) and (2.5). In addition, it is a Hilbert space endowed with inner product and norm

$$(u, v)_{\mu, V} = \int_0^\infty \mu(r)(u(r), v(r))_V \, dr, \quad \|u\|_{\mu, V}^2 = (u, u)_{\mu, V} = \int_0^\infty \mu(r)\|u(r)\|_V^2 \, dr,$$

Finally we introduce the phase space

$$\mathscr{H} = V \times H \times L^2_\mu(\mathbf{R}^+; V),$$

equipped with the norm

$$\|(u, v, \xi)\|_{\mathscr{H}}^2 = \|\Delta u\|_2^2 + \|v\|_2^2 + \|\xi\|_{\mu, V}^2.$$

In order to obtain the global attractors of the problem (1.6)–(1.9), we need the following theorem. The well-posedness of the problem (1.6)–(1.9) is given by Faedo-Galerkin method (see [2, 7, 9, 11, 13]).

THEOREM 2.1. *Assume that assumptions* (2.1)–(2.6) *hold and consider* $g \in L^2(\Omega)$. *Then we have*

(i) *If initial data* $(u_0, u_1, \eta_0) \in \mathscr{H}$, *then problem* (1.6)–(1.9) *has a weak solution*

$$(u, u_t, \eta) \in C([0, T]; \mathscr{H}), \quad \forall T > 0,$$

satisfying

$$u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; H), \quad \eta \in L^\infty(0, T; L^2_\mu(\mathbf{R}^+; V)).$$

(ii) *Let* $z_i = (u^i, u^i_t, \eta^i)$ *be weak solutions of problem* (1.6)–(1.9) *corresponding to initial data* $z_i(0) = (u^i_0, u^i_1, \eta^i_0)$, $i = 1, 2$. *Then one has*

$$\|z_1(t) - z_2(t)\|_{\mathscr{H}} \le e^{ct}\|z_1(0) - z_2(0)\|_{\mathscr{H}}, \quad t \in [0, T],$$

for some constant $c > 0$.

Remark 2.1. The well-posedness of problem (1.6)–(1.9) given by Theorem 2.1 implies that the one-parameter family of operators $S(t) : \mathscr{H} \to \mathscr{H}$ defined by

$$S(t)(u_0, u_1, \eta_0) = (u(t), u_t(t), \eta^t), \quad t \ge 0, \tag{2.7}$$

where $(u(t), u_t(t), \eta^t)$ is the unique weak solution of the system (1.6)–(1.9), and satisfies the semigroup properties

$$S(0) = I \quad \text{and} \quad S(t + s) = S(t) \circ S(s), \quad t, s \ge 0,$$

and defines a nonlinear C_0-semigroup, which is locally Lipschitz continuous on the phase space $\mathscr{H}$. Then the dynamics of problem (1.6)–(1.9) can be studied through the dynamical system $(\mathscr{H}, S(t))$.

Our main result in the present paper is the following.

THEOREM 2.2. *Assume that assumptions* (2.1)–(2.6) *hold and* $g \in L^2(\Omega)$, *then the dynamical system* $(\mathscr{H}, S(t))$ *corresponding to the system* (1.6)–(1.9) *has a compact global attractor* $\mathscr{A} \subset \mathscr{H}$.

The proof of Theorem 2.2 is given in Sec. 4.

3. Infinite-dimensional dynamical systems

To make the paper more self contained we recall some fundamentals of the theory of infinite dimensional dynamical systems in mathematical physics. Below we follow more closely the book by Chueshov and Lasiecka [4] (Chap. 7).

Let $S(t)$ be a C_0-semigroup defined in a Banach space $\mathscr{H}$. A global attractor for $(\mathscr{H}, S(t))$ is a bounded closed set $\mathscr{A} \subset \mathscr{H}$ which is fully invariant and uniformly attracting, that is, $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$ and for every bounded subset $B \subset \mathscr{H}$,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathscr{H}}(S(t)B, \mathscr{A}) = 0,$$

where $\mathrm{dist}_{\mathscr{H}}$ is the Hausdorff semidistance in $\mathscr{H}$. We say that a dynamical system $(\mathscr{H}, S(t))$ is dissipative if it possesses a bounded absorbing set, that is, a bounded set $\mathscr{B} \subset \mathscr{H}$ such that for any bounded set $B \subset \mathscr{H}$ there exists $t_B \geq 0$ satisfying

$$S(t)B \subset \mathscr{B}, \quad \forall t \geq t_B.$$

In addition, we say that $(\mathscr{H}, S(t))$ is asymptotically smooth if for any bounded positively invariant set $B \subset \mathscr{H}$, there exists a compact set $K \subset \bar{B}$ such that

$$\lim_{t\to\infty} \mathrm{dist}_{\mathscr{H}}(S(t)B, K) = 0.$$

The following result is well-known. See for instance Chueshov and Lasiecka [4] (Theorem 7.2.3).

