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NONEXISTENCE AND EXISTENCE RESULTS FOR A 2*n*th-ORDER DISCRETE DIRICHLET BOUNDARY VALUE PROBLEM*

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Abstract

This paper is concerned with a $2n$th-order nonlinear difference equation. By making use of the critical point method, we establish various sets of sufficient conditions for the nonexistence and existence of solutions for Dirichlet boundary value problem and give some new results. The existing results are generalized and significantly complemented.

1. Introduction

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, ecology, cybernetics, etc. For the general background of difference equations, one can refer to monographs [1, 29, 36]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity and results on oscillation and other topics, see [9–14, 16–18, 23–25, 28, 30, 35, 41–43].

Below $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. k is a positive integer. For any $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a < b$. Besides, $*$ denotes the transpose of a vector.

The present paper considers the $2n$th-order nonlinear difference equation

$$(1.1) \quad \Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \quad n \in \mathbf{Z}(1), \quad i \in \mathbf{Z}(1, k),$$

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with boundary value conditions

$$u_{1-n} = u_{2-n} = \cdots = u_0 = 0, \quad u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0, \tag{1.2}$$

where Δ is the forward difference operator $\Delta u_i = u_{i+1} - u_i$, $\Delta^n u_i = \Delta^{n-1}(\Delta u_i)$, γ_i is nonzero and real valued for each $i \in \mathbf{Z}(2-n, k+1)$, $f \in C(\mathbf{R}^4, \mathbf{R})$.

We may think of (1.1) as a discrete analogue of the following $2n$th-order functional differential equation

$$\frac{d^n}{dt^n}\left[\gamma(t)\frac{d^n u(t)}{dt^n}\right] = (-1)^n f(t, u(t+1), u(t), u(t-1)), \quad t \in [a,b], \tag{1.3}$$

with boundary value conditions

$$u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0, \quad u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0. \tag{1.4}$$

Equations similar in structure to (1.3) arise in the study of the existence of solitary waves [38] of lattice differential equations and periodic solutions [20, 22] of functional differential equations.

In recent years, the study of boundary value problems for differential equations develops at relatively rapid rate. By using various methods and techniques, such as Schauder fixed point theory, topological degree theory, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in literatures, we refer to [2, 5, 8, 26, 39, 40]. And critical point theory is also an important tool to deal with problems on differential equations [15, 21, 31, 34]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [23–25] and Shi *et al.* [37] have successfully proved the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [41, 42] for the discrete boundary value problems. Compared to first-order or second-order difference equations, the study of higher-order equations, and in particular, $2n$th-order equations, has received considerably less attention (see, for example, [6, 10–14, 16–19, 27, 30, 43] and the references contained therein). Ahlbrandt and Peterson [3] in 1994 studied the $2n$th-order difference equation of the form,

$$\sum_{j=0}^{n} \Delta^j(\gamma_j(i-j)\Delta^j u(i-j)) = 0 \tag{1.5}$$

in the context of the discrete calculus of variations, and Peil and Peterson [33] studied the asymptotic behavior of solutions of (1.5) with $\gamma_j(i) \equiv 0$ for $1 \le j \le n-1$. In 1998, Anderson [4] considered (1.5) for $i \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n,n)-disconjugacy for (1.5). Migda [32] in 2004 studied an mth-order linear difference equation. In 2007, Cai and Yu [7] have obtained some criteria for the existence of periodic solutions of a $2n$th-order difference equation

$$\Delta^n(\gamma_{i-n}\Delta^n u_{i-n}) + f(i, u_i) = 0, \quad n \in \mathbf{Z}(3), \quad i \in \mathbf{Z}, \tag{1.6}$$

for the case where f grows superlinearly at both 0 and ∞.

The boundary value problem (BVP) for determining the existence of solutions of difference equations has been a very active area of research in the last twenty years, and for surveys of recent results, we refer the reader to the monographs by Agarwal *et al.* [1, 29, 36]. However, to the best of our knowledge, the results on solutions to boundary value problems of higher-order nonlinear difference equations are very scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few manuscripts dealing with this subject. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above results, we use the critical point theory to give some sufficient conditions for the nonexistence and existence of solutions for the BVP (1.1) with (1.2). We shall study the superlinear and sublinear cases. The main idea in this paper is to transfer the existence of the BVP (1.1) with (1.2) into the existence of the critical points of some functional. The proof is based on the notable Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we shall further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we shall complement existing results. The motivation for the present work stems from the recent papers in [12, 15].

