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ON THE PROFILE OF SOLUTIONS WITH TIME-DEPENDENT SINGULARITIES FOR THE HEAT EQUATION

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Abstract

Let $N \geq 2$, $T \in (0, \infty]$ and $\xi \in C(0, T; \mathbf{R}^N)$. Under some regularity condition for ξ, it is known that the heat equation

$$u_t - \Delta u = 0, \quad x \in \mathbf{R}^N \backslash \{\xi(t)\}, \quad t \in (0, T)$$

has a solution behaving like the fundamental solution of the Laplace equation as $x \to \xi(t)$ for any fixed t. In this paper we construct a singular solution whose behavior near $x = \xi(t)$ suddenly changes from that of the fundamental solution of the Laplace equation at some t.

1. Introduction

This paper is concerned with the following inhomogeneous linear heat equation.

$$u_t - \Delta u = w(t)\delta_{\xi(t)} \quad \text{in } \mathbf{R}^N \times (0, T). \tag{1.1}$$

Here $N \geq 2$, $T \in (0, \infty]$, w is a weight function satisfying $w \in L^1(0, t)$ for each $t \in (0, T)$, $\xi : (0, T) \to \mathbf{R}^N$ is a continuous curve and $\delta_{\xi(t)}$ is the Dirac distribution concentrated at $\xi(t) \in \mathbf{R}^N$. The main purpose of this paper is to investigate the behavior of a solution of (1.1) near $x = \xi(t)$.

Removability of singularities and existence of singular solutions in partial differential equations have been studied as interesting problems. As a simple example, let us consider the Laplace equation in $\Omega\backslash\{0\}$, where Ω is a neighborhood of 0 in $\mathbf{R}^N$. It is known that the singularity of a solution u at 0 is removable, which means that u can be extended as a solution in Ω, if $u(x) = o(|x|^{2-N})$ for $N \geq 3$ and $u(x) = o(\log|x|)$ for $N = 2$ as $x \to 0$. We immediately see that this condition is optimal since a fundamental solution of the Laplace equation is given by

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$$\Psi(x) := \begin{cases} \dfrac{1}{N(N-2)\omega_N}|x|^{2-N} = \dfrac{\Gamma(N/2-1)}{4\pi^{N/2}}|x|^{2-N} & \text{if } N \geq 3, \\ \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{if } N = 2, \end{cases}$$

where ω_N is the volume of the unit ball in $\mathbf{R}^N$ and Γ denotes the gamma function. For nonlinear elliptic equations, this kind of problems were examined by many authors, see, e.g., [1, 12, 8, 2, 13] and references therein.

In the recent works by [5, 6, 4], a condition for removability of singularities was considered in a certain class of nonlinear parabolic equations including the Fujita equation

$$u_t = \Delta u + |u|^{p-1}u \tag{1.2}$$

with $N \geq 3$ and $0 \leq p < N/(N-2)$. More precisely, it was shown that a solution u of (1.2) in $(\Omega\backslash\{0\}) \times (0,T)$ can be extended as a solution in $\Omega \times (0,T)$ if u satisfies $u(x,t) = o(|x|^{2-N})$ locally uniformly for $t \in (0,T)$ as $x \to 0$.

An interesting problem on singular solutions of parabolic equations is the existence of solutions with a time-dependent singularity. Here, by time-dependent singularity, we mean a singularity with respect to the space variable whose position depends on the time variable. The existence of such solutions were revealed in [9, 10] for the equation (1.2) with $N \geq 3$ and p in a certain range. Recently, solutions with a time-dependent singularity were constructed also in the Navier-Stokes equation [7].

In this paper, we focus on the linear heat equation

$$u_t - \Delta u = 0, \quad x \in \mathbf{R}^N\backslash\{\xi(t)\}, \quad t \in (0,T). \tag{1.3}$$

For this equation, the following were shown in [11, 7]. A condition for the removability of singularities is given by $u(x,t) = o(\Psi(x-\xi(t)))$ locally uniformly for $t \in (0,T)$ as $x \to \xi(t)$, if ξ is 1/2-Hölder continuous. At the same time, under the condition that ξ has α-Hölder continuity with $\alpha > 1/2$, there is a solution $\tilde{u}$ satisfying

$$\tilde{u}(x,t) = (1+o(1))\Psi(x-\xi(t)) \quad \text{for every } t \in (0,T) \text{ as } x \to \xi(t).$$

These results are analogous to the case of the Laplace equation. Our interest here is a solution which has a singularity at $x = \xi(t)$ but does not always behave like $\Psi(x-\xi(t))$ as $x \to \xi(t)$. In order to construct such a solution, we consider the equation (1.1). Then a desired solution can be found by considering two effects. One is due to the quick movement of ξ. We will observe that the profile of a solution of (1.1) is distorted when ξ loses α-Hölder continuity with $\alpha > 1/2$. The other one is the weight w. We will show that if w is forced to vanish or diverge as $t \uparrow t_0$, then the strength of a singularity suddenly changes at t_0.

This paper is organized as follows. In the next section, we set up our problem precisely and state main results. Section 3 is devoted to proving the results. In Section 4, we discuss the precise behavior at $x = \xi(t)$ in the case where $w(t) \equiv 1$ and ξ is α-Hölder continuous with $\alpha > 1/2$.

2. Main results

We introduce some definitions and notation before stating our results. Let F be defined by

$$F(x,t) := \int_0^t w(s)\Phi(x - \xi(s), t - s)\, ds, \tag{2.1}$$

where Φ is the heat kernel, that is, $\Phi(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$. Note that F can be defined for $x \in \mathbf{R}^N \setminus \{\xi(t)\}$ and $t \in (0,T)$, since

$$\sup_{s \in (0,t)} \Phi(x - \xi(s), t - s) < +\infty$$

provided $x \neq \xi(t)$. Moreover,

$$\|F(\cdot,t)\|_{L^1(\mathbf{R}^N)} \le \int_0^t |w(s)| \left(\int_{\mathbf{R}^N} \Phi(x - \xi(s), t - s)\, dx \right) ds = \int_0^t |w(s)|\, ds,$$

and so $F \in L^\infty(0,t; L^1(\mathbf{R}^N))$ for any $t \in (0,T)$. We will prove that F satisfies (1.1) in the distributional sense, that is, the equality

$$\int_0^T \int_{\mathbf{R}^N} F(x,t)(-\varphi_t(x,t) - \Delta\varphi(x,t))\, dxdt = \int_0^T w(t)\varphi(\xi(t),t)\, dt \tag{2.2}$$

holds for all $\varphi \in C_0^\infty(\mathbf{R}^N \times (0,T))$ (see Proposition 3.1 in Section 3). In particular, by the Weyl lemma for the heat equation (see, e.g., [3, Section 6]), we see that F satisfies (1.3) in the classical sense. For the same reason, we also find that if $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N \times (0,T))$ satisfies (1.1) in the distributional sense, then $u - F$ is smooth in $\mathbf{R}^N \times (0,T)$. This implies that the singularity of any solution of (1.1) is determined by F, and therefore our focus is on the behavior of F as $x \to \xi(t)$.

Let $t_0 \in (0,T)$ be fixed. As instantaneous quickness of $\xi(t)$ and weight of $w(t)$ at $t = t_0$, we define

$$v_\alpha := \lim_{s \uparrow t_0} \frac{\xi(t_0) - \xi(s)}{(t_0 - s)^\alpha} \quad (\alpha > 0), \qquad w_\beta := \lim_{s \uparrow t_0} \frac{w(s)}{(t_0 - s)^\beta} \quad (\beta > -1)$$

if their respective limits exist. Throughout this paper, we suppose that

$$v_\alpha \text{ and } w_\beta \text{ exist for some } \alpha > 0 \text{ and } \beta > -1.$$

The goal of this study is to describe how the magnitude and the direction of v_α and the weight w_β influence the singularity of $F(x,t_0)$.