Theorem 3.1. *A dissipative dynamical system $(\mathscr{H}, S(t))$ has a compact global attractor if and only if it is asymptotically smooth.*

Theorem 3.2. *Suppose that for any bounded positively invariant set $B \subset \mathscr{H}$ and for $\forall \varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\|_{\mathscr{H}} \leq \varepsilon + \phi_T(x, y), \quad \forall x, y \in B,$$

where $\phi_T : B \times B \to \mathbf{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z_n, z_m) = 0, \tag{3.1}$$

for any sequence $\{z_n\}$ in B. Then $S(t)$ is asymptotically smooth in $\mathscr{H}$.

4. Proof of the main result

In order to prove Theorem 2.2 we will apply the abstract results presented in Sec. 3. The first step is to show that the dynamical system $(\mathscr{H}, S(t))$ is dissipative. The second step is to verify the asymptotic smoothness. Then the existence of a compact global attractor is guaranteed by Theorem 3.1.

In this section, we first prove the existence of the bounded absorbing set in $\mathscr{H}$.

Lemma 4.1. *Under assumptions* (2.1)–(2.6), *the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to problems* (1.6)–(1.9) *has a bounded absorbing set in $\mathscr{H}$.*

Proof. We set $v = u_t + \delta u$ and rewrite the equation of (1.6) as follows:

$$\begin{aligned}(4.1)\qquad & v_t + (\alpha - \delta)u_t + \varrho\Delta^2 u + \int_0^\infty \mu(s)\Delta^2\eta^t(s)\,ds + \lambda u \\ & + (p - \beta\|\nabla u\|_2^2)\Delta u + f(u) = g(x) \quad \text{in } \mathbf{\Omega} \times \mathbf{R}^+.\end{aligned}$$

Taking the scalar product in H of (4.1) with v and integrating over Ω, we obtain

$$\begin{aligned}(4.2)\qquad & \frac{d}{dt}\left[\frac{1}{2}\|v\|_2^2 + \frac{\varrho}{2}\|\Delta u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 + \int_\Omega F(u(t))\,dx\right] + (\alpha - \delta)(u_t, v) \\ & + \delta\varrho\|\Delta u\|_2^2 + ((p - \beta\|\nabla u\|_2^2)\Delta u, v) + (\eta, u_t)_{\mu, V} \\ & + \delta(\eta, u)_{\mu, V} + \lambda\delta\|u\|_2^2 + \delta(f(u), u) = (g, v),\end{aligned}$$

where $F(s) = \int_0^s f(\tau)\,d\tau$.

Exploiting (1.7), (2.4), (2.6) and Hölder inequality, we have

$$(\alpha - \delta)(u_t, v) = (\alpha - \delta)\|v\|_2^2 - \delta(\alpha - \delta)(u, v),$$

$$\begin{aligned}(\eta, u_t)_{\mu, V} = (\eta, \eta_t + \eta_s)_{\mu, V} &= \frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu, V}^2 + \int_0^\infty \mu(s)(\eta(s), \eta_s(s))_V\,ds \\ &= \frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu, V}^2 + \frac{1}{2}\int_0^\infty \mu(s) d\|\eta(s)\|_V^2 \\ &= \frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu, V}^2 - \frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|_V^2\,ds \\ &\geq \frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu, V}^2 + \frac{k_1}{2}\int_0^\infty \mu(s)\|\eta(s)\|_V^2\,ds \\ &= \frac{1}{2}\frac{d}{dt}\|\eta\|_{\mu, V}^2 + \frac{k_1}{2}\|\eta\|_{\mu, V}^2,\end{aligned}$$

$$\delta(\eta, u)_{\mu, V} \geq -\frac{k_1}{4}\|\eta\|_{\mu, V}^2 - \frac{\mu_0\delta^2}{k_1}\|\Delta u\|_2^2.$$

