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ORDERS OF MEROMORPHIC MAPPINGS INTO HOPF AND INOUE SURFACES

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Abstract

In a late paper of J. Noguchi and J. Winkelmann [7] (J. Math. Soc. Jpn., Vol. **64** No. 4 (2012), 1169–1180) they gave the first instance where Kähler or non-Kähler conditions of the image spaces make a difference in the value distribution theory. In this paper, we will investigate orders of meromorphic mappings into a Hopf surface which is more general than dealt with by Noguchi-Winkelmann, and an Inoue surface. They are non-Kähler surfaces and belong to VII_0-class. For a general Hopf surface S, we prove that there exists a differentiably non-degenerate holomorphic mapping $f : \mathbf{C}^2 \to S$ with order at most one. For any Inoue surface S', we prove that every non-constant meromorphic mapping $f : \mathbf{C}^n \to S'$ is holomorphic and its order satisfies $\rho_f \geq 2$.

1. Main results

In Nevanlinna theory, there are many studies on value distributions of meromorphic mappings whose image spaces are Kähler, especially complex projective algebraic manifolds. On the other hand, however, little are known for non-Kähler cases. The first instance where Kähler or non-Kähler conditions of the image spaces make a difference in the value distribution theory was given by J. Noguchi and J. Winkelmann [7]. They proved a theorem on order of meromorphic mappings and rationality of the image space under a Kähler condition, and showed that without the Kähler condition, there is a counter-example by constructing a holomorphic mapping to a special Hopf surface with low order. The purpose of this paper is to investigate orders of meromorphic mappings into Hopf surfaces which are more general than dealt with by them and Inoue surfaces. They are non-Kähler surfaces and belong to VII_0-class. The two main theorems are as follows.

Main Theorem 1.1. *Let $S_{a,b}$ be a Hopf surface defined by the action,*

$$n : (x, y) \in \mathbf{C}^2 \setminus \{(0,0)\} \mapsto (a^n x, b^n y) \in \mathbf{C}^2 \setminus \{(0,0)\}, \quad n \in \mathbf{Z},$$

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where a, b are complex numbers with $|a|, |b| > 1$. Then there exists a differentiably non-degenerate holomorphic mapping $f : \mathbf{C}^2 \to S_{a,b}$ with order at most one.

N.B. In general, whether there exists a differentiably non-degenerate meromorphic mapping to a compact complex manifold with order less than two or not are big difference. Because if there is such a map, every global covariant holomorphic tensor on the manifold must vanish [7].

Main Theorem 1.2. *Let S be an Inoue surface. Let $n \geq 1$ be an arbitrary natural number. Then every non-constant meromorphic mapping $f : \mathbf{C}^n \to S$ is holomorphic and its order satisfies $\rho_f \geq 2$. In addition to this, when $n \geq \dim_{\mathbf{C}} S (= 2)$, f is differentiably degenerate.*

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2. Preliminaries

2.1. Notation

We fix the following notation.

- Let $f : \mathbf{C}^n \to X$ be a meromorphic mapping to a complex manifold. We denote by $I(f)$ the *indeterminancy locus of* f.
- If the *rank*(df) is equal to the dimension of the image space generically, f is said to be *differentiably non-degenerate*.
- For $z = (z_j) \in \mathbf{C}^n$, we set

$$\alpha = dd^c \|z\|^2, \tag{2.1}$$

$$\zeta = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1}, \tag{2.2}$$

  where $d^c = \dfrac{i}{4\pi}(\bar{\partial} - \partial)$ and $\|z\|^2 = \sum_{j=1}^n |z_j|^2$.
- $B(r) = \{z \in \mathbf{C}^n : \|z\| < r\}$, $S(r) = \{z \in \mathbf{C}^n : \|z\| = r\}$ $(r > 0)$.

Definition 2.3. Let $f : \mathbf{C}^n \to X$ be a meromorphic mapping to a compact complex manifold and let ω be a Hermitian metric form on X. We define a function

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}$$

which is called the characteristic function of f with respect to ω.

Definition 2.4. In above setting we define the order of f as follows,

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r,\omega)}{\log r}. \tag{2.5}$$

Since X is compact, ρ_f is independent of the choice of a Hermitian metric form ω on X.

2.2. Relations between orders and one-dimensional image spaces

Possible values of orders are affected by image spaces. To get a better comprehension of our results, we recall the following facts of one dimensional case.

Fact 2.6. *Let X be a closed Riemann surface of genus g.*

(i) *Let $g \geq 2$ and let f be a holomorphic mapping from* $\mathbf{C}$ *into X. Then f is constant.*

(ii) *Let $g = 1$ and let f be a non-constant holomorphic mapping from* $\mathbf{C}$ *into X. Then the order satisfies $\rho_f \geq 2$.*

(iii) *Let $g = 0$ and let $s \geq 0$ be a given real number. Then there exists a non-constant holomorphic mapping $f : \mathbf{C} \to X$ with order s.* ([5], *Theorem* 7.5.9, *p.* 241.)

Here it is noted that every meromorphic mapping from $\mathbf{C}$ *into a compact complex manifold is holomorphic since* codim $I(f) \geq 2$ ($I(f) = \emptyset$ *in this case*).

2.3. Difference between Kähler and non-Kähler surfaces

J. Noguchi and J. Winkelmann proved the following theorems, giving the first instance where Kähler or non-Kähler conditions of image spaces make a difference in value distribution.

Theorem 2.7 (J. Noguchi-J. Winkelmann [7]). *Let X be a compact Kähler surface. Assume that there is a differentiably non-degenerate meromorphic mapping $f : \mathbf{C}^2 \to X$. If $\rho_f < 2$, then X is rational.*

The Kähler condition is necessary by the following:

Theorem 2.8 (J. Noguchi-J. Winkelmann [7]). *Let a be a complex number with $|a| > 1$. Let $S_{a,a}$ be a Hopf surface defined as the quotient of $\mathbf{C}^2 \backslash \{(0,0)\}$ by a* $\mathbf{Z}$*-action $n : (x,y) \mapsto (a^n x, a^n y)$. Then there exists a differentiably non-degenerate holomorphic mapping $f : \mathbf{C}^2 \to S_{a,a}$ such that $\rho_f \leq 1$.*

3. General Hopf surfaces: Proof of Main Theorem 1.1

Our Main Theorem 1.1 asserts that Theorem 2.8 still holds for more general Hopf surfaces.