Set $p := N/2 - \beta$. If w_β exists for some β with $p < 1$, then we have

$$|w(s)\Phi(x - \xi(s), t_0 - s)| \le (4\pi)^{-N/2} |w(s)| (t_0 - s)^{-N/2} \le C(t_0 - s)^{-p}$$

for all $x \in \mathbf{R}^N$ and $s \in (0, t_0)$. Here $C > 0$ is a constant. This shows that the value of $F(x, t_0)$ at $x = \xi(t_0)$ can be defined as a finite value. Furthermore, Lebesgue's dominated convergence theorem yields

$$\lim_{x \to \xi(t_0)} F(x, t_0) = F(\xi(t_0), t_0).$$

Therefore, in what follows, we only consider the case $p \geq 1$.

We perform the change of variables $z = x - \xi(t_0)$ and $\tau = t_0 - s$. Then

$$F(x, t_0) = (4\pi)^{-N/2} \int_0^{t_0} w(t_0 - \tau)\tau^{-N/2} \exp\left(-\frac{1}{4\tau}|z + \tau^\alpha \gamma_\alpha(\tau)|^2\right) d\tau,$$

where we write $\xi(t_0) - \xi(t_0 - \tau) = \tau^\alpha \gamma_\alpha(\tau)$. Put $\rho_\alpha := |v_\alpha|$ and $\nu_\alpha := v_\alpha/|v_\alpha|$. We write $r = |z|$, $\omega = z/|z|$ and denote $\theta \in [0, \pi]$ by the angle between ω and $-\nu_\alpha$, that is, $\cos\theta = -\omega \cdot \nu_\alpha$. With this notation, we have the decomposition $\omega = -(\cos\theta)\nu_\alpha + (\sin\theta)n$ for some $n \in \mathbf{R}^N$ with $|n| = 1$ and $n \cdot \nu_\alpha = 0$.

The effect of v_α is considerably different depending on α. When $\alpha = 1/2$, the effect of v_α appears in the coefficient of the leading term and $F(x, t_0)$ can lose asymptotic radial symmetry as $x \to \xi(t_0)$.

THEOREM 2.1. *Let* $\alpha = 1/2$. *Then the following* (i) *and* (ii) *hold as* $z = x - \xi(t_0) \to 0$.

(i) *If* $p = 1$,

$$F(x, t_0) = 2^{1-N}\pi^{-N/2} w_\beta e^{-\rho_\alpha^2/4} \log\frac{1}{r} + o\left(\log\frac{1}{r}\right).$$

(ii) *If* $p > 1$,

$$F(x, t_0) = (4\pi)^{-N/2} w_\beta \left(\int_0^\infty \sigma^{p-2} e^{-(1/4)|\sqrt{\sigma}\omega + v_\alpha|^2}\, d\sigma\right) r^{-2(p-1)} + o(r^{-2(p-1)}).$$

Remark 2.1. Suppose that $w_\beta = 1$ and $v_{1/2} = 0$. Then Theorem 2.1 implies that, as $x \to \xi(t_0)$,

$$F(x, t_0) = \begin{cases} (2\pi)^{-1} \log\dfrac{1}{r} + o\left(\log\dfrac{1}{r}\right) & (N = 2) \\ (4\pi)^{-N/2}\left(\displaystyle\int_0^\infty \sigma^{N/2-2} e^{-(1/4)\sigma}\, d\sigma\right) r^{2-N} + o(r^{2-N}) & (N \geq 3) \end{cases}$$
$$= (1 + o(1))\Psi(z).$$

Next, let us consider the case $\alpha < 1/2$. We remark that if $v_\alpha \neq 0$ with some $\alpha < 1/2$, the integral

$$\int_0^{t_0} |w(s)|(t_0 - s)^{-N/2} e^{-|\xi(t_0) - \xi(s)|^2/4(t_0 - s)}\, ds = \int_0^{t_0} |w(t_0 - \tau)|\tau^{-N/2} e^{-(1/4)\tau^{-(1-2\alpha)}|\gamma_\alpha(\tau)|^2}\, d\tau$$

is finite, because the integrand is bounded by $C|w(t_0 - \tau)|$ for some constant $C > 0$. Therefore the value of $F(x, t_0)$ at $x = \xi(t_0)$ can be defined as a finite value. This fact suggests that there is some region $\mathcal{N}$ containing the point $\xi(t_0)$ such that $F(\cdot, t_0)$ is bounded in $\mathcal{N}$. The problems in this case are to find such a region $\mathcal{N}$ and to determine the asymptotic behavior of $F(x, t_0)$ as $x \notin \mathcal{N}$, $x \to \xi(t_0)$. In order to state our result, we define, for $\varepsilon > 0$ and $M > 0$,

$$S_\varepsilon := \left\{ z \in \mathbf{R}^N \backslash \{0\}; 1 - \cos\theta \geq 2\rho_\alpha^{-1/\alpha}\left(\frac{2p-3}{2\alpha} + 1 + \varepsilon\right) r^{1/\alpha - 2} \log\frac{1}{r} \right\},$$
$$T_M := \{ z \in \mathbf{R}^N \backslash \{0\}; 1 - \cos\theta \leq M r^{1/\alpha - 2} \}.$$

Theorem 2.2. *Let $\alpha \in (0, 1/2)$ and $v_\alpha \neq 0$. Then the following* (i) *and* (ii) *hold.*

(i) *Suppose that $1 \leq p < 3/2 - \alpha$ and*

$$\xi(t_0) - \xi(s) = (t_0 - s)^\alpha v_\alpha + o((t_0 - s)^{\alpha + p - 1}) \tag{2.3}$$

as $s \uparrow t_0$. Then, as $z = x - \xi(t_0) \to 0$,

$$F(x, t_0) = F(\xi(t_0), t_0) + o(1).$$

(ii) *Suppose that $p \geq 3/2 - \alpha$ and*

$$\xi(t_0) - \xi(s) = (t_0 - s)^\alpha v_\alpha + (t_0 - s)^{1/2}\hat{v} + o((t_0 - s)^{1/2}) \tag{2.4}$$

for some $\hat{v} \in \mathbf{R}^N$ as $s \uparrow t_0$. Then, for any $\varepsilon > 0$ and $M > 0$,

$$F(x, t_0) = F(\xi(t_0), t_0) + o(1) \tag{2.5}$$

as $z = x - \xi(t_0) \in S_\varepsilon$, $z \to 0$, and

$$F(x, t_0) = \begin{cases} F(\xi(t_0), t_0) + (4\pi)^{-(N-1)/2} w_\beta \alpha^{-1} \rho_\alpha^{-1} & \text{if } p = \frac{3}{2} - \alpha, \\ \quad \times e^{-(1/4)c_\alpha - (1/4)J(z)} + o(1) & \\ (4\pi)^{-(N-1)/2} w_\beta \alpha^{-1} \rho_\alpha^{(2p-3)/2\alpha} e^{-(1/4)c_\alpha - (1/4)J(z)} & \text{if } p > \frac{3}{2} - \alpha \\ \quad \times r^{-(2p-3)/2\alpha - 1} + o(r^{-(2p-3)/2\alpha - 1}) & \end{cases} \tag{2.6}$$

as $z = x - \xi(t_0) \in T_M$, $z \to 0$. Here $J(z) := 2\rho_\alpha^{1/\alpha} r^{-(1/\alpha - 2)}(1 - \cos\theta) + 2\rho_\alpha^{1/2\alpha}(n \cdot \hat{v}) r^{-(1/2\alpha - 1)} \sin\theta$ and $c_\alpha := |\hat{v}|^2 - (v_\alpha \cdot \hat{v})^2$. Furthermore,

$$\liminf_{x \to \xi(t_0)} F(x, t_0) = F(\xi(t_0), t_0), \tag{2.7}$$

$$\limsup_{x \to \xi(t_0)} (r^{(2p-3)/2\alpha + 1} F(x, t_0)) \tag{2.8}$$
$$= \begin{cases} F(\xi(t_0), t_0) + (4\pi)^{-(N-1)/2} w_\beta \alpha^{-1} \rho_\alpha^{-1} & \text{if } p = \frac{3}{2} - \alpha, \\ (4\pi)^{-(N-1)/2} w_\beta \alpha^{-1} \rho_\alpha^{(2p-3)/2\alpha} & \text{if } p > \frac{3}{2} - \alpha. \end{cases}$$