$$\begin{aligned}((p - \beta\|\nabla u\|_2^2)\Delta u, v) &= ((p - \beta\|\nabla u\|_2^2)\Delta u, u_t + \delta u) \\ &= (p - \beta\|\nabla u\|_2^2)(\Delta u, u_t) + (p - \beta\|\nabla u\|_2^2)(\Delta u, \delta u) \\ &\geq \frac{1}{2}\frac{d}{dt}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2 - \frac{p}{\sqrt{2\beta}}\right)^2 \\ &\quad + \delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2 - \frac{p}{\sqrt{2\beta}}\right)^2 - \frac{\delta p^2}{2\beta}.\end{aligned}$$

By the assumption (2.3) we know that there are $\lambda > \lambda' > 0$ and C_0 such that

$$(f(u), u) > -\lambda' \|u\|_2^2 - C_0 \operatorname{meas}(\Omega), \qquad \int_\Omega F(u)\, dx > -\frac{\lambda'}{2}\|u\|_2^2 - C_0 \operatorname{meas}(\Omega).$$

Hence we conclude from (4.2) that

$$\begin{aligned}(4.3)\qquad & \frac{d}{dt}\left[\frac{1}{2}\|v\|_2^2 + \frac{\varrho}{2}\|\Delta u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 - \frac{\lambda'}{2}\|u\|_2^2 + \frac{1}{2}\|\eta\|_{\mu, V}^2\right.\\ & \left. + \frac{1}{2}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2 - \frac{p}{\sqrt{2\beta}}\right)^2\right] + (\alpha - \delta)\|v\|_2^2 - \delta(\alpha - \delta)(u, v) + \delta\varrho\|\Delta u\|_2^2\\ & + \delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2 - \frac{p}{\sqrt{2\beta}}\right)^2 - \frac{\delta p^2}{2\beta} + \frac{k_1}{2}\|\eta\|_{\mu, V}^2 - \frac{k_1}{4}\|\eta\|_{\mu, V}^2 - \frac{\mu_0\delta^2}{k_1}\|\Delta u\|_2^2\\ & + \lambda\delta\|u\|_2^2 - \lambda'\delta\|u\|_2^2 - C_0\delta \operatorname{meas}(\Omega) \le (g, v).\end{aligned}$$

Moreover using Poincaré inequality in Sec. 2, Hölder inequality and Young inequality, when δ small enough, such that

$$\frac{\alpha}{2} - \delta > \frac{\alpha}{4}, \quad 1 - \frac{\mu_0\delta}{k_1\varrho} - \frac{\delta\alpha}{2\lambda_1\varrho} > 1 - \delta.$$

Then we obtain

$$\begin{aligned}(4.4)\qquad & (\alpha - \delta)\|v\|_2^2 - \delta(\alpha - \delta)(u, v) + \left(\delta\varrho - \frac{\mu_0\delta^2}{k_1}\right)\|\Delta u\|_2^2\\ & \ge (\alpha - \delta)\|v\|_2^2 - \left(\frac{\delta^2\alpha}{2\lambda_1}\|\Delta u\|_2^2 + \frac{\alpha}{2}\|v\|_2^2\right) + \delta\varrho\left(1 - \frac{\mu_0\delta}{k_1\varrho}\right)\|\Delta u\|_2^2\\ & \ge \left(\frac{\alpha}{2} - \delta\right)\|v\|_2^2 + \delta\varrho\left(1 - \frac{\mu_0\delta}{k_1\varrho} - \frac{\delta\alpha}{2\lambda_1\varrho}\right)\|\Delta u\|_2^2\\ & \ge \frac{\alpha}{4}\|v\|_2^2 + \delta\varrho(1 - \delta)\|\Delta u\|_2^2\end{aligned}$$

and

$$(4.5)\qquad (g, v) \le \|g\|_2\|v\|_2 \le \frac{2}{\alpha}\|g\|_2^2 + \frac{\alpha}{8}\|v\|_2^2.$$