Proof. We may assume $1 < |b| \le |a|$ without loss of generality. We prove the theorem in two steps. In the first step, we prove it under the additional condition that $|a|$ and $|b|$ are close to each other. Under this condition, we can apply some estimates introduced by J. Noguchi-J. Winkelmann [7] to prove the case of $a = b$. In the second step, we construct a branched covering to remove this additional condition.

3.1. The first step

We assume that

$$1 < |b| \le |a| \le |b|^{3/2}. \tag{3.1}$$

We prove under this condition that the holomorphic mapping $f : \mathbf{C}^2 \to S_{a,b}$ induced by

$$\tilde{f} : \mathbf{C}^2 \to \mathbf{C}^2 \backslash \{(0,0)\} \quad (z, w) \mapsto (z, 1 + zw)$$

is diffetentiably non-degenerate and its order satisfies $\rho_f \le 1$.

Let α be as in (2.1). Setting $\gamma = \dfrac{\log|a|}{\log|b|} - 1$ and $\delta = 1 - \dfrac{\log|b|}{\log|a|}$, we have $0 \le \delta \le \gamma \le \frac{1}{2}$ by (3.1). We define a continuous positive Hermitian form on $\mathbf{C}^2 \backslash \{(0,0)\}$ which is invariant under the above $\mathbf{Z}$-action as follows,

$$\tilde{\omega} = \frac{i}{2\pi} \cdot \frac{dx \wedge d\bar{x} + (|y|^{2\gamma} + |x|^{2\delta})\, dy \wedge d\bar{y}}{|x|^2 + (|y|^{2\gamma} + |x|^{2\delta})|y|^2},$$

and denote by ω the induced continuous positive Hermitian form on the quotient space $S_{a,b}$. Although this induced Hermitian form is not always smooth Hermitian metric form, it it sufficient to calculate orders by the compactness of $S_{a,b}$.

We show the following inequality

$$\varlimsup_{r \to \infty} \frac{1}{\log r} \log \int_1^r \frac{dt}{t^3} \int_{B(t)} f^*\omega \wedge \alpha \le 1.$$

Note that

$$f^*\omega \wedge \alpha = \frac{1 + (|z|^2 + |w|^2)(|1 + zw|^{2\gamma} + |z|^{2\delta})}{|z|^2 + |1 + zw|^2(|1 + zw|^{2\gamma} + |z|^{2\delta})} \alpha^2.$$

We define

$$I_r' = \int_{S(r)} \frac{r^2 + \dfrac{1}{|1 + zw|^{2\gamma} + |z|^{2\delta}}}{\dfrac{|z|^2}{|1 + zw|^{2\gamma} + |z|^{2\delta}} + |1 + zw|^2}\, dV, \quad r = \|(z, w)\|,$$

$$I_r = \int_{S(r)} \frac{r^2}{\dfrac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1+zw|^2} \, dV, \quad r = \|(z,w)\|.$$

Here dV is the euclidean volume element on $S(r)$.

Then we have

$$I_r' \le 2I_r \tag{3.2}$$

for all large r. Indeed,

- When $|z| \ge r^{-1/\delta}$, we have $|1+zw|^{2\gamma} + |z|^{2\delta} \ge |z|^{2\delta} \ge r^{-2}$.
- When $|z| \le r^{-1/\delta}$, we have $|zw| \le r^{1-1/\delta} \le r^{-1}$. This implies

  $$|1+zw|^{2\gamma} + |z|^{2\delta} \ge |1+zw|^{2\gamma} \ge (1-r^{1-1/\delta})^{2\gamma} \ge (1-r^{1-1/\delta}) \ge r^{-2}$$

  for all large r.

In both cases, $\dfrac{1}{|1+zw|^{2\gamma} + |z|^{2\delta}} \le r^2$ for all large r, implying (3.2). Hence it is sufficient to show

$$I_r = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0. \tag{3.3}$$

In fact, from this and (3.2), we obtain

$$\int_{B(r)} \frac{r^2 + \dfrac{1}{|1+zw|^{2\gamma} + |z|^{2\delta}}}{\dfrac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1+zw|^2} \alpha^2 = O\left(\int^r I_r' \, dr\right) = O(r^{3+\varepsilon}), \quad \forall \varepsilon > 0,$$

implying

$$T_f(r,\omega) = \int_1^r \frac{dt}{t^3} \int_{B(t)} \frac{r^2 + \dfrac{1}{|1+zw|^{2\gamma} + |z|^{2\delta}}}{\dfrac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1+zw|^2} \alpha^2 = O(r^{1+\varepsilon}), \quad \forall \varepsilon > 0,$$

implying

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r,\omega)}{\log r} \le 1.$$

To show (3.3), we set

$$\eta = \frac{r^2}{\dfrac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1+zw|^2}.$$

To estimate $I_r = \int_{S(r)} \eta \, dV$, we divide $S(r)$ into eleven regions, A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H which are defined later, and estimate the volume and the integrand on each region. We introduce some geometric and arithmetic estimates used in [7].

Geometric estimates.

For $(z,w)\in\mathbf{C}^2$ with $z\neq 0$ and $w\neq 0$, set $\theta\in[0,2\pi)$ by $e^{i\theta}|zw|=zw$. For $K>0$, $-\infty<\lambda<1$ and $\mu\geq 0$, we set

$$\Omega_{K,\lambda,\mu}=\{(z,w)\in S(r)\mid z=0 \text{ or } (0<|z|\leq Kr^{\lambda}, |\sin\theta|\leq r^{-\mu})\}.$$