Remark 2.2. We can also discuss a solution which has a singularity for all $t \in \mathbf{R}$. Let $W \in L^1_{\mathrm{loc}}(\mathbf{R})$ satisfy $(-t)^{-N/2} W(t) \in L^1(-\infty, -1)$ and let Ξ be a continuous curve in $\mathbf{R}$. We set

$$G(x,t) := \int_{-\infty}^{t} W(s)\Phi(x - \Xi(s), t - s)\, ds.$$

Then it can be shown that G satisfies $G_t - \Delta G = W(t)\delta_{\Xi(t)}$ in $\mathbf{R}^N \times (-\infty, \infty)$ and the same assertions as in Theorems 2.1 and 2.2 hold by replacing F with G. We note that in the special case $N \geq 3$, $W \equiv 1$ and $\Xi \equiv 0$, G coincides Ψ.

3. Proofs of theorems

First of all, we show that the function F defined by (2.1) is a solution of (1.1). Although this fact is essentially proved in [11, Section 4], we give a proof for the completion.

Proposition 3.1. *F satisfies* (1.1) *in the distributional sense.*

Proof. We fix $\varphi \in C_0^\infty(\mathbf{R}^N \times (0,T))$ and take $0 < \underline{t} < \bar{t} < T$ such that the support of φ is contained in $\mathbf{R}^N \times (\underline{t}, \bar{t})$. It is enough to verify (2.2) under the assumption that w is smooth. Indeed, choosing $\{w_n\}_{n=1}^\infty \subset C_0^\infty(\mathbf{R})$ such that $w_n \to w$ in $L^1(0,\bar{t})$ as $n \to \infty$ and putting $\tilde{F}_n(x,t) := \int_0^t w_n(s)\Phi(x - \xi(s), t - s)\, ds$, we have

$$\begin{aligned}\sup_{t \in (\underline{t},\bar{t})} \|F(\cdot,t) - \tilde{F}_n(\cdot,t)\|_{L^1(\mathbf{R}^N)} &\leq \sup_{t \in (\underline{t},\bar{t})} \int_0^t |w(s) - w_n(s)|\, \|\Phi(\cdot - \xi(s), t - s)\|_{L^1(\mathbf{R}^N)}\, ds \\ &\leq \|w - w_n\|_{L^1(0,\bar{t})} \to 0 \quad (n \to \infty).\end{aligned}$$

Therefore once we show that $\tilde{F}_n$ satisfies (2.2), the proof is finished by letting $n \to \infty$.

From now on we assume $w \in C_0^\infty(\mathbf{R})$. Let $h > 0$ and define

$$F_h(x,t) := \int_0^{t-h} w(s)\Phi(x - \xi(s), t - s)\, ds.$$

Then we easily see that F_h satisfies $(F_h)_t - \Delta F_h = w(t-h)\Phi(x - \xi(t-h), h)$ in the classical sense in $\mathbf{R}^N \times (h,T)$. Hence integration by parts yields

$$\int_0^T \int_{\mathbf{R}^N} F_h(-\varphi_t - \Delta\varphi)\, dxdt = \int_0^T w(t-h)\left(\int_{\mathbf{R}^N} \varphi(x,t)\Phi(x - \xi(t-h), h)\, dx\right) dt$$

provided $h < \underline{t}$. Let us take the limit as $h \downarrow 0$. Then the left-hand side convergences to $\int_0^T \int_{\mathbf{R}^N} F(-\varphi_t - \Delta\varphi)\, dxdt$, because

$$\begin{aligned}\sup_{t \in (\underline{t},\bar{t})} \|F(\cdot,t) - F_h(\cdot,t)\|_{L^1(\mathbf{R}^N)} &\leq \sup_{t \in (\underline{t},\bar{t})} \int_{t-h}^t |w(s)|\, \|\Phi(\cdot - \xi(s), t - s)\|_{L^1(\mathbf{R}^N)}\, ds \\ &\leq \|w\|_{L^\infty(\mathbf{R})} h \to 0 \quad (h \downarrow 0).\end{aligned}$$

It is straightforward to show that $w(t-h) \to w(t)$ and $\int_{\mathbf{R}^N} \varphi(x,t)\Phi(x-\xi(t-h),h)\,dx \to \varphi(\xi(t),t)$ locally uniformly for $t \in (0,T)$ as $h \downarrow 0$. Hence we see that the right-hand side tends to $\int_0^T w(t)\varphi(\xi(t),t)\,dt$. Thus the proof is complete. ■

3.1. The case $\alpha = 1/2$. Let us consider the case $\alpha = 1/2$ and prove Theorem 2.1.

Proof of Theorem 2.1. We fix $\delta \in (0,1)$ and take $\tau_0 > 0$ such that $(w_\beta - \delta)\tau^\beta \le w(t_0 - \tau) \le (w_\beta + \delta)\tau^\beta$ and $\rho_\alpha^2 - \delta \le |\gamma_\alpha(\tau)|^2 \le \rho_\alpha^2 + \delta$ for $\tau \in (0,\tau_0]$. Then

$$(w_\beta - \delta)I(z) \le (4\pi)^{N/2}F(x,t_0) \le C + (w_\beta + \delta)I(z), \tag{3.1}$$

where $C > 0$ is a constant and

$$I(z) := \int_0^{\tau_0} \tau^{-p} e^{-(1/4\tau)|z+\tau^\alpha\gamma_\alpha(\tau)|^2}\,d\tau.$$

First, we assume $p = 1$ and prove (i). Since

$$r^2 - 2(\rho_\alpha^2+1)r\tau^{1/2} + (\rho_\alpha^2 - \delta)\tau \le |z + \tau^\alpha\gamma_\alpha(\tau)|^2 \le r^2 + 2(\rho_\alpha^2+1)r\tau^{1/2} + (\rho_\alpha^2+\delta)\tau$$

for $\tau \in (0,\tau_0]$, we have

$$\begin{aligned} &e^{-(\rho_\alpha^2+\delta)/4}\int_0^{\tau_0} \tau^{-1}e^{-r^2/4\tau-(\rho_\alpha^2+1)r/2\sqrt{\tau}}\,d\tau \\ &\qquad \le I(z) \le e^{-(\rho_\alpha^2-\delta)/4}\int_0^{\tau_0}\tau^{-1}e^{-r^2/4\tau+(\rho_\alpha^2+1)r/2\sqrt{\tau}}\,d\tau. \end{aligned} \tag{3.2}$$

By the change of variables $\sigma = r^2/\tau$ and integration by parts, we deduce that

$$\begin{aligned} &\int_0^{\tau_0} \tau^{-1}e^{-r^2/4\tau\pm(\rho_\alpha^2+1)r/2\sqrt{\tau}}\,d\tau \\ &= \int_{r^2/\tau_0}^\infty \sigma^{-1}e^{-(1/4)\sigma\pm(1/2)(\rho_\alpha^2+1)\sqrt{\sigma}}\,d\sigma \\ &= \left(\log\frac{\tau_0}{r^2}\right)\exp\left(-\frac{r^2}{4\tau_0} \pm \frac{(\rho_\alpha^2+1)r}{2\sqrt{\tau_0}}\right) \\ &\quad - \int_{r^2/\tau_0}^\infty \left(-\frac{1}{4} \pm \frac{\rho_\alpha^2+1}{4\sqrt{\sigma}}\right)e^{-(1/4)\sigma\pm(1/2)(\rho_\alpha^2+1)\sqrt{\sigma}}\log\sigma\,d\sigma \\ &= 2\log\frac{1}{r} + O(1) \quad (z \to 0). \end{aligned} \tag{3.3}$$

Thus, combining (3.1)–(3.3), we obtain (i).