Combining with (4.4) and (4.5), we deduce from (4.3)

$$\frac{d}{dt}\left[\frac{1}{2}\|v\|_2^2+\frac{\varrho}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\eta\|_{\mu,V}^2+\frac{1}{2}\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2\right]$$
$$+\frac{\alpha}{8}\|v\|_2^2+\delta\varrho(1-\delta)\|\Delta u\|_2^2+\frac{k_1}{4}\|\eta\|_{\mu,V}^2+\delta\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2$$
$$\leq C_0\delta\,\mathrm{meas}(\Omega)+\frac{2}{\alpha}\|g\|_2^2+\frac{\delta p^2}{2\beta}.$$

Setting

$$\alpha_1=\min\left\{\frac{\alpha}{4},2\delta(1-\delta),\frac{k_1}{2}\right\}.$$

We conclude that

$$\frac{d}{dt}W(t)+\alpha_1W(t)\leq\frac{4}{\alpha}\|g\|_2^2+\frac{\delta p^2}{\beta}+2C_0\delta\,\mathrm{meas}(\Omega):=C_1,$$

where

$$W(t)=\|v\|_2^2+\varrho\|\Delta u\|_2^2+\|\eta\|_{\mu,V}^2+\left(\frac{\beta}{\sqrt{2\beta}}\|\nabla u\|_2^2-\frac{p}{\sqrt{2\beta}}\right)^2\geq 0.$$

By the Gronwall Lemma, we get

$$W(t)\leq W(0)e^{-\alpha_1t}+\frac{C_1}{\alpha_1}(1-e^{-\alpha_1t}).$$

In view of (2.4), we conclude

$$\|(u,v,\eta)\|_{\mathscr{H}}^2=\|\Delta u\|_2^2+\|v\|_2^2+\|\eta\|_{\mu,V}^2\leq\frac{1}{\varrho}W(0)e^{-\alpha_1t}+\frac{C_1}{\alpha_1\varrho}(1-e^{-\alpha_1t}),$$

This shows that any closed ball $\mathscr{B}=\bar{B}(0,R)$ with $R>\sqrt{\dfrac{C_1}{\alpha_1\varrho}}$ is a bounded absorbing set of $(\mathscr{H},S(t))$. □

Remark 4.1. The existence of a bounded absorbing set implies that for initial data lying in bounded sets $B\subset\mathscr{H}$, the solutions of problem (1.6)–(1.9) are globally bounded. More precisely, let (u,u_t,η) be a solution of (1.6)–(1.9) with initial data (u_0,u_1,η_0) in a bounded set B, then one has

$$\|(u(t),u_t(t),\eta^t)\|_{\mathscr{H}}\leq C_B,\quad\forall t\geq 0,\tag{4.6}$$

where $C_B>0$ is a constant depending on B. Lemma 4.1 also ensures the existence of bounded positively invariant sets.

We show an essential inequality to our proof of Theorem 2.2.

LEMMA 4.2. *Under the hypotheses of Theorem 2.2, given a bounded set $B \subset \mathcal{H}$, let $z_1 = (u, u_t, \eta)$ and $z_2 = (v, v_t, \xi)$ be two weak solutions of problem (1.6)–(1.9) such that $z_1(0) = (u_0, u_1, \eta_0)$ and $z_2(0) = (v_0, v_1, \xi_0)$ are in B. Then*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}}^2 \le Ce^{-\alpha_2 t} + C\int_0^t e^{-\alpha_2(t-s)}(\|w_t(s)\|_2^2 + \|w(s)\|_{2(\rho+1)}^2)\, ds, \tag{4.7}$$

for any $t \ge 0$, where $C > 0$ and $\alpha_2 > 0$ are constants.

Proof. Let us fix a bounded set $B \subset \mathcal{H}$. We set $w = u - v$ and $\zeta = \eta - \xi$. Then (w, ζ) satisfy

$$w_{tt} + \alpha w_t + \varrho\Delta^2 w + \int_0^\infty \mu(s)\Delta^2\zeta^t(s)\, ds + \lambda w + (p - \beta\|\nabla u\|_2^2)\Delta u \tag{4.8}$$
$$- (p - \beta\|\nabla v\|_2^2)\Delta v + f(u) - f(v) = 0,$$
$$\zeta_t = -\zeta_s + w_t, \tag{4.9}$$

with initial condition

$$w(0) = u_0 - v_0, \quad w_t(0) = u_1 - v_1, \quad \zeta^0 = \eta_0 - \xi_0.$$

Taking the scalar product in H of (4.8) with $\varphi = w_t + \sigma w$ and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2 + \frac{\varrho}{2}\|\Delta w\|_2^2\right) + (\alpha - \sigma)(w_t, \varphi) + \sigma\varrho\|\Delta w\|_2^2 \tag{4.10}$$
$$+ ((p - \beta\|\nabla u\|_2^2)\Delta u, \varphi) - ((p - \beta\|\nabla v\|_2^2)\Delta v, \varphi) + \lambda(w, \varphi)$$
$$+ (\zeta, w_t)_{\mu, V} + \sigma(\zeta, w)_{\mu, V} + (f(u) - f(v), \varphi) = 0.$$