We define a mapping $\Phi:\mathbf{C}^2\backslash\{z=0 \text{ or } w=0\}\to\mathbf{C}\times\mathbf{R}^2$ as follows,

$$\Phi:(z,w)\mapsto(z, r\arg(zw), r)$$

where $r=\|(z,w)\|=\sqrt{|z|^2+|w|^2}$. To show the Jacobian of Φ is identically -1 we set $z=x+\sqrt{-1}y$, $w=u+\sqrt{-1}v$ and write Φ with real coordinates as follows,

$$\Phi:(x,y,u,v)\mapsto(x,y,r(\arg z+\arg w),r)\in\mathbf{R}^4$$

with $r=\sqrt{x^2+y^2+u^2+v^2}$. The Jacobian of Φ is

$$\begin{aligned}|J_\Phi|&=\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ * & * & \dfrac{u}{r}(\arg z+\arg w)+r\dfrac{\partial}{\partial u}\arg w & \dfrac{v}{r}(\arg z+\arg w)+r\dfrac{\partial}{\partial v}\arg w\\ * & * & \dfrac{u}{r} & \dfrac{v}{r}\end{vmatrix}\\ &=\begin{vmatrix} r\dfrac{\partial}{\partial u}\arg w & \dfrac{u}{r}\\ r\dfrac{\partial}{\partial v}\arg w & \dfrac{v}{r}\end{vmatrix}\\ &\equiv -1\end{aligned}$$

Furthermore the gradient $\mathrm{grad}(r)$ is of length one and normal on the level set $S(r)$.

Hence the euclidean volume of $\Omega_{K,\lambda,\mu}$ is the same as the euclidean volume of

$$\{z\in\mathbf{C}:|z|\leq Kr^{\lambda}\}\times\{\theta r:\theta\in[0,2\pi), |\sin\theta|\leq r^{-\mu}\}\times\{r\}$$

becuase

$$\begin{aligned}\mathrm{vol}(\Omega_{K,\lambda,\mu})&=\mathrm{vol}(\Omega_{K,\lambda,\mu}\backslash\{z=0\})\\ &=\mathrm{vol}(\Phi(\Omega_{K,\lambda,\mu}\backslash\{z=0\}))\\ &=\mathrm{vol}(\{z\in\mathbf{C}:0<|z|\leq Kr^{\lambda}\}\times\{\theta r:\theta\in[0,2\pi), |\sin\theta|\leq r^{-\mu}\}\times\{r\})\\ &=\mathrm{vol}(\{z\in\mathbf{C}:|z|\leq Kr^{\lambda}\}\times\{\theta r:\theta\in[0,2\pi), |\sin\theta|\leq r^{-\mu}\}\times\{r\})\end{aligned}$$

Using $\sin(\theta) \geq \frac{2}{\pi}\theta \ \left(\theta \in \left[0, \frac{\pi}{2}\right)\right)$, it follows that for $r \geq 1$ the volume of $\Omega_{K,\lambda,\mu}$ is bounded from above by

$$\pi(Kr^{\lambda})^2 \cdot 2r^{-\mu}\pi r = 2K^2\pi^2 r^{2\lambda+1-\mu}.$$

In particular,

$$\mathrm{vol}(\Omega_{K,\lambda,\mu}) = O(r^{2\lambda+1-\mu}). \tag{3.4}$$

Arithmetic estimates.

Besides the Landau O-symbols we also use the notation $\gtrsim$: If f, g are functions of a real parameter r, then $f(r) \gtrsim g(r)$ indicates that

$$\varliminf_{r\to\infty} \frac{f(r)}{g(r)} \geq 1.$$

Similarly $f \sim g$ indicates

$$\lim_{r\to\infty} \frac{f(r)}{g(r)} = 1.$$

In the sequel, we will work with domains $\Omega \subset S(r)$ (i.e. for each $r > 0$ some subset $\Omega = \Omega_r \subset S(r)$ is chosen). In this context, given functions f, g on $\mathbf{C}^2$ we say $f(z,w) \gtrsim g(z,w)$ holds on Ω if for every sequence $(z_n, w_n) \in \Omega_r$ $(r = \|(z_n, w_n)\|)$ with

$$\lim_{n\to\infty} \|(z_n, w_n)\| = +\infty$$

and we have

$$\varliminf_{n\to\infty} \frac{f(z_n, w_n)}{g(z_n, w_n)} \geq 1.$$

We show some estimates for $\eta = \dfrac{r^2}{\dfrac{|z|^2}{|1+zw|^{2\gamma} + |z|^{2\delta}} + |1+zw|^2}$. Fix $-\infty < \lambda < 1$.

(i) Suppose $(z,w) \in S(r)$ and $|z| \leq \dfrac{1}{2r}$. Since $|w| \leq r$, we have $|zw| \leq \frac{1}{2}$, implying $|1+zw| \geq \frac{1}{2}$. Therefore $\eta \leq \dfrac{r^2}{|1+zw|^2} \leq 4r^2$.

(ii) Suppose $|z| \leq r^{\lambda}$. Then we have $|w| \sim r$.

(iii) Suppose $|z| \geq \dfrac{3}{2r}$ and $|z| \leq r^{\lambda}$. Using (ii), we obtain $|zw| \gtrsim \frac{3}{2}$ (equivalently, $1 \lesssim \frac{2}{3}|zw|$), which implies $|1+zw| \geq |zw| - 1 \gtrsim \frac{1}{3}|zw|$. Hence $\eta \leq \dfrac{cr^2}{|zw|^2}$. (Here c is a positive constant greater than nine)

(iv) For all z and w, $\dfrac{|z|^2}{|1+zw|^{2\gamma}+|z|^{2\delta}}+|1+zw|^2 \geq |\mathrm{Im}(1+zw)|^2 = (|zw|\sin\theta)^2$.

Estimates on each regions.

We are going to prove the following claim

$$I_r = O(r^{2+\varepsilon}), \quad \forall \varepsilon > 0$$

by dividing $S(r)$ into eleven regions A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H, each of which is investigated separately.

- $A = \left\{(z,w) \in S(r) \,\middle|\, |z| \leq \dfrac{1}{2r}\right\}$, i.e., $A = \Omega_{1/2,-1,0}$. By (3.4), we have $\mathrm{vol}(A) = O(r^{-1})$. Due to (i), restriction of integrand η to A is $\eta|_A = O(r^2)$. Thus

$$\int_A \eta\, dV \leq \mathrm{vol}(A) \cdot \sup_{(z,w)\in A} \eta(z,w) \leq O(r).$$

Hence the contribution of A to the integral $I_r = \int_{S(r)} \eta\, dV$ is bounded by $O(r)$.