Let

$$\bar{\gamma} = \max\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}, \quad \underline{\gamma} = \min\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}.$$

Our main results are as follows.

THEOREM 1.1. *Assume that the following hypotheses are satisfied*:
(γ) *for any* $i \in \boldsymbol{Z}(2-n, k+1)$, $\gamma_i < 0$;
(F_1) *there exists a functional* $F(i, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0, \cdot) = 0$ *such that*

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3), \quad \forall i \in \boldsymbol{Z}(1, k);$$

(F_2) *there exists a constant* $M_0 > 0$ *such that for all* $(i, v_1, v_2) \in \boldsymbol{Z}(1, k) \times \boldsymbol{R}^2$

$$\left|\frac{\partial F(i, v_1, v_2)}{\partial v_1}\right| \le M_0, \quad \left|\frac{\partial F(i, v_1, v_2)}{\partial v_2}\right| \le M_0.$$

Then the BVP (1.1) *with* (1.2) *possesses at least one solution.*

Remark 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$ such that
(F_2') $|F(i, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|)$, $\forall (i, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2$.

THEOREM 1.2. *Suppose that* (F_1) *and the following hypotheses are satisfied*:
(γ') *for any* $i \in \boldsymbol{Z}(2-n, k+1)$, $\gamma_i > 0$;
(F_3) *there exists a functional* $F(i, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *such that*

$$\lim_{r \to 0} \frac{F(i, v_1, v_2)}{r^2} = 0, \quad r = \sqrt{v_1^2 + v_2^2}, \quad \forall i \in \boldsymbol{Z}(1, k);$$

(F_4) *there exists a constant* $\beta > 2$ *such that for any* $i \in \boldsymbol{Z}(1,k)$,

$$0 < \frac{\partial F(i,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(i,v_1,v_2)}{\partial v_2} v_2 < \beta F(i,v_1,v_2), \quad \forall (v_1,v_2) \neq 0.$$

Then the BVP (1.1) *with* (1.2) *possesses at least two nontrivial solutions.*

Remark 1.2. Assumption (F_4) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that
(F_4') $F(i,v_1,v_2) > a_1(\sqrt{v_1^2+v_2^2})^{\beta} - a_2, \ \forall i \in \mathbf{Z}(1,k)$.

THEOREM 1.3. *Suppose that* (γ'), (F_1) *and the following assumption are satisfied*:
(F_5) *there exist constants* $R > 0$ *and* $1 < \alpha < 2$ *such that for* $i \in \boldsymbol{Z}(1,k)$ *and* $\sqrt{v_1^2+v_2^2} \geq R$,

$$0 < \frac{\partial F(i,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(i,v_1,v_2)}{\partial v_2} v_2 \leq \alpha F(i,v_1,v_2).$$

Then the BVP (1.1) *with* (1.2) *possesses at least one solution.*

Remark 1.3. Assumption (F_5) implies that for each $i \in \mathbf{Z}(1,k)$ there exist constants $a_3 > 0$ and $a_4 > 0$ such that
(F_5') $F(i,v_1,v_2) \leq a_3(\sqrt{v_1^2+v_2^2})^{\alpha} + a_4, \ \forall (i,v_1,v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2$.

THEOREM 1.4. *Suppose that* (γ), (F_1) *and the following assumption are satisfied*:
(F_6) $v_2 f(i,v_1,v_2,v_3) > 0$, *for* $v_2 \neq 0$, $\forall i \in \mathbf{Z}(1,k)$.
Then the BVP (1.1) *with* (1.2) *has no nontrivial solutions.*

Remark 1.4. In the existing literature, results on the nonexistence of solutions of discrete boundary value problems are very scarce. Hence, Theorem 1.4 complements existing ones.

The remainder of this paper is organized as follows. First, in Section 2, we shall establish the variational framework for the BVP (1.1) with (1.2) and transfer the problem of the existence of the BVP (1.1) with (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give three examples to illustrate the main results.