Noting that similar procedure used in Lemma 4.1 we obtain

$$(\alpha - \sigma)(w_t, \varphi) = (\alpha - \sigma)\|\varphi\|_2^2 - \sigma(\alpha - \sigma)(w, \varphi),$$
$$(\zeta, w_t)_{\mu, V} \ge \frac{1}{2}\frac{d}{dt}\|\zeta\|_{\mu, V}^2 + \frac{k_1}{2}\|\zeta\|_{\mu, V}^2,$$
$$\sigma(\zeta, w)_{\mu, V} \ge -\frac{k_1}{4}\|\zeta\|_{\mu, V}^2 - \frac{\mu_0\sigma^2}{k_1}\|\Delta w\|_2^2.$$

Then

$$(\alpha - \sigma)\|\varphi\|_2^2 - \sigma(\alpha - \sigma)(w, \varphi) + \left(\sigma\varrho - \frac{\mu_0\sigma^2}{k_1}\right)\|\Delta w\|_2^2 \ge \frac{\alpha}{4}\|\varphi\|_2^2 + \frac{\sigma\varrho}{2}\|\Delta w\|_2^2. \tag{4.11}$$

Combining with (4.11), we deduce from (4.10)

$$
\begin{aligned}
(4.12)\qquad & \frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2+\frac{\varrho}{2}\|\Delta w\|_2^2+\frac{1}{2}\|\zeta\|_{\mu,V}^2\right)+\frac{\sigma\varrho}{2}\|\Delta w\|_2^2+\frac{\alpha}{4}\|\varphi\|_2^2 \\
&\quad +\frac{k_1}{4}\|\zeta\|_{\mu,V}^2+((p-\beta\|\nabla u\|_2^2)\Delta u,\varphi)-((p-\beta\|\nabla v\|_2^2)\Delta v,\varphi) \\
&\le -\lambda(w,\varphi)-(f(u)-f(v),\varphi).
\end{aligned}
$$

From the Young inequality, we obtain

$$
\begin{aligned}
(4.13)\qquad |-\lambda(w,\varphi)| &\le \frac{\lambda^2}{\sigma}\|w\|_2^2+\frac{\sigma}{4}\|\varphi\|_2^2 \\
&\le \frac{\lambda^2 c_0}{\sigma}\|w\|_{2(\rho+1)}^2+\frac{\sigma}{4}\|\varphi\|_2^2,
\end{aligned}
$$

where $c_0>0$ is an embedding constant for $L^{2(\rho+1)}(\Omega)\hookrightarrow L^2(\Omega)$.

Using generalized Hölder inequality with $\dfrac{\rho}{2(\rho+1)}+\dfrac{1}{2(\rho+1)}+\dfrac{1}{2}=1$, assumption (2.1), estimate (4.6) and Young inequality, we have

$$
\begin{aligned}
&\left|-\int_\Omega (f(u(t))-f(v(t)))\varphi(t)\,dx\right| \\
&\quad\le k_0\int_\Omega (1+|u(t)|^\rho+|v(t)|^\rho)|w(t)|\,|\varphi(t)|\,dx \\
&\quad\le k_0(|\Omega|^{\rho/(2(\rho+1))}+\|u\|_{2(\rho+1)}^\rho+\|v\|_{2(\rho+1)}^\rho)\|w\|_{2(\rho+1)}\|\varphi\|_2 \\
&\quad\le C_B\|w\|_{2(\rho+1)}\|\varphi\|_2 \\
&\quad\le \frac{C_B^2}{\sigma}\|w\|_{2(\rho+1)}^2+\frac{\sigma}{4}\|\varphi\|_2^2.
\end{aligned}
$$

Now we estimate $((p-\beta\|\nabla u\|_2^2)\Delta u,\varphi)-((p-\beta\|\nabla v\|_2^2)\Delta v,\varphi)$.