- $B = \left\{(z,w) \in S(r) \,\middle|\, \dfrac{1}{2r} \leq |z| \leq \dfrac{3}{2r} \text{ and } |\sin\theta| < \dfrac{1}{r}\right\}$. Thus $B \subset \Omega_{3/2,-1,1}$. Due to (3.4), we have $\mathrm{vol}(B) = O(r^{-2})$. Since $|zw| \leq \frac{3}{2}$, the function $|1+zw|^{2\gamma}$ is bounded on B. Therefore we obtain

$$\eta|_B \leq \frac{r^2}{|z|^2}(|1+zw|^{2\gamma}+|z|^{2\delta}) = O(r^4).$$

At the last estimate we used the inequality $|z| \geq \dfrac{1}{2r}$. Hence we have

$$\int_B \eta\, dV \leq \mathrm{vol}(B) \cdot \sup_{(z,w)\in B} \eta(z,w) = O(r^2),$$

which implies the contribution of B to the integral I_r is bounded by $O(r^2)$.

- $C = \left\{(z,w) \in S(r) \,\middle|\, \dfrac{1}{2r} \leq |z| \leq \dfrac{3}{2r} \text{ and } |\sin\theta| > \dfrac{1}{r}\right\}$. Then its image by Φ is

$$\Phi(C) = \left\{z \in \mathbf{C} \,\middle|\, \frac{1}{2r} \leq |z| \leq \frac{3}{2r}\right\} \times \left\{\theta r \,\middle|\, \theta \in [0,2\pi), |\sin\theta| > \frac{1}{r}\right\} \times \{r\}.$$

For $z \in \mathbf{C}$ with $\dfrac{1}{2r} \leq |z| \leq \dfrac{3}{2r}$, we define

$$J_r(z) := \int_{0<\theta<2\pi,\, |\sin\theta|>1/r} \eta(\Phi^{-1}(z, r\theta, r))r\, d\theta.$$

Since $|w| \sim r$, we obtain $\frac{1}{2} \lesssim |zw| \lesssim \frac{3}{2}$. Using arithmetic estimate (iv), we get

$$\eta \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{|\sin^2 \theta|\,|zw|^2} \leq \frac{c \cdot r^2}{|\sin^2 \theta|}.$$

Here c is a constant greater than four. Hence we obtain

$$\begin{aligned} J_r(z) &\leq \int_{0<\theta<2\pi,\,|\sin \theta|>1/r} \frac{c \cdot r^2}{|\sin^2 \theta|} r \, d\theta \\ &= 4\int_{\arcsin 1/r}^{\pi/2} \frac{c \cdot r^3}{|\sin^2 \theta|} \, d\theta = 4c \cdot r^4 \sqrt{1 - \frac{1}{r^2}} \leq 4c \cdot r^4. \end{aligned}$$

Therefore it follows that

$$\int_C \eta \, dV = \int_{1/2r \leq |z| \leq 3/2r} J_r \frac{\sqrt{-1}}{2} \, dz \wedge d\bar{z} \leq c' r^2 \tag{3.5}$$

where c' is a positive constant. Thus the contribution of C to the integral I_r is bounded by $O(r^2)$.

- For $n \in \{-2, -1, 0, 1\}$, set $D_n = \left\{ (z, w) \in S(r) \,\middle|\, |z| \geq \frac{3}{2r}, |z| \leq r^{1-\varepsilon} \text{ and } r^{n/2} \leq |z| \leq r^{(n+1)/2} \right\}$. For each n, the integrand η is bounded by $O(r^{-n})$ on D_n due to (ii) and (iii), and $\mathrm{vol}(D_n) = O(r^{2+n})$ because $D_n \subset \Omega_{1,(n+1)/2,0}$. Thus the contribution of D_n to the integral I_r is bounded by $O(r^2)$.
- $E = \{(z, w) \in S(r) \mid |z| \geq r^{1-\varepsilon}, |w| \geq r^{1/2}\}$. Since $|zw| \geq r^{3/2-\varepsilon}$, we have

$$\eta|_E \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{(|zw|-1)^2} \leq \frac{r^2}{(r^{3/2-\varepsilon}-1)^2} = O(r^{2\varepsilon-1}).$$

Because $\mathrm{vol}(E)$ is bounded by the total volume of $S(r)$, $\mathrm{vol}(E) = O(r^3)$. Thus the contribution of E to I_r is bounded by $O(r^{2+2\varepsilon})$.

- $F = \{(z, w) \in S(r) \mid 1 \leq |w| \leq r^{1/2}\}$. Since $|z| = \sqrt{r^2 - |w|^2} \geq \sqrt{r^2 - r} > 1$, we have

$$\eta|_F \leq \frac{r^2}{|1+zw|^2} \leq \frac{r^2}{(\sqrt{r^2-r}-1)^2} = O(1).$$

Because the volume of F agrees with the volume of $\{(z, w) \in S(r) \mid 1 \leq |z| \leq r^{1/2}\} \subset \Omega_{1,1/2,0}$, we obtain

$$\mathrm{vol}(F) \leq \mathrm{vol}(\Omega_{1,1/2,0}) = O(r^2).$$

Thus the contribution of F to I_r is bounded by $O(r^2)$.

- $G = \{(z,w) \in S(r) \mid r^{-1} \le |w| \le 1\}$. Since $|z| \le r$, we have $|zw| \le r$. This implies $|1+zw|^{2\gamma} \le (r^2+2r+1)^{\gamma}$. Hence we obtain
$$\eta|_G \le \frac{r^2}{|z|^2}(|1+zw|^{2\gamma}+|z|^{2\delta}) \le O(r^{2\gamma}) \le O(r).$$
Here we used $|z| \sim r$ and $0 \le \delta \le \gamma \le \frac{1}{2}$. Because $\mathrm{vol}(G) \le \mathrm{vol}(\Omega_{1,0,0}) = O(r)$, the contribution of G to I_r is bounded by $O(r^2)$.
- $H = \{(z,w) \in S(r) \mid 0 \le |w| \le r^{-1}\}$. Since $|w| \le r^{-1}$, we have $|z| \sim r$ and $|zw| \le 1$. Hence we obtain
$$\eta|_H \le \frac{r^2}{|z|^2}(|1+zw|^{2\gamma}+|z|^{2\delta}) \le O(r^{2\delta}) \le O(r).$$
Because $\mathrm{vol}(H) \le O(r^{-1})$, the contribution of H to the integral I_r is bounded by $O(1)$.