About the basic knowledge for variational methods, we refer the reader to [31, 34, 44].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1.1) with (1.2) and give some lemmas which will be of fundamental importance in proving our main results. First, we state some basic notations.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v\rangle = \sum_{j=1}^{k} u_j v_j, \quad \forall u, v \in \mathbf{R}^k, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{1/2}, \quad \forall u \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, we define the norm $\|\cdot\|_s$ on $\mathbf{R}^k$ as follows:

$$\|u\|_s = \left(\sum_{j=1}^{k} |u_j|^s\right)^{1/s}, \tag{2.3}$$

for all $u \in \mathbf{R}^k$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \quad \forall u \in \mathbf{R}^k. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1.1) with (1.2), consider the functional J defined on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}\sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i)^2 - \sum_{i=1}^{k} F(i, u_{i+1}, u_i), \quad \forall u = (u_1, u_2, \dots, u_k)^* \in \mathbf{R}^k, \tag{2.5}$$

where

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3),$$

$$u_{1-n} = u_{2-n} = \cdots = u_0 = 0, \quad u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0.$$

It is easy to see that $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_i\}_{i=1}^{k} = (u_1, u_2, \dots, u_k)^*$, by using $u_{1-n} = u_{2-n} = \cdots = u_0 = 0$, $u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}) - f(i, u_{i+1}, u_i, u_{i-1}), \quad \forall i \in \mathbf{Z}(1, k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \quad \forall i \in \mathbf{Z}(1, k).$$

We reduce the existence of the BVP (1.1) with (1.2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.1) with (1.2).

Let D be the $(k+n)\times(k+n)$ matrix defined by

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, D is positive definite. Let $\lambda_{1-n}, \lambda_{2-n}, \ldots, \lambda_k$ be the eigenvalues of D. Applying matrix theory, we know $\lambda_j > 0$, $j = 1-n, 2-n, \ldots, k$. Without loss of generality, we may assume that

$$0 < \lambda_{1-n} \le \lambda_{2-n} \le \cdots \le \lambda_k. \tag{2.6}$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(l)}\} \subset E$ for which $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)}) \to 0$ $(l \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [34]). *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

(J_1) *there exist constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \ge a$, and*

(J_2) *there exists $e \in E \backslash B_\rho$ such that $J(e) \le 0$.*

Then J possesses a critical value $c \ge a$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} J(g(s)), \tag{2.7}$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = e\}. \tag{2.8}$$

Lemma 2.2. *Suppose that (γ'), (F_1), (F_3) and (F_4) are satisfied. Then the functional J satisfies the P.S. condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k$, $l \in \mathbf{Z}(1)$ be such that $\{J(u^{(l)})\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2 \le J(u^{(l)}) \le M_2, \quad \forall l \in \mathbf{N}.$$

By (F_4'), we have

$$
\begin{aligned}
-M_2 \le J(u^{(l)}) &= \frac{1}{2}\sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i^{(l)})^2 - \sum_{i=1}^{k} F(i, u_{i+1}^{(l)}, u_i^{(l)}) \\
&\le \frac{\overline{\gamma}}{2}(x^{(l)})^* D x^{(l)} - a_1 \sum_{i=1}^{k} [\sqrt{(u_{i+1}^{(l)})^2 + (u_i^{(l)})^2}]^{\beta} + a_2 k \\
&\le \frac{\overline{\gamma}}{2}\lambda_k \|x^{(l)}\|^2 - a_1 c_1^{\beta}\|u^{(l)}\|^{\beta} + a_2 k,
\end{aligned}
$$

where $x^{(l)} = (\Delta^{n-1}u_{1-n}^{(l)}, \Delta^{n-1}u_{2-n}^{(l)}, \dots, \Delta^{n-1}u_k^{(l)})^*$. Since

$$
\|x^{(l)}\|^2 = \sum_{i=1-n}^{k} (\Delta^{n-2}u_{i+1}^{(l)} - \Delta^{n-2}u_i^{(l)})^2 \le \lambda_k \sum_{i=1-n}^{k} (\Delta^{n-2}u_i^{(l)})^2 \le \lambda_k^{n-1}\|u^{(l)}\|^2,
$$

we have

$$
J(u^{(l)}) \le \frac{\overline{\gamma}}{2}\lambda_k^n \|u^{(l)}\|^2 - a_1 c_1^{\beta}\|u^{(l)}\|^{\beta} + a_2 k.
$$

That is,

$$
a_1 c_1^{\beta}\|u^{(l)}\|^{\beta} - \frac{\overline{\gamma}}{2}\lambda_k^n \|u^{(l)}\|^2 \le M_2 + a_2 k.
$$

Since $\beta > 2$, there exists a constant $M_3 > 0$ such that

$$
\|u^{(l)}\| \le M_3, \quad \forall l \in \mathbf{N}.
$$

Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergence subsequence in $\mathbf{R}^k$. Thus the P.S. condition is verified. □