Setting

$$
\begin{aligned}
(4.14)\qquad & ((p-\beta\|\nabla u\|_2^2)\Delta u,\varphi)-((p-\beta\|\nabla v\|_2^2)\Delta v,\varphi) \\
&= \int_\Omega [(p-\beta\|\nabla u\|_2^2)\Delta u-(p-\beta\|\nabla v\|_2^2)\Delta v]\varphi(t)\,dx \\
&= I_1+I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_\Omega (p\Delta u-p\Delta v)\varphi(t)\,dx, \\
I_2 &= \int_\Omega (-\beta\|\nabla u\|_2^2\Delta u+\beta\|\nabla v\|_2^2\Delta v)\varphi(t)\,dx.
\end{aligned}
$$

We derive from Hölder inequality, Young inequality and the estimate (4.6) the following estimates:

$$\begin{aligned}
|I_1| &= \left|\int_\Omega (p\Delta u - p\Delta v)\varphi(t)\,dx\right| \\
&\le \int_\Omega |p|\,|\Delta w|\,|\varphi(t)|\,dx \\
&\le \sigma^2 p^2 \|\Delta w\|_2^2 + \frac{1}{4\sigma^2}\|\varphi\|_2^2 \\
&\le \sigma^2 p^2 \|\Delta w\|_2^2 + \frac{1}{4\sigma^2}\|w_t + \sigma w\|_2^2 \\
&\le \sigma^2 p^2 \|\Delta w\|_2^2 + \frac{1}{2\sigma^2}\|w_t\|_2^2 + \frac{1}{2}\|w\|_2^2 \\
&\le \sigma^2 p^2 \|\Delta w\|_2^2 + \frac{1}{2\sigma^2}\|w_t\|_2^2 + \frac{c_0}{2}\|w\|_{2(\rho+1)}^2,
\end{aligned}$$

$$\begin{aligned}
|I_2| &= \left|\int_\Omega (\beta\|\nabla u\|_2^2\Delta u - \beta\|\nabla v\|_2^2\Delta v)\varphi(t)\,dx\right| \\
&\le \int_\Omega |(\beta\|\nabla u\|_2^2\Delta u - \beta\|\nabla u\|_2^2\Delta v + \beta\|\nabla u\|_2^2\Delta v - \beta\|\nabla v\|_2^2\Delta v)\varphi(t)|\,dx \\
&\le \beta\int_\Omega \|\nabla u\|_2^2|\Delta w|\,|\varphi(t)|\,dx + \beta\int_\Omega (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)|\Delta v|\,|\varphi(t)|\,dx \\
&\le \frac{C_B\beta}{2}\int_\Omega |\Delta w|\,|\varphi(t)|\,dx + \sqrt{\frac{C_B}{2}}\beta\int_\Omega |\Delta v|\,|\varphi(t)|\,dx \\
&\le \frac{C_B\beta}{2}\|\Delta w\|_2\|\varphi\|_2 + \sqrt{\frac{C_B}{2}}\beta\|\Delta v\|_2\|\varphi\|_2 \\
&\le C_B\beta\left(\sigma^2\|\Delta w\|_2^2 + \frac{1}{4\sigma^2}\|\varphi\|_2^2\right) \\
&\le C_B\beta\sigma^2\|\Delta w\|_2^2 + \frac{C_B\beta}{2\sigma^2}\|w_t\|_2^2 + \frac{C_B\beta c_0}{2}\|w\|_{2(\rho+1)}^2,
\end{aligned}$$

where we have used the fact that $\|\triangle v\|_2 = \|\triangle u - \triangle w\|_2 \le \|\triangle u\|_2 + \|\triangle w\|_2 \le \sqrt{\frac{C_B}{2}}\|\triangle w\|_2$. Inserting above two inequalities into (4.14) we obtain

$$\begin{aligned}
&((p - \beta\|\nabla u\|_2^2)\Delta u, \varphi) - ((p - \beta\|\nabla v\|_2^2)\Delta v, \varphi) \\
&\quad \ge -(\sigma^2 p^2 + C_B\beta\sigma^2)\|\Delta w\|_2^2 - \left(\frac{1}{2\sigma^2} + \frac{C_B\beta}{2\sigma^2}\right)\|w_t\|_2^2 \\
&\quad\quad - \left(\frac{c_0}{2} + \frac{C_B\beta c_0}{2}\right)\|w\|_{2(\rho+1)}^2.
\end{aligned} \tag{4.15}$$