Eleven regions A, B, C, D_{-2}, D_{-1}, D_0, D_1, E, F, G, H cover the sphere $S(r)$. On each such region Ω we have verified
$$\int_{\Omega} \eta \, dV = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$
Therefore those establish our claim
$$I_r = O(r^{2+\varepsilon}), \quad \varepsilon > 0.$$
As a consequence, the holomorphic mapping $f : \mathbf{C}^2 \to S_{a,b}$ induced by $\tilde{f} : (z,w) \mapsto (z, 1+zw)$ is of order at most one.

3.2. The second step: To remove assumption (3.1)

We show by constructing a covering that for every $a,b \in \mathbf{C}$ with $1 < |b| \le |a|$, there exists a differentiably non-degenerate meromorphic mapping from $\mathbf{C}^2$ into Hopf surface $S_{a,b}$ with order at most one.

Take $a,b \in \mathbf{C}$ with $1 < |b| \le |a|$. Then there exist $p,q \in \mathbf{N}$ such that $|b|^q \le |a|^p \le |b|^{(3/2)q}$. Let $\Pi_{a,b}$ be the universal covering of $S_{a,b}$, and Π_{a^p,b^q} be the one of S_{a^p,b^q}. We define a holomorphic mapping $\tilde{\Psi}$ as follows,
$$\tilde{\Psi} : \mathbf{C}^2 \backslash \{(0,0)\} \to \mathbf{C}^2 \backslash \{(0,0)\}, \quad (x,y) \mapsto (x^q, y^p).$$
Then $\tilde{\Psi}$ induces a branched covering Ψ,
$$\begin{array}{ccc} \mathbf{C}^2\backslash\{(0,0)\} & \xrightarrow{\tilde{\Psi}} & \mathbf{C}^2\backslash\{(0,0)\} \\ {\scriptstyle \Pi_{a^p,b^q}}\downarrow & & {\scriptstyle \Pi_{a,b}}\downarrow \\ S_{a^p,b^q} & \xrightarrow[\Psi]{} & S_{a,b} \end{array}.$$
Note that a^p and b^q satisfy (3.1). By the first step, there exists a differentiably non-degenerate holomorphic mapping $g : \mathbf{C}^2 \to S_{a^p,b^q}$ with order at most one.

Then $\Psi \circ g$ is also a differentiably non-degenrate holomorphic mapping from $\mathbf{C}^2$ into $S_{a,b}$ with order at most one since $d\Psi$ is generically rank 2. $\square$

4. Inoue surfaces: Proof of the Main Theorem 1.2

M. Inoue constructed in [2], three type of surfaces S_M, $S^{(+)}_{N,p,q,r;t}$ and $S^{(-)}_{N,p,q,r}$, which are called Inoue surfaces. It is known that a VII_0 surface with second betti number zero is either an Inoue surface or a Hopf surface, and that an Inoue surface contains no closed curve. In this section we recall the definition of S_M, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ and prove the Main Theorem 1.2 as $S = S_M$, $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$ respectively.

The case of $S = S_M$: Let $\mathbf{H} = \{x \in \mathbf{C} \mid \operatorname{Im} x > 0\}$ be the upper half plane. Let $M = (m_{ij}) \in SL(3, \mathbf{Z})$ be a unimodular matrix with one real eigenvalue $\lambda_1 > 1$ and two complex conjugate eigenvalues $\lambda_2 \neq \overline{\lambda}_2$. Note that $\lambda_1 |\lambda_2|^2 = 1$ and that real number λ_1 is necessarily irrational. Let (a_1, a_2, a_3) be a real eigenvector with eigenvalue λ_1 and let (b_1, b_2, b_3) be an eigen vector with eigen value λ_2. Since (a_1, a_2, a_3), (b_1, b_2, b_3), $(\overline{b}_1, \overline{b}_2, \overline{b}_3)$ are $\mathbf{C}$-linearly independent, it follows that (a_1, b_1), (a_2, b_2), and (a_3, b_3) are $\mathbf{R}$-linearly independent. Let G_M be the group of analytic automorphisms of $\mathbf{H} \times \mathbf{C}$ generated by

$$\begin{aligned} g_0(x, y) &= (\lambda_1 x, \lambda_2 y), \\ g_j(x, y) &= (x + a_j, y + b_j), \quad 1 \leq j \leq 3. \end{aligned}$$

Then G_M acts on $\mathbf{H} \times \mathbf{C}$ properly discontinuously without fixed points. Hence

$$S_M = (\mathbf{H} \times \mathbf{C})/G_M$$

is a complex surface. Furthermore by the definition of the action, S_M becomes a compact complex surface, which is diffeomorphic to a 3-torus bundle over a circle. Relations between the generators g_0, g_1, g_2, g_3 of G_M are as follows:

$$\begin{aligned} g_i g_j &= g_j g_i \quad \text{for } i, j = 1, 2, 3, \\ g_0 g_i g_0^{-1} &= g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}} \quad \text{for } i = 1, 2, 3. \end{aligned}$$

It follows that

$$H_1(S_M, \mathbf{Z}) \cong \pi_1(S_M)/[\pi_1(S_M), \pi_1(S_M)] \cong G_M/[G_M, G_M] = \mathbf{Z} \oplus \mathbf{Z}_{e_1} \oplus \mathbf{Z}_{e_2} \oplus \mathbf{Z}_{e_3},$$

where $e_1, e_2, e_3 \neq 0$ are the elementary divisors of $M - I$. Hence $b_1(S_M) = 1$. Thus we deduce $b_2(S_M) = 0$, since Euler characteristic of S_M is zero.