Next, we show (ii). In the case $p > 1$, the change of variables $\sigma = r^2/\tau$ yields

$$|r^{2(p-1)} I(z)| \le \int_{r^2/\tau_0}^{\infty} \sigma^{p-2} e^{-(1/4)|\sqrt{\sigma}\omega + \gamma_\alpha(r^2/\sigma)|^2}\, d\sigma \le C \tag{3.4}$$

for all $r > 0$ with some constant $C > 0$, and

$$\begin{aligned} &r^{2(p-1)} I(z) - \int_0^\infty \sigma^{p-2} e^{-(1/4)|\sqrt{\sigma}\omega + v_\alpha|^2}\, d\sigma \\ &\quad = \int_0^\infty \sigma^{p-2} \big(e^{-(1/4)|\sqrt{\sigma}\omega + \gamma_\alpha(r^2/\sigma)|^2} \chi_{[r^2/\tau_0,\,\infty)}(\sigma) - e^{-(1/4)|\sqrt{\sigma}\omega + v_\alpha|^2}\big)\, d\sigma, \end{aligned}$$

where χ_A denotes the indicator function of a set A. The integrand of the right-hand side convergences to 0 for each $\sigma \in (0, \infty)$ and is bounded by $C\sigma^{p-2} e^{-(1/8)\sigma}$. Lebesgue's dominated convergence theorem, (3.1) and (3.4) show

$$\limsup_{z \to 0} \left| \frac{(4\pi)^{N/2} F(x, t_0)}{r^{-2(p-1)}} - w_\beta \int_0^\infty \sigma^{p-2} e^{-(1/4)|\sqrt{\sigma}\omega + v_\alpha|^2}\, d\sigma \right| \le C\delta.$$

This gives (ii). ∎

3.2. The case $\alpha < 1/2$. Under the assumption that $\alpha \in (0, 1/2)$ and $v_\alpha \ne 0$ ($\rho_\alpha > 0$), we separate $F(x, t_0)$ into two parts as follows.

$$\begin{aligned} (4\pi)^{N/2} F(x, t_0) &= \int_{(0,\tau_+) \cup (\tau_-,\, t_0)} + \int_{\tau_+}^{\tau_-} w(t_0 - \tau) \tau^{-N/2} e^{-(1/4\tau)|z + \tau^\alpha \gamma_\alpha(\tau)|^2}\, d\tau \\ &=: I_1(z) + I_2(z), \end{aligned}$$

where $\tau_\pm := \{r(1 \pm \delta)^{-1} \rho_\alpha^{-1}\}^{1/\alpha}$ and $\delta \in (0, 1)$.

To prove Theorem 2.2, we prepare some lemmas. The first one gives the behavior of I_1.

Lemma 3.1. *For any fixed* $\delta \in (0, 1)$,

$$\lim_{z \to 0} I_1(z) = (4\pi)^{N/2} F(\xi(t_0), t_0). \tag{3.5}$$

Proof. Note that $\tau_+ \le \tau \le \tau_-$ is equivalent to $\rho_\alpha(1 - \delta)\tau^\alpha \le r \le \rho_\alpha(1 + \delta)\tau^\alpha$. Hence, taking $\tau_0 > 0$ so that $\rho_\alpha(1 - \delta/2) \le |\gamma_\alpha(\tau)| \le \rho_\alpha(1 + \delta/2)$ for $\tau \in (0, \tau_0]$, we have

$$|z + \tau^\alpha \gamma_\alpha(\tau)|^2 \ge (r - \tau^\alpha |\gamma_\alpha(\tau)|)^2 \ge \frac{\rho_\alpha^2 \delta^2}{4} \tau^{2\alpha}$$

provided that $\tau \in (0, \tau_+] \cup [\tau_-, \tau_0]$. This implies that for all $z \in \mathbf{R}^N$ and $\tau \in (0, t_0)$,

$$|w(t_0 - \tau)|\tau^{-N/2} e^{-(1/4\tau)|z+\tau^\alpha \gamma_\alpha(\tau)|^2} \chi_{(0,\tau_+)\cup(\tau_-,t_0)}(\tau) \le C|w(t_0-\tau)|\tau^{-N/2} e^{-(\rho_\alpha^2\delta^2/16)\tau^{-(1-2\alpha)}}$$
$$\le C|w(t_0-\tau)|.$$

Here $C > 0$ is a constant independent of z and τ. Since $w(t_0 - \cdot) \in L^1(0, t_0)$, we obtain (3.5) by Lebesgue's dominated convergence theorem. ∎

Next we consider I_2 for $1 \le p < 3/2 - \alpha$.

Lemma 3.2. *Assume* $1 \le p < 3/2 - \alpha$ *and* (2.3). *Then* $\lim_{z\to 0} I_2(z) = 0$.

Proof. For $\tau \in [\tau_+, \tau_-]$ and $q > 0$,

$$\rho_\alpha^{q/\alpha}(1-\delta)^{q/\alpha} r^{-q/\alpha} \le \tau^{-q} \le \rho_\alpha^{q/\alpha}(1+\delta)^{q/\alpha} r^{-q/\alpha},$$

and so

$$|I_2(z)| \le C\int_{\tau_+}^{\tau_-} \tau^{-p} \exp\left(-\frac{1}{4}\tau^{-1+2\alpha}|\tau^{-\alpha}z + \gamma_\alpha(\tau)|^2\right) d\tau$$
$$\le Cr^{-p/\alpha}\int_{\tau_+}^{\tau_-} \exp\left(-C^{-1}r^{-(1/\alpha-2)}|\tau^{-\alpha}z + \gamma_\alpha(\tau)|^2\right) d\tau,$$

where $C > 0$ denotes a generic constant independent of z and τ. Putting

$$\eta(r) := \max\left\{\sup_{\tau\in[\tau_+,\tau_-]} |\gamma_\alpha(\tau) - v_\alpha|, r^{1/2\alpha-1}\right\},$$

we have

$$|\tau^{-\alpha}z + \gamma_\alpha(\tau)| \ge (|\tau^{-\alpha}z + v_\alpha| - |\gamma_\alpha(\tau) - v_\alpha|)\chi_{\{|\tau^{-\alpha}r-\rho_\alpha| \ge \eta(r)\}}(\tau)$$
$$\ge (|\tau^{-\alpha}r - \rho_\alpha| - \eta(r))\chi_{\{|\tau^{-\alpha}r-\rho_\alpha| \ge \eta(r)\}}(\tau)$$

for all $\tau \in [\tau_+, \tau_-]$. Therefore, by the change of variables $\tau = \{r(\rho_\alpha + \eta(r)\sigma)^{-1}\}^{1/\alpha}$,

$$|I_2(z)| \le Cr^{-p/\alpha}\int_{\tau_+}^{\tau_-} \exp(-C^{-1}r^{-(1/\alpha-2)}(|\tau^{-\alpha}r - \rho_\alpha| - \eta(r))^2\chi_{\{|\tau^{-\alpha}r-\rho_\alpha|\ge\eta(r)\}}(\tau))\, d\tau$$
$$\le Cr^{-(p-1)/\alpha}\eta(r)\int_{-\rho_\alpha\delta/\eta(r)}^{\rho_\alpha\delta/\eta(r)} (\rho_\alpha + \eta(r)\sigma)^{-1/\alpha-1}$$
$$\times \exp(-C^{-1}r^{-(1/\alpha-2)}\eta(r)^2(|\sigma|-1)^2\chi_{\{|\sigma|\ge 1\}}(\sigma))\, d\sigma$$
$$\le Cr^{-(p-1)/\alpha}\eta(r)\int_{-\infty}^{\infty} \exp(-C^{-1}(|\sigma|-1)^2\chi_{\{|\sigma|\ge 1\}}(\sigma))\, d\sigma.$$