Combining with (4.13), (4.14) and (4.15), we deduce from (4.12)

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\varphi\|_2^2+\frac{\varrho}{2}\|\Delta w\|_2^2+\frac{1}{2}\|\zeta\|_{\mu,V}^2\right)\\
&\qquad+\left[\frac{\sigma\varrho}{2}-\sigma^2p^2-C_B\beta\sigma^2\right]\|\Delta w\|_2^2+\left(\frac{\alpha}{4}-\frac{\sigma}{2}\right)\|\varphi\|_2^2+\frac{k_1}{4}\|\zeta\|_{\mu,V}^2\\
&\qquad\le\left(\frac{1}{2\sigma^2}+\frac{C_B\beta}{2\sigma^2}\right)\|w_t\|_2^2+\left(\frac{\lambda^2c_0}{\sigma}+\frac{C_B^2}{\sigma}+\frac{c_0}{2}+\frac{C_B\beta c_0}{2}\right)\|w\|_{2(\rho+1)}^2.
\end{aligned}$$

We can choose σ so small such that

$$\frac{\sigma\varrho}{2}-\sigma^2p^2-C_B\beta\sigma^2=\sigma\left(\frac{\varrho}{2}-\sigma p^2-C_B\beta\sigma\right)>0,\quad \frac{\alpha}{4}-\frac{\sigma}{2}>0.$$

Then we set

$$E_W(t)=\|\varphi\|_2^2+\varrho\|\Delta w\|_2^2+\|\zeta\|_{\mu,V}^2,$$

and we have

$$\frac{d}{dt}E_W(t)+\alpha_2E_W(t)\le C_2(\|w_t\|_2^2+\|w\|_{2(\rho+1)}^2),$$

where

$$\alpha_2=\min\left\{\sigma-\frac{2\sigma^2p^2}{\varrho}-\frac{2C_B\beta\sigma^2}{\varrho},\frac{\alpha}{2}-\sigma,\frac{k_1}{2}\right\},$$

and

$$C_2=\max\left\{\frac{1}{\sigma^2}+\frac{C_B\beta}{\sigma^2},\frac{2\lambda^2c_0}{\sigma}+\frac{2C_B^2}{\sigma}+c_0+C_B\beta c_0\right\}.$$

By the Gronwall Lemma, we get

$$E_W(t)\le E_W(0)e^{-\alpha_2t}+C_2\int_0^te^{-\alpha_2(t-s)}(\|w_t(s)\|_2^2+\|w(s)\|_{2(\rho+1)}^2)\,ds.$$

Since

$$\begin{aligned}
\|z_1(t)-z_2(t)\|_{\mathscr{H}}^2&=\|\Delta w\|_2^2+\|w_t\|_2^2+\|\zeta\|_{\mu,V}^2\\
&=\|\Delta w\|_2^2+\|\varphi-\sigma w\|_2^2+\|\zeta\|_{\mu,V}^2\\
&\le\|\Delta w\|_2^2+2\|\varphi\|_2^2+2\sigma^2\|w\|_2^2+\|\zeta\|_{\mu,V}^2\\
&\le\left(1+\frac{2\sigma^2}{\lambda_1}\right)\|\Delta w\|_2^2+2\|\varphi\|_2^2+\|\zeta\|_{\mu,V}^2\\
&\le C_3(\|\varphi\|_2^2+\varrho\|\Delta w\|_2^2+\|\zeta\|_{\mu,V}^2),
\end{aligned}$$

where $C_3=\max\left\{\frac{1}{\varrho}+\frac{2\sigma^2}{\varrho\lambda_1},2\right\}$.