Proof. We first prove that meromorphic mapping $f : \mathbf{C}^n \to S$ is holomorphic. Let $p : \mathbf{H} \times \mathbf{C} \to S$ be the universal covering mapping. Since $\operatorname{codim} I(f) \geq 2$, $\mathbf{C}^n \backslash I(f)$ is simply connected. Then we get a holomorphic lift

$$\widetilde{f_{\mathbf{C}^n \backslash I(f)}} : \mathbf{C}^n \backslash I(f) \to \mathbf{H} \times \mathbf{C}$$

of

$$f|_{\mathbf{C}^n \backslash I(f)} : \mathbf{C}^n \backslash I(f) \to S.$$

Since $\operatorname{codim} I(f) \geq 2$, the holomorphic mapping $\widetilde{f_{\mathbf{C}^n \backslash I(f)}} : \mathbf{C}^n \backslash I(f) \to \mathbf{H} \times \mathbf{C}$ extends to a holomorphic mapping $\tilde{f} : \mathbf{C}^n \to \mathbf{H} \times \mathbf{C}$. Because $f = p \circ \tilde{f}$, we deduce that f is holomorphic.

We now calculate the order of f. Since S_M is compact, the order is independent of the choice of Hermitian metric forms on S_M. We define a Hermitian metric form on $\mathbf{H} \times \mathbf{C}$ which is invariant under the action of G_M as follows

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \left(\frac{1}{(\operatorname{Im} x)^2} \, dx \wedge d\bar{x} + (\operatorname{Im} x) \, dy \wedge d\bar{y} \right).$$

Here we used $\lambda_1 |\lambda_2|^2 = 1$.

Let $\tilde{f} = (f_1, f_2) : \mathbf{C}^n \to \mathbf{H} \times \mathbf{C}$ be a holomorphic lift of f. Then f_1 is constant. Set $\operatorname{Im} f_1 = c$. Since

$$\tilde{f}^* \tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \left(\frac{1}{c^2} \, df_1 \wedge d\bar{f_1} + c \, df_2 \wedge d\bar{f_2} \right) = \frac{\sqrt{-1}}{2\pi} (c \, df_2 \wedge d\bar{f_2}) = \frac{\sqrt{-1}}{2\pi} (c \partial f_2 \wedge \bar{\partial} \bar{f_2}),$$

we obtain

$$\tilde{f}^* \tilde{\omega} \wedge \alpha^{n-1} = c \, dd^c |f_2|^2 \wedge \alpha^{n-1}.$$

Therefore we have

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^* \omega \wedge \alpha^{n-1} = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} c \, dd^c |f_2|^2 \wedge \alpha^{n-1}.$$

From Jensen's formula we obtain

$$\int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} c \, dd^c |f_2|^2 \wedge \alpha^{n-1} = \frac{c}{2} \int_{S(r)} |f_2|^2 \zeta - \frac{c}{2} \int_{S(1)} |f_2|^2 \zeta.$$

Let $f_2(z) = \sum_{k \geq 0} P_k(z_1, \ldots, z_n)$ be the expansion with homogeneous polynomials P_k of degree k. Since f_2 is not constant, there exists $k_0 \geq 1$ such that $P_{k_0} \neq 0$. Hence we obtain

$$\frac{c}{2} \int_{S(r)} |f_2|^2 \zeta = \frac{c}{2} \sum_{k \geq 0} r^{2k} \int_{S(1)} |P_k|^2 \zeta \geq \frac{c \cdot r^{2k_0}}{2} \int_{S(1)} |P_{k_0}|^2 \zeta \geq \frac{c \cdot r^2}{2} \int_{S(1)} |P_{k_0}|^2.$$

Therefore we deduce the order of f satisfies $\rho_f \geq 2$, since $c \neq 0$ and $\int_{S(1)} |P_{k_0}|^2 \neq 0$.

When $n \geq \dim_{\mathbf{C}} S(= 2)$, arguing on differentiably degeneracy of f make sense. Since the first component f_1 of a holomorphic lift of f is constant, f must differentiably degenerate. $\square$

The case of $S = S^{(+)}_{N,p,q,r;t}$: Here we study Inoue surface $S^{(+)}_{N,p,q,r;t}$. Let $N = (n_{ij}) \in SL(2,\mathbf{Z})$ be a matrix with two real eigenvalues $\lambda > 1$ and $\frac{1}{\lambda}$. Let (a_1, a_2) and (b_1, b_2) be two real eigen vectors of N corresponding to λ and $\frac{1}{\lambda}$ respectively (λ is necessarily irrational).

Fix integers p, q, r with $r \neq 0$ and a complex number t. Set real numbers (c_1, c_2) as the solution of the following linear equation

$$(c_1, c_2) = (c_1, c_2) \cdot {}^t N + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q),$$

where

$$e_i = \frac{1}{2} n_{i1}(n_{i1} - 1) a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2, \quad i = 1, 2.$$

Let $G^{(+)}_{N,p,q,r,t}$ be the group of analytic automorphisms of $\mathbf{H} \times \mathbf{C}$ generated by

$$\begin{aligned} g_0(x, y) &= (\lambda x, y + t), \\ g_j(x, y) &= (x + a_j, y + b_j x + c_j), \quad j = 1, 2, \\ g_3(x, y) &= \left(x, y + \frac{b_1 a_2 - b_2 a_1}{r}\right). \end{aligned}$$

They satisfy the following relations:

$$\begin{aligned} g_3 g_i &= g_i g_3 \quad \text{for } i = 0, 1, 2, \\ g_1 g_2 &= g_2 g_1 g_3^r, \\ g_0 g_1 g_0^{-1} &= g_1^{n_{11}} g_2^{n_{12}} g_3^p, \\ g_0 g_2 g_0^{-1} &= g_1^{n_{21}} g_2^{n_{22}} g_3^q. \end{aligned} \tag{4.1}$$

Then $S^{+}_{N,p,q,r;t} = (\mathbf{H} \times \mathbf{C})/G^{(+)}_{N,p,q,r,t}$ is an Inoue surface. Since the action is properly discontinuously with no fixed points, $S^{(+)}_{N,p,q,r;t}$ becomes a complex surface. Moreover it is a compact complex surface. It is known that $S^{(+)}_{N,p,q,r;t}$ is diffeomorphic to a fiber bundle over a circle whose fiber is a circle bundle over a two torus ([2]). It is known that $b_1(S^{+}_{N,p,q,r;t}) = 1$ and $b_2(S^{+}_{N,p,q,r;t}) = 0$.