Conditions $1 \le p < 3/2 - \alpha$ and (2.3) yield $(2p-3)/(2\alpha) + 1 < 0$ and $\lim_{\tau \downarrow 0} (\tau^{-(p-1)} |\gamma_\alpha(\tau) - v_\alpha|) = 0$, and hence

$$\begin{aligned} r^{-(p-1)/\alpha} \eta(r) &\le r^{-(p-1)/\alpha} \left\{ \tau_-^{p-1} \sup_{\tau \in [\tau_+, \tau_-]} (\tau^{-(p-1)} |\gamma_\alpha(\tau) - v_\alpha|) + r^{1/2\alpha - 1} \right\} \\ &\le C \left\{ \sup_{\tau \in [\tau_+, \tau_-]} (\tau^{-(p-1)} |\gamma_\alpha(\tau) - v_\alpha|) + r^{-(2p-3)/2\alpha - 1} \right\} \\ &\to 0 \quad (r \to 0). \end{aligned}$$

Thus the lemma follows. ■

We prepare another estimate of I_2.

Lemma 3.3. *Assume that*

$$\xi(t_0) - \xi(s) = (t_0 - s)^\alpha v_\alpha + O((t_0 - s)^{1/2}) \quad \text{as } s \uparrow t_0. \tag{3.6}$$

Then, for any $\delta \in (0,1)$, *there exists a constant* $C > 0$ *such that*

$$I_2(z) \le C r^{-(2p-3)/2\alpha - 1} \exp\left(-\frac{1}{2} \rho_\alpha^{1/\alpha} (1-\delta)^{1/\alpha} r^{-(1/\alpha - 2)} (1 - \cos\theta) \right) \tag{3.7}$$

for all $z \in \mathbf{R}^N \backslash \{0\}$.

Proof. One can easily show that the inequality $|a+b|^2 \ge (1-c)(|a|^2 - |b|^2/c)$ holds for $a, b \in \mathbf{R}^N$ and $c > 0$. From this and (3.6), we have

$$\begin{aligned} \frac{1}{\tau} |z + \xi(t_0) - \xi(t_0 - \tau)|^2 &\ge \frac{1-\delta}{\tau} |z + \tau^\alpha v_\alpha|^2 - \frac{1-\delta}{\delta\tau} |\xi(t_0) - \xi(t_0 - \tau) - \tau^\alpha v_\alpha|^2 \\ &\ge \frac{1-\delta}{\tau} \{ (r - \tau^\alpha \rho_\alpha)^2 + 2\tau^\alpha \rho_\alpha r (1 - \cos\theta) \} - C \end{aligned}$$

for some constant $C > 0$ independent of z and τ. Moreover, if $\tau \le \tau_-$,

$$\tau^{-1+\alpha} r \ge \tau_-^{-1+\alpha} r = \rho_\alpha^{1/\alpha - 1} (1-\delta)^{1/\alpha - 1} r^{-(1/\alpha - 2)}.$$

Thus

$$\begin{aligned} |I_2(z)| &\le C \exp\left(-\frac{1}{2} \rho_\alpha^{1/\alpha} (1-\delta)^{1/\alpha} r^{-(1/\alpha - 2)} (1 - \cos\theta) \right) \\ &\quad \times \int_{\tau_+}^{\tau_-} \tau^{-p} \exp\left(-\frac{1-\delta}{4\tau} (r - \tau^\alpha \rho_\alpha)^2 \right) d\tau. \end{aligned}$$

Making the substitution $\tau = \{ \rho_\alpha^{-1} r (1 + r^\mu \sigma)^{-1} \}^{1/\alpha}$ $(\mu := 1/(2\alpha) - 1)$ yields

$$\int_{\tau_+}^{\tau_-} \tau^{-p} \exp\left(-\frac{1-\delta}{4\tau}(r-\tau^\alpha\rho_\alpha)^2\right) d\tau$$
$$= \alpha^{-1}\rho_\alpha^{(p-1)/\alpha} r^{-(2p-3)/2\alpha-1} \int_{-\delta/r^\mu}^{\delta/r^\mu} (1+r^\mu\sigma)^{(p-1)/\alpha-1}$$
$$\times \exp\left(-\frac{1-\delta}{4}\rho_\alpha^{1/\alpha}(1+r^\mu\sigma)^{1/\alpha-2}\sigma^2\right) d\sigma$$
$$\le Cr^{-(2p-3)/2\alpha-1} \int_{-\infty}^{\infty} \exp(-C^{-1}\sigma^2)\, d\sigma.$$

Therefore we obtain (3.7). ∎

Let $\kappa \in (\mu, 2\mu)$ $(\mu := 1/(2\alpha)-1)$ and define a set $\tilde{T}$ by

$$\tilde{T} := \{z \in \mathbf{R}^N \backslash \{0\}; 1-\cos\theta \le r^\kappa\}.$$

When ξ satisfies (2.4), we write $\xi(t_0)-\xi(t_0-\tau) = \tau^\alpha v_\alpha + \tau^{1/2}\hat{v} + \tau^{1/2}\zeta(\tau)$ and define $d(z) := \min\{\sup_{\tau\in[\tau_+,\tau_-]} |\zeta(\tau)|, 1/2\}$. Then (2.4) yields $d(z) \to 0$ as $z \to 0$.

In the next lemma, we examine the behavior of I_2 on $\tilde{T}$.

LEMMA 3.4. *Under the condition* (2.4), *the following hold.*

$$\liminf_{z\in\tilde{T}, z\to 0} [r^{(2p-3)/2\alpha+1} e^{(1/4)(1+d(z))J(z)} I_2(z)] \ge (4\pi)^{1/2} w_\beta \alpha^{-1} \rho_\alpha^{(2p-3)/2\alpha} e^{-(1/4)c_\alpha},$$
$$\limsup_{z\in\tilde{T}, z\to 0} [r^{(2p-3)/2\alpha+1} e^{(1/4)(1-d(z))J(z)} I_2(z)] \le (4\pi)^{1/2} w_\beta \alpha^{-1} \rho_\alpha^{(2p-3)/2\alpha} e^{-(1/4)c_\alpha}.$$

Here J *and* c_α *are defined in Theorem* 2.2.