Namely

$$\|z_1(t) - z_2(t)\|_{\mathscr{H}}^2 \le C_3 E_W(t).$$

Hence

$$\|z_1(t) - z_2(t)\|_{\mathscr{H}}^2 \le C_3 E_W(0) e^{-\alpha_2 t} + C_2 C_3 \int_0^t e^{-\alpha_2 (t-s)} (\|w_t(s)\|_2^2 + \|w(s)\|_{2(\rho+1)}^2)\, ds,$$

we have (4.7) with $C = \max\{C_3 E_W(0), C_2 C_3\}$. $\square$

Lemma 4.3. *Under assumptions of Theorem* 2.2, *the dynamical system* $(\mathscr{H}, S(t))$ *is asymptotically smooth.*

Proof. Let B be a bounded subset of $\mathscr{H}$ positively invariant with respect to $S(t)$. Denote by C_B several positive constants that are dependent on B but not on t. For $z_0^1, z_0^2 \in B$, $S(t)z_0^1 = (u(t), u_t(t), \eta^t)$ and $S(t)z_0^2 = (v(t), v_t(t), \xi^t)$ are the solutions of (1.6)–(1.9). Then given $\varepsilon > 0$, from inequality (4.7), we take $T > 0$ such that $Ce^{(-\alpha_2 T)/2} < \varepsilon$ and

$$\begin{aligned} (4.16) \quad & \|S(T)z_0^1 - S(T)z_0^2\|_{\mathscr{H}} \\ & \le \varepsilon + C_B \left(\int_0^T (\|u(s) - v(s)\|_{2(\rho+1)}^2 + \|(u_t(s) - v_t(s))\|_2^2)\, ds \right)^{1/2}, \end{aligned}$$

where $C_B > 0$ is a constant which depends only on the size of B.

Now we note that condition (2.2) implies that $2 < 2(\rho+1) < \infty$ if $1 \le N \le 4$ and $2 < 2(\rho+1) \le \dfrac{2N}{N-4}$ if $N \ge 5$. Taking $\theta = \dfrac{N}{4}\left(1 - \dfrac{1}{\rho+1}\right)$ we obtain from interpolation theorem

$$\|u(t) - v(t)\|_{2(\rho+1)} \le C \|\Delta(u(t) - v(t))\|_2^{\theta} \|u(t) - v(t)\|_2^{1-\theta} \le C_B \|u(t) - v(t)\|_2^{1-\theta}.$$

Since $\|\Delta u(t)\|_2$ and $\|\Delta v(t)\|_2$ are uniformly bounded, there exists a constant $C_B > 0$ such that

$$\|u(t) - v(t)\|_{2(\rho+1)}^2 \le C_B \|u(t) - v(t)\|_2^{2(1-\theta)}.$$

Then we can rewrite (4.16) as

$$\|S(T)z_0^1 - S(T)z_0^2\|_{\mathscr{H}} \le \varepsilon + \Phi_T(z_0^1, z_0^2),$$

with

$$\Phi_T(z_0^1, z_0^2) = C_B \left(\int_0^T (\|u(s) - v(s)\|_2^{2(1-\theta)} + \|(u_t(s) - v_t(s))\|_2^2)\, ds \right)^{1/2}.$$

Let us show that Φ_T satisfies (3.1). Indeed, given a sequence of initial data $z_n = (u_0^n, u_1^n, \eta_0^n)$ in B, as before, we write $S(t)z_n = (u^n(t), u_t^n(t), \eta^{n,t})$. Since B is invariant by $S(t)$, $t \geq 0$, it follows that $(u^n(t), u_t^n(t), \eta^{n,t})$ are uniformly bounded in $\mathscr{H}$. In particular,

$$(u^n, u_t^n) \text{ is bounded in } C([0,T], V \times H), \quad T > 0.$$

Then from the compact embedding of V into H, the Aubins lemma (see Simon [17] (Corollary 4)) implies that there exist subsequences $\{u^{n_k}\}$ and $\{u_t^{n_k}\}$ that converge strongly in $C([0,T], H)$. Therefore, we see that

$$\lim_{k\to\infty} \lim_{l\to\infty} \int_0^T (\|u^{n_k}(s) - u^{n_l}(s)\|_2^{2(1-\theta)} + \|(u_t^{n_k}(s) - u_t^{n_l}(s))\|_2^2)\, ds = 0,$$

and consequently (3.1) holds. The asymptotic smoothness property of $(\mathscr{H}, S(t))$ follows from Theorem 3.2. $\square$

Proof of Theorem 2.2. We note that Lemmas 4.1 and 4.3 imply that $(\mathscr{H}, S(t))$ is a dissipative dynamical system which is asymptotically smooth. Then from Theorem 3.1 it has compact global attractor $\mathscr{A}$ in the phase space $\mathscr{H}$. $\square$

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