Proof. Let $p : \mathbf{H} \times \mathbf{C} \to S^{(+)}_{N,p,q,r;t}$ be the universal covering. As in the case of S_M, every meromorphic mapping $f : \mathbf{C}^n \to S^{(+)}_{N,p,q,r;t}$ is holomorphic. We construct an Hermitian metric form on $\mathbf{H} \times \mathbf{C}$ which is invariant under the action of $G^{(+)}_{N,p,q,r,t}$ and which makes it easier to calculate the order of f. Take an arbitrary Hermitian metric form ω on $S^{(+)}_{N,p,q,r,t}$. Let $\tilde{\omega}$ be the pull-back $p^*\omega$. Then $\tilde{\omega}$ is invariant under the action. Write $\tilde{\omega}$ in coordinates,

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}(h_{11}\, dx \wedge d\bar{x} + h_{12}\, dx \wedge d\bar{y} + h_{21}\, dy \wedge d\bar{x} + h_{22}\, dy \wedge d\bar{y}).$$

Then $h_{22} \neq 0$ since $\tilde{\omega}$ is a positive Hermitian metric form. Therefore we can define a Hermitian metric form $\tilde{\sigma} = \dfrac{\tilde{\omega}}{h_{22}}$. Note that the coefficient of $dy \wedge d\bar{y}$ of $\tilde{\sigma}$ is one. Since $g_i^*\tilde{\omega} = \tilde{\omega}$, we obtain $h_{22}(g_i(x, y)) = h_{22}(x, y)$ for $i = 0, 1, 2, 3$. This implies

$$g_i^*\tilde{\sigma} = g_i^*\left(\frac{\tilde{\omega}}{h_{22}}\right) = \frac{\tilde{\omega}}{h_{22}} = \tilde{\sigma}.$$

Let $\tilde{f} = (f_1, f_2) : \mathbf{C}^n \to \mathbf{H} \times \mathbf{C}$ be a holomorphic lift of $f : \mathbf{C}^n \to S^{(+)}_{N,p,q,r;t}$. We calculate the order of $\tilde{f}$ with respect to $\tilde{\sigma}$. Since f_1 is constant, we have

$$\begin{aligned}\tilde{f}^*\tilde{\sigma} &= \frac{\sqrt{-1}}{2\pi}\left(\frac{h_{11}}{h_{22}}(\tilde{f})\, df_1 \wedge d\bar{f}_1 + \frac{h_{12}}{h_{22}}(\tilde{f})\, df_1 \wedge d\bar{f}_2\right.\\ &\qquad\left. + \frac{h_{21}}{h_{22}}(\tilde{f})\, df_2 \wedge d\bar{f}_1 + \frac{h_{22}}{h_{22}}(\tilde{f})\, df_2 \wedge d\bar{f}_2\right)\\ &= \frac{\sqrt{-1}}{2\pi}(df_2 \wedge d\bar{f}_2).\end{aligned}$$

Hence we obtain

$$T_{\tilde{f}}(r; \tilde{\sigma}) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} \tilde{f}^*\tilde{\sigma} \wedge \alpha^{n-1} = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} dd^c|f_2|^2 \wedge \alpha^{n-1}. \tag{4.2}$$

Note that f_2 is not constant. As in the case of S_M, we deduce from (4.2) that the order of f satisfies $\rho_f \geq 2$.

In addition to this, when $n \geq \dim_{\mathbf{C}} S^{(+)}_{N,p,q,r;t} (= 2)$, f must differentiably degenerate for the same reason as in the case of S_M. $\square$

The case of $S = S^{(-)}_{N,p,q,r}$: We define an Inoue surface $S^{(-)}_{N,p,q,r}$ as follows. Let $N = (n_{ij}) \in GL(2, \mathbf{Z})$ be a matrix with $\det N = -1$ and with two real eigenvalues λ and $-\dfrac{1}{\lambda}$. Let (a_1, a_2) and (b_1, b_2) be two real eigenvectors for N with eigenvalues λ and $-\dfrac{1}{\lambda}$ respectively. Fix integers p, q, r, with $r \neq 0$. Define two real numbers (c_1, c_2) as the solution of the following linear equation

$$-(c_1, c_2) = (c_1, c_2)^t N + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q),$$

where e_i are the same as for the surface $S^{(+)}_{N,p,q,r;t}$. Let $G^{(-)}_{N,p,q,r}$ be a group of analytic automorphisms of $\mathbf{H} \times \mathbf{C}$ generated by

$$g_0(x, y) = (\lambda x, -y),$$
$$g_j(x, y) = (x + a_j, y + b_j x + c_j), \quad j = 1, 2,$$
$$g_3(x, y) = \left(x, y + \frac{b_1 a_2 - b_2 a_1}{r}\right).$$

Then $S^{(-)}_{N,p,q,r} = (\mathbf{H} \times \mathbf{C})/G^{(-)}_{N,p,q,r}$ is an Inoue surface.

Proof. As we have seen in other Inoue surfaces, meromorphic mapping f is holomorphic. As in the case of $S^{(+)}_{N,p,q,r;t}$, we can construct a Hermitian metric form $\tilde{\sigma}$ on $\mathbf{H} \times \mathbf{C}$ which is invariant under the action of $G^{(-)}_{(N,p,q,r)}$ and is written in coordinates as follows,

$$\tilde{\sigma} = \frac{\sqrt{-1}}{2\pi}(h_{11}\, dx \wedge d\bar{x} + h_{12}\, dx \wedge d\bar{y} + h_{21}\, dy \wedge d\bar{x} + dy \wedge d\bar{y}).$$

Note that the coefficient of $dy \wedge d\bar{y}$ is one. This implies that the order of a non-constant holomorphic mapping $f : \mathbf{C}^n \to S^{(-)}_{N,p,q,r}$ satisfies $\rho_f \geq 2$.

In addition to this, when $n \geq \dim_{\mathbf{C}} S^{(-)}_{N,p,q,r} (= 2)$, f must differentiably degenerate as we have seen in other Inoue surfaces. □

5. Inoue surfaces: Restriction of the universal covering to a leaf

We now prove that the restriction of the universal covering mapping to a leaf $\{x_0\} \times \mathbf{C}$ $(\forall x_0 \in \mathbf{H})$ is of order two.