Proof. From the inequality $(1-c)(|a|^2-|b|^2/c) \le |a+b|^2 \le (1+c)(|a|^2+|b|^2/c)$ $(a,b\in\mathbf{R}^N, c>0)$, we have

$$\frac{1}{\tau}|z+\xi(t_0)-\xi(t_0-\tau)|^2$$
$$= \frac{1}{\tau}|r\omega + \tau^\alpha\rho_\alpha v_\alpha + \tau^{1/2}\hat{v} + \tau^{1/2}\zeta(\tau)|^2$$
$$\ge \frac{1-d(z)}{\tau}|r\omega + \tau^\alpha\rho_\alpha v_\alpha + \tau^{1/2}\hat{v}|^2 - \frac{1-d(z)}{d(z)}|\zeta(\tau)|^2$$
$$\ge (1-d(z))\{(\tau^{-1/2}r - \tau^{-1/2+\alpha}\rho_\alpha - v_\alpha\cdot\hat{v})^2$$
$$+ 2\tau^{-1+\alpha}\rho_\alpha r(1-\cos\theta) + 2\tau^{-1/2}(n\cdot\hat{v})r\sin\theta\}$$
$$+ (1-d(z))\{|\hat{v}|^2 - (v_\alpha\cdot\hat{v})^2\}$$
$$+ 2(1-d(z))\tau^{-1/2}(v_\alpha\cdot\hat{v})r(1-\cos\theta) - (1-d(z))d(z)$$

and, in a similar way,

$$\begin{aligned}
&\frac{1}{\tau}|z+\xi(t_0)-\xi(t_0-\tau)|^2\\
&\quad\le (1+d(z))\{(\tau^{-1/2}r-\tau^{-1/2+\alpha}\rho_\alpha-v_\alpha\cdot\hat{v})^2\\
&\qquad+2\tau^{-1+\alpha}\rho_\alpha r(1-\cos\theta)+2\tau^{-1/2}(n\cdot\hat{v})r\sin\theta\}\\
&\qquad+(1+d(z))\{|\hat{v}|^2-(v_\alpha\cdot\hat{v})^2\}\\
&\qquad+2(1+d(z))\tau^{-1/2}(v_\alpha\cdot\hat{v})r(1-\cos\theta)+(1+d(z))d(z).
\end{aligned}$$

Since the inequality

$$\tau^{-1/2}r(1-\cos\theta)\le\tau_+^{-1/2}r^{1+\kappa}=\rho_\alpha^{1/2\alpha}(1+\delta)^{1/2\alpha}r^{\kappa-\mu}$$

holds for any $z\in\tilde{T}$ and $\tau\ge\tau_+$, we deduce that

$$\liminf_{z\in\tilde{T},z\to0}(I_2(z)/I_3^+(z))\ge w_\beta e^{-(1/4)c_\alpha},\quad \limsup_{z\in\tilde{T},z\to0}(I_2(z)/I_3^-(z))\le w_\beta e^{-(1/4)c_\alpha}, \tag{3.8}$$

where

$$\begin{aligned}
I_3^\pm(z):=\int_{\tau_+}^{\tau_-}\tau^{-p}\exp\biggl(-\frac{1}{4}(1\pm d(z))\{(\tau^{-1/2}r-\tau^{-1/2+\alpha}\rho_\alpha-v_\alpha\cdot\hat{v})^2\\
+2\tau^{-1+\alpha}\rho_\alpha r(1-\cos\theta)+2\tau^{-1/2}(n\cdot\hat{v})r\sin\theta\}\biggr)\,d\tau.
\end{aligned}$$

By the change of variables $\tau=\{\rho_\alpha^{-1}r(1+r^\mu\sigma)^{-1}\}^{1/\alpha}$, we have

$$\begin{aligned}
I_3^\pm(z)&=\alpha^{-1}\rho_\alpha^{(p-1)/\alpha}r^{-(2p-3)/2\alpha-1}e^{-(1/4)(1\pm d(z))J(z)}\\
&\quad\times\int_{-\delta/r^\mu}^{\delta/r^\mu}(1+r^\mu\sigma)^{(p-\alpha-1)/\alpha}e^{-(1/4)(1\pm d(z))(J_1+J_2)}\,d\sigma,\\
J_1&=J_1(z,\sigma):=\{\rho_\alpha^{1/2\alpha}(1+r^\mu\sigma)^\mu\sigma-v_\alpha\cdot\hat{v}\}^2,\\
J_2&=J_2(z,\sigma):=2\rho_\alpha^{1/\alpha}\{(1+r^\mu\sigma)^{1/\alpha-1}-1\}r^{-2\mu}(1-\cos\theta)\\
&\quad+2\rho_\alpha^{1/2\alpha}(n\cdot\hat{v})\{(1+r^\mu\sigma)^{1/2\alpha}-1\}r^{-\mu}\sin\theta.
\end{aligned}$$

We easily see that $\lim_{z\to0}J_1(z,\sigma)=(\rho_\alpha^{1/2\alpha}\sigma-v_\alpha\cdot\hat{v})^2$ for each $\sigma\in\mathbf{R}$ and

$$J_1(z,\sigma)\ge\frac{1}{2}\rho_\alpha^{1/\alpha}(1+r^\mu\sigma)^{2\mu}\sigma^2-(v_\alpha\cdot\hat{v})^2\ge\frac{1}{2}\rho_\alpha^{1/\alpha}(1-\delta)^{2\mu}\sigma^2-(v_\alpha\cdot\hat{v})^2$$

provided that $\sigma\ge-\delta/r^\mu$. For $z\in\tilde{T}$ and $\sigma\in[-\delta/r^\mu,\delta/r^\mu]$, J_2 is estimated as

$$\begin{aligned}|J_2(z,\sigma)| &\le Cr^{\mu}|\sigma|\cdot r^{-2\mu}(1-\cos\theta)+Cr^{\mu}|\sigma|\cdot r^{-\mu}|\sin\theta| \\ &\le C\{r^{-\mu}(1-\cos\theta)+(1-\cos\theta)^{1/2}\}|\sigma| \\ &\le C(r^{\kappa-\mu}+r^{\kappa/2})|\sigma|,\end{aligned}$$

where $C>0$ is a constant. This particularly implies that $\lim_{z\in\tilde{T},z\to 0} J_2(z,\sigma)=0$ for fixed $\sigma\in\mathbf{R}$, and furthermore, there is a constant $C>0$ such that

$$(1+r^{\mu}\sigma)^{(p-\alpha-1)/\alpha}e^{-(1/4)(1\pm d(z))(J_1(z,\sigma)+J_2(z,\sigma))}\chi_{[-\delta/r^{\mu},\delta/r^{\mu}]}(\sigma)\le Ce^{-C^{-1}(\sigma^2-\sigma)}$$

for all $z\in\tilde{T}$ with $r\le 1$ and $\sigma\in\mathbf{R}$. The right-hand side is integrable on $\mathbf{R}$, and so Lebesgue's dominated convergence theorem gives

$$\begin{aligned}&\lim_{\substack{z\in\tilde{T}\\ z\to 0}}\{r^{(2p-3)/2\alpha+1}e^{(1/4)(1\pm d(z))J(z)}I_3^{\pm}(z)\} \\ &\quad=\alpha^{-1}\rho_\alpha^{(p-1)/\alpha}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}\left(\rho_\alpha^{1/2\alpha}\sigma-v_\alpha\cdot\hat{v}\right)^2\right)d\sigma \\ &\quad=(4\pi)^{1/2}\alpha^{-1}\rho_\alpha^{(2p-3)/2\alpha}.\end{aligned}$$

From this and (3.8), the lemma follows. ■

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. (i) is a direct consequence of Lemmas 3.1 and 3.2, and therefore it suffices to consider (ii).

In what follows, we suppose $p\ge 3/2-\alpha$ and (2.4). First we derive (2.5). For given $\varepsilon>0$, we take δ so that $(1-\delta)^{1/\alpha}((2p-3)/(2\alpha)+1+\varepsilon)\ge(2p-3)/(2\alpha)+1+\varepsilon/2$. Then, by Lemma 3.3,

$$\begin{aligned}|I_2(z)| &\le Cr^{-(2p-3)/2\alpha-1}\exp\left(-\frac{1}{2}\rho_\alpha^{1/\alpha}(1-\delta)^{1/\alpha}r^{-(1/\alpha-2)}(1-\cos\theta)\right) \\ &\le Cr^{-(2p-3)/2\alpha-1}\exp\left(-\left(\frac{2p-3}{2\alpha}+1+\frac{\varepsilon}{2}\right)\log\frac{1}{r}\right) \\ &= Cr^{\varepsilon/2}\end{aligned}$$

for all $z\in S_\varepsilon$. From this and Lemma 3.1, we conclude (2.5).

Next we show (2.6). Since $\sin\theta\le\{2(1-\cos\theta)\}^{1/2}$, we have $\sup_{z\in T_M}|J(z)|<+\infty$. Note that $T_M\cap\{|z|\le\eta\}\subset\tilde{T}$ provided that $\eta>0$ is small. Hence we see from Lemma 3.4 that

$$I_2(z)=(1+o(1))(4\pi)^{1/2}w_\beta\alpha^{-1}\rho_\alpha^{(2p-3)/2\alpha}e^{-(1/4)c_\alpha-(1/4)J(z)}r^{-(2p-3)/2\alpha-1} \tag{3.9}$$

as $z\in T_M$, $z\to 0$. This and Lemma 3.1 give (2.6).

(2.7) immediately follows from Fatou's lemma and (2.5), and so we only have to show (2.8). It is easily seen that for any $\varepsilon > 0$, $(\mathbf{R}^N \backslash \tilde{T}) \cap \{|z| \leq \eta\} \subset S_\varepsilon$ provided that η is small. Therefore (2.5) and (3.5) yield

$$\text{(3.10)} \qquad \begin{aligned} &\limsup_{x \to \xi(t_0)} (r^{(2p-3)/2\alpha+1} F(x, t_0)) \\ &= \limsup_{\substack{x - \xi(t_0) \in \tilde{T} \\ x \to \xi(t_0)}} (r^{(2p-3)/2\alpha+1} F(x, t_0)) \\ &= \begin{cases} F(\xi(t_0), t_0) + (4\pi)^{-N/2} \displaystyle\limsup_{z \in \tilde{T}, z \to 0} I_2(z) & \text{if } p = \frac{3}{2} - \alpha, \\ (4\pi)^{-N/2} \displaystyle\limsup_{z \in \tilde{T}, z \to 0} (r^{(2p-3)/2\alpha+1} I_2(z)) & \text{if } p > \frac{3}{2} - \alpha. \end{cases} \end{aligned}$$

Let us first consider the estimate of the above quantity from below. Set $n_\alpha := -\{\hat{v} - (v_\alpha \cdot \hat{v}) v_\alpha\} / |\hat{v} - (v_\alpha \cdot \hat{v}) v_\alpha|$. This is defined unless $c_\alpha = 0$ and satisfies $|n_\alpha| = 1$, $n_\alpha \cdot v_\alpha = 0$ and $n_\alpha \cdot \hat{v} = -c_\alpha^{1/2}$. We define

$$T_* := \begin{cases} \{z \in \mathbf{R}^N \backslash \{0\}; n = n_\alpha, 2\rho_\alpha^{1/\alpha}(1 - \cos\theta) = c_\alpha r^{1/\alpha - 2}\} & \text{if } c_\alpha \neq 0, \\ \{z \in \mathbf{R}^N \backslash \{0\}; \theta = 0\} = \{z \in \mathbf{R}^N \backslash \{0\}; \omega = -v_\alpha\} & \text{if } c_\alpha = 0. \end{cases}$$

Then it is easy to see that $T_* \subset T_M$ for large M and $T_* \cap \{|z| \leq \eta\} \subset \tilde{T}$ for small η. Furthermore, since $\lim_{\phi \to 0} \{2(1 - \cos\phi)/\sin^2\phi\} = 1$, we have

$$\lim_{\substack{z \in T_* \\ z \to 0}} J(z) = c_\alpha + 2(n_\alpha \cdot \hat{v}) c_\alpha^{1/2} = -c_\alpha.$$

From this and (3.9), we deduce that

$$\text{(3.11)} \qquad \begin{aligned} &\limsup_{z \in \tilde{T}, z \to 0} (r^{(2p-3)/2\alpha+1} I_2(z)) \\ &\geq \lim_{z \in T_*, z \to 0} (r^{(2p-3)/2\alpha+1} I_2(z)) = (4\pi)^{1/2} w_\beta \alpha^{-1} \rho_\alpha^{(2p-3)/2\alpha}. \end{aligned}$$

Next we derive an upper bound. It is elementary to show that $n \cdot \hat{v} \geq -c_\alpha^{1/2} (= n_\alpha \cdot \hat{v})$ and that $a \cos\phi + b \sin\phi \leq (a^2 + b^2)^{1/2}$ for $a, b, \phi \in \mathbf{R}$. Hence we have

$$\begin{aligned} J(z) &\geq 2\rho_\alpha^{1/\alpha} r^{-(1/\alpha - 2)}(1 - \cos\theta) - 2c_\alpha^{1/2} \rho_\alpha^{1/2\alpha} r^{-(1/2\alpha - 1)} \sin\theta \\ &\geq 2\rho_\alpha^{1/\alpha} r^{-(1/\alpha - 2)}\{1 - (1 + c_\alpha \rho_\alpha^{-1/\alpha} r^{1/\alpha - 2})^{1/2}\} \\ &\to -c_\alpha \quad (r \to 0). \end{aligned}$$

This and Lemma 3.4 give

$$\limsup_{z\in\tilde{T}, z\to 0}(r^{(2p-3)/2\alpha+1}I_2(z)) \le e^{(1/4)c_\alpha} \limsup_{z\in\tilde{T}, z\to 0}(r^{(2p-3)/2\alpha+1}e^{(1/4)(1-d(z))J(z)}I_2(z)) \le (4\pi)^{1/2}w_\beta\alpha^{-1}\rho_\alpha^{(2p-3)/2\alpha}. \tag{3.12}$$

Combining (3.10)–(3.12), we obtain (2.8). Thus the proof is complete. ■

4. The profile of F for $\alpha > 1/2$

In this section we discuss the effect of v_α for $\alpha > 1/2$. As mentioned in Section 1, it is known that if $w(t) \equiv 1$ and $\alpha > 1/2$, the leading term of the expansion of $F(x, t_0)$ as $x \to \xi(t_0)$ is $\Psi(x - \xi(t_0))$. The aim in this section is to obtain the second-order term. More precisely, we prove the following theorem.