Proposition 5.1. *Let S be an Inoue surface and let $p : \mathbf{H} \times \mathbf{C} \to S$ be the universal covering mapping. Fix an arbitrary $x_0 \in \mathbf{H}$. Let $\tilde{f}$ be a holomorphic mapping $w \in \mathbf{C} \mapsto (x_0, w) \in \mathbf{H} \times \mathbf{C}$. Then $p \circ \tilde{f}$ has order two.*

Proof. The case of $S = S_M$. Take the following Hermitian metric form on $\mathbf{H} \times \mathbf{C}$

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}\left(\frac{1}{(\operatorname{Im} x)^2}\, dx \wedge d\bar{x} + (\operatorname{Im} x)\, dy \wedge d\bar{y}\right).$$

Let ω be the induced Hermitian metric form on S by $\tilde{\omega}$. We calculate the characteristic function of $p \circ \tilde{f}$ with respect to ω. Since $\tilde{f}^* \tilde{\omega} = (\operatorname{Im} x_0)\alpha$,

$$T_{p \circ \tilde{f}}(r, \omega) = T_{\tilde{f}}(r, \tilde{\omega}) = \int_1^r \frac{dt}{t} \int_{B(t)} \tilde{f}^* \tilde{\omega} = \frac{1}{2}(\operatorname{Im} x_0) r^2 - \frac{1}{2}(\operatorname{Im} x_0).$$

Hence we obtain $\rho_{p \circ \tilde{f}} = 2$.

The case of $S = S^{(+)}_{N,p,q,r;t}, S^{(-)}_{N,p,q,r}$. Take the following Hermitian metric form on $\mathbf{H} \times \mathbf{C}$

$$\tilde{\sigma} = \frac{\sqrt{-1}}{2\pi}(h_{11}\, dx \wedge d\bar{x} + h_{12}\, dx \wedge d\bar{y} + h_{21}\, dy \wedge d\bar{x} + dy \wedge d\bar{y}).$$

Let σ be the induced Hermitian metric form on S. We calculate the characteristic function of $p \circ \tilde{f}$ with respect to σ. Since $\tilde{f}^*\tilde{\sigma} = \alpha$, we have

$$T_{p\circ\tilde{f}}(r,\sigma) = T_{\tilde{f}}(r,\tilde{\sigma}) = \int_1^r \frac{dt}{t}\int_{B(t)} \tilde{f}^*\tilde{\sigma} = \frac{1}{2}r^2 - \frac{1}{2}.$$

Hence we deduce $\rho_{p\circ\tilde{f}} = 2$. □

Remark 5.2. By similar calculations, we get the order of the holomorphic mapping from $\mathbf{C}^n$ to an Inoue surface S induced by $(z_1, \dots, z_{n-1}, w) \in \mathbf{C}^n \mapsto (x_0, w^d) \in \mathbf{H} \times \mathbf{C}$ is $2d$.

Remark 5.3. Let S be an Inoue surface. Let $p : \mathbf{H} \times \mathbf{C} \to S$ be the universal covering mapping. Fix an arbitrary $x_0 \in \mathbf{H}$. Then its image $p(\{x_0\} \times \mathbf{C}) \subset S$ is Zariski dense, for there are no closed curves on an Inoue surface (see [2]), but not dense with respect to the differential topology. The differential structure of an Inoue surface S is as follows:

If $S = S_M$, S is diffeomorphic to a real 3-torus bundle over a circle parametrized by the imaginary part $\operatorname{Im} x$ of $x \in \mathbf{H}$.

If $S = S^{(+)}_{N,p,q,r;t}$, S is diffeomorphic to a fiber bundle over a circle parametrized by $\operatorname{Im} x$, whose fiber is a real three dimensional compact manifold. According to [2], this three dimensional compact manifold is a circle bundle over a real 2-torus.

If $S = S^{(-)}_{N,p,q,r}$, S is diffeomorphic to a fiber bundle over a circle parametrized by $\operatorname{Im} x$, whose fiber is a real three dimensional compact manifold.

6. Problems

Finally we pose some interesting questions related to characteristic functions of meromorphic mappings from $\mathbf{C}^2$ into Hopf surfaces.

Problem 6.1. Let $S_{a,b}$ be a Hopf surface defined in Main Theorem 1.1. We define a non-negative number $\rho(S_{a,b})$ as follows,

$$\rho(S_{a,b}) = \inf\{\rho_f \mid f : \mathbf{C}^2 \to S_{a,b} \text{ differentiably non-degenerate meromorphic mapping}\}.$$

Which number is $\rho(S_{a,b})$? Since there exists a holomorphic mapping from $\mathbf{C}^2$ into $S_{a,b}$ with order at most one, we have at least $\rho(S_{a,b}) \le 1$.

PROBLEM 6.2. Let $S_{a,a}$ be a Hopf surface defined in theorem 2.8. Let $f : \mathbf{C}^2 \to S_{a,a}$ be a holomorphic mapping, and let $\tilde{f} = (f_1, f_2) : \mathbf{C}^2 \to \mathbf{C}^2 \setminus \{(0,0)\}$ be its lift. Let $\tilde{\omega} = \frac{\sqrt{-1}}{2\pi} \frac{dx \wedge d\bar{x} + dy \wedge d\bar{y}}{|x|^2 + |y|^2}$ be a Hermitian metric form on $\mathbf{C}^2 \setminus \{(0,0)\}$ and let ω be the induced Hermitian metric form on $S_{a,a}$. Let ω_0 be Fubini-Study metric form on $\mathbf{P}^1(\mathbf{C})$ and let $\pi : \mathbf{C}^2 \setminus \{(0,0)\} \to \mathbf{P}^1(\mathbf{C})$, $(x, y) \mapsto [x : y]$ be the Hopf mapping. Set $F = \pi \circ \tilde{f}$. Then we found the following decomposition of the characteristic function of f with respect to ω,

$$T_f(r, \omega) = T_F(r, \omega_0) + \int_1^r \frac{dt}{t^3} \int_{B(t)} d \log(|f_1|^2 + |f_2|^2) \wedge d^c \log(|f_1|^2 + |f_2|^2) \wedge \alpha.$$

Let $R_f(r)$ denote the second term of the above formula. It is interesting to compare the growth of $T_F(r, \omega_0)$ and $R_f(r)$ as $r \to \infty$ or the growth of $T_F(r, \omega_0)$ and $T_f(r, \omega)$ as $r \to \infty$.

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