Theorem 4.1. *Assume $w(t) \equiv 1$ and $\alpha \in (1/2, 1]$. Then the following* (i), (ii) *and* (iii) *hold as $z = x - \xi(t_0) \to 0$.*

(i) *If $N = 2$,*

$$F(x, t_0) = \Psi(z) + (4\pi)^{-1} \times \left\{\log(4t_0) + \Gamma'(1) - \int_0^{t_0} \tau^{-1}(1 - e^{-(1/4)\tau^{2\alpha-1}|\gamma_\alpha(\tau)|^2})\, d\tau\right\} + o(1).$$

(ii) *If $N = 3$ and $\alpha = 1$,*

$$F(x, t_0) = \Psi(z) + (4\pi)^{-3/2} \times \left[-\frac{2}{\sqrt{t_0}} + \Gamma\left(\frac{1}{2}\right)\rho_\alpha \cos\theta - \int_0^{t_0} \tau^{-3/2}(1 - e^{-(1/4)\tau|\gamma_1(\tau)|^2})\, d\tau\right] + o(1).$$

(iii) *If $N = 3$ and $\alpha \ne 1$, or $N \ge 4$,*

$$F(x, t_0) = \Psi(z) + 4^{-(\alpha+1/2)}\pi^{-N/2}\Gamma\left(\frac{N}{2} - \alpha\right)\rho_\alpha(\cos\theta)r^{2\alpha+1-N} + o(r^{2\alpha+1-N}).$$

Proof. We write

$$\begin{aligned}(4\pi)^{N/2}F(x, t_0) &= \int_0^{t_0} \tau^{-N/2}e^{-r^2/4\tau}\, d\tau - \int_0^{t_0} \tau^{-N/2}e^{-r^2/4\tau}(1 - e^{-(1/4)\tau^{2\alpha-1}|\gamma_\alpha(\tau)|^2})\, d\tau \\ &\quad + \int_0^{t_0} \tau^{-N/2}e^{-r^2/4\tau-(1/4)\tau^{2\alpha-1}|\gamma_\alpha(\tau)|^2}(e^{-(1/2)r\tau^{\alpha-1}(\omega\cdot\gamma_\alpha(\tau))} - 1)\, d\tau \\ &=: I_1(z) - I_2(z) + I_3(z).\end{aligned}$$

We first consider $I_1(z)$. The change of variables $\tau = r^2/\sigma$ and integration by parts show that as $z \to 0$,

(4.1) $$I_1(z) = \int_{r^2/t_0}^{\infty} \sigma^{-1} e^{-(1/4)\sigma}\, d\sigma = \left(\log \frac{t_0}{r^2}\right) e^{-r^2/4t_0} + \frac{1}{4}\int_{r^2/t_0}^{\infty} e^{-(1/4)\sigma} \log \sigma\, d\sigma$$
$$= 2 \log \frac{1}{r} + \log t_0 + \Gamma'(1) + \log 4 + O\left(r^2 \log \frac{1}{r}\right)$$

if $N = 2$, and

(4.2) $$I_1(z) = \int_0^{\infty} \tau^{-N/2} e^{-r^2/4\tau}\, d\tau - \int_{t_0}^{\infty} \tau^{-N/2} e^{-r^2/4\tau}\, d\tau$$
$$= r^{2-N} \int_0^{\infty} \sigma^{N/2-2} e^{-(1/4)\sigma}\, d\sigma - \frac{2}{(N-2)t_0^{N/2-1}} + O(r^2)$$

if $N \geq 3$.

Next we examine $I_2(z)$. Since the integrand of $I_2(z)$ is positive and monotone decreasing with respect to r, we have,

(4.3) $$\lim_{z\to 0} I_2(z) = \int_0^{t_0} \tau^{-N/2}(1 - e^{-(1/4)\tau^{2\alpha-1}|\gamma_\alpha(\tau)|^2})\, d\tau$$

by the monotone convergence theorem. The right-hand side of the above is finite if $N = 2$, or $N = 3$ and $\alpha > 3/4$. If this is not the case, we have $N/2 - 2\alpha - 1 \geq -1$, and so

(4.4) $$|I_2(z)| \leq C \int_0^{t_0} \tau^{-N/2+2\alpha-1} e^{-r^2/4\tau}\, d\tau = C r^{-N+4\alpha} \int_{r^2/t_0}^{\infty} \sigma^{N/2-2\alpha-1} e^{-(1/4)\sigma}\, d\sigma$$
$$\leq C r^{-N+4\alpha}\left(1 + \log \frac{1}{r}\right),$$

where C denotes a positive constant independent of z. In particular, $I_2(z) = o(r^{-N+2\alpha+1})$ as $z \to 0$ if $N \geq 3$ and $\alpha \neq 1$, or $N \geq 4$.

Finally let us consider $I_3(z)$. We derive

(4.5) $$\lim_{z\to 0}\left(r^{N-2\alpha-1} I_3(z) - \frac{1}{2}\rho_\alpha \cos\theta \int_0^{\infty} \sigma^{N/2-\alpha-1} e^{-(1/4)\sigma}\, d\sigma\right) = 0$$

unless $N = 2$ and $\alpha = 1$. By the change of variables, we rewrite

$$I_3(z) = -\frac{1}{2} r \int_0^{t_0} \tau^{-N/2+\alpha-1} e^{-r^2/4\tau-(1/4)\tau^{2\alpha-1}|\gamma_\alpha(\tau)|^2}\left(\int_0^1 e^{-(\eta/2) r\tau^{\alpha-1}\omega\cdot\gamma_\alpha(\tau)}\, d\eta\right)\omega\cdot\gamma_\alpha(\tau)\, d\tau$$
$$= -\frac{1}{2} r^{-N+2\alpha+1} \int_{r^2/t_0}^{\infty} \sigma^{N/2-\alpha-1} e^{-(1/4)\sigma-(1/4)g(\sigma,z)}$$
$$\times \left(\int_0^1 e^{-(\eta/2)f(\sigma,z)}\, d\eta\right)\omega\cdot\gamma_\alpha\left(\frac{r^2}{\sigma}\right) d\sigma,$$

where $f(\sigma, z) := r^{2\alpha-1}\sigma^{1-\alpha}\omega \cdot \gamma_\alpha(r^2/\sigma)$ and $g(\sigma, z) := |\gamma_\alpha(r^2/\sigma)|^2 r^{4\alpha-2}\sigma^{1-2\alpha}$. Then,

$$\begin{aligned}
&r^{N-2\alpha-1}I_3(z) - \frac{1}{2}\rho_\alpha \cos\theta \int_0^\infty \sigma^{N/2-\alpha-1}e^{-(1/4)\sigma}\,d\sigma \\
&\quad= \frac{1}{2}\int_0^\infty \sigma^{N/2-\alpha-1}e^{-(1/4)\sigma} \\
&\qquad\times \left\{ -e^{-(1/4)g(\sigma,z)}\left(\int_0^1 e^{-(\eta/2)f(\sigma,z)}\,d\eta\right)\omega\cdot\gamma_\alpha\left(\frac{r^2}{\sigma}\right)\chi_{[r^2/t_0,\,\infty)}(\sigma) + \omega\cdot v_\alpha \right\} d\sigma.
\end{aligned}$$

Since $\lim_{z\to 0} f(\sigma, z) = \lim_{z\to 0} g(\sigma, z) = 0$ and $\lim_{z\to 0} \gamma_\alpha(r^2/\sigma) = v_\alpha$, the integrand of the right-hand side convergences to 0 for any $\sigma \in (0, \infty)$ as $z \to 0$. Moreover, one easily see that $|f(\sigma, z)| \le C\sigma^{1-\alpha}$ and $g(\sigma, r) \ge 0$ for $r \in (0, 1)$ and $\sigma \in (0, \infty)$, and therefore the integrand is bounded by $C\sigma^{N/2-\alpha-1}e^{-(1/4)\sigma}(e^{C\sigma^{1-\alpha}} + 1)$, which is integrable on $(0, \infty)$ unless $N = 2$ and $\alpha = 1$. Thus, by applying Lebesgue's dominated convergence theorem, we obtain (4.5).

In the case $N = 2$ and $\alpha = 1$, $I_3(z)$ is estimated as

$$|I_3(z)| \le Cr\int_{r^2/t_0}^\infty \sigma^{-1}e^{-(1/4)\sigma + C\sigma^{1-\alpha}}\,d\sigma \le Cr\left(1 + \log\frac{1}{r}\right) \tag{4.6}$$

with some constant $C > 0$. In particular, we see from (4.5) and this computation that $\lim_{z\to 0} I_3(z) = 0$ provided that $N = 2$.

(i), (ii) and (iii) follow from (4.1)–(4.6). Thus the proof is complete. ■

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