3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

Proof. By (F_2'), for any $u = (u_1, u_2, \dots, u_k)^* \in \mathbf{R}^k$, we have

$$
\begin{aligned}
J(u) &= \frac{1}{2}\sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i)^2 - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\
&\le \frac{\overline{\gamma}}{2} x^* D x + M_0 \sum_{i=1}^{k} (|u_{i+1}| + |u_i|) + M_1 k \\
&\le \frac{\overline{\gamma}}{2}\lambda_{1-n}\|x\|^2 + 2M_0 \sum_{i=1}^{k} |u_i| + M_1 k,
\end{aligned}
$$

where $x = (\Delta^{n-1}u_{1-n}, \Delta^{n-1}u_{2-n}, \ldots, \Delta^{n-1}u_k)^*$. Since

$$\|x\|^2 = \sum_{i=1-n}^{k} (\Delta^{n-2}u_{i+1} - \Delta^{n-2}u_i)^2 \geq \lambda_{1-n} \sum_{i=1-n}^{k} (\Delta^{n-2}u_i)^2 \geq \lambda_{1-n}^{n-1}\|u\|^2,$$

we have

$$J(u) \leq \frac{\bar{\gamma}}{2}\lambda_{1-n}^{n}\|u\|^2 + 2M_0\sqrt{k}\|u\| + M_1 k \to -\infty \quad \text{as } \|u\| \to +\infty.$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, and we denote it $\check{u}$, that is,

$$J(\check{u}) = \max\{J(u) \mid u \in \mathbf{R}^k\}.$$

Clearly, $\check{u}$ is a critical point of the functional J. This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2

Proof. By (F_3), for any $\varepsilon = \frac{\gamma}{4}\lambda_{1-n}^{n}$ (λ_{1-n} can be referred to (2.6)), there exists $\rho > 0$, such that

$$|F(i, v_1, v_2)| \leq \frac{\gamma}{4}\lambda_{1-n}^{n}(v_1^2 + v_2^2), \quad \forall i \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2 + v_2^2} \leq \sqrt{2}\rho$.

For any $u = (u_1, u_2, \ldots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \leq \rho$, we have $|u_i| \leq \rho$, $i \in \mathbf{Z}(1, k)$.

From the proof of the Theorem 1.1, for any $u \in \mathbf{R}^k$,

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i)^2 - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\ &\geq \frac{\gamma}{2}\lambda_{1-n}^{n}\|u\|^2 - \frac{\gamma}{4}\lambda_{1-n}^{n}\sum_{i=1}^{k}(u_{i+1}^2 + u_i^2) \\ &\geq \frac{\gamma}{2}\lambda_{1-n}^{n}\|u\|^2 - \frac{\gamma}{4}\lambda_{1-n}^{n}\|u\|^2 \\ &= \frac{\gamma}{4}\lambda_{1-n}^{n}\|u\|^2. \end{aligned}$$

Take $a = \frac{\gamma}{4}\lambda_{1-n}^{n}\rho^2 > 0$. Therefore,

$$J(u) \geq a > 0, \quad \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \geq a$. That is to say, J satisfies the condition (J_1) of the Mountain Pass Lemma.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify all other conditions of the Mountain Pass Lemma. By Lemma 2.2, J satisfies the P.S. condition in Lemma 2.2. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition, we know

$$J(u) \leq \frac{\overline{\gamma}}{2}\lambda_k^n\|u\|^2 - a_1 c_1^\beta \|u\|^\beta + a_2 k.$$

Since $\beta > 2$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

By the Mountain Pass Lemma, J possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0, h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s \in [0,1]} J(h(s)).$$

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = c_{\max} = \max_{s \in [0,1]} J(h(s))$. That is,

$$\sup_{u \in \mathbf{R}^k} J(u) = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s \in [0,1]} J(h(s)), \quad \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that

$$J(h(s_0)) = c_{\max}.$$

Choose $h_1, h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1, s_2 \in (0,1)$ such that

$$J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}.$$

Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1(s_1), \quad u^2 = h_2(s_2).$$

The above argument implies that the BVP (1.1) with (1.2) possesses at least two nontrivial solutions. The proof of Theorem 1.2 is finished. □

3.3. Proof of Theorem 1.3

Proof. We only need to find at least one critical point of the functional J defined as in (2.5).

By (F_5'), for any $u = (u_1, u_2, \dots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i)^2 - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\
&\geq \frac{\gamma}{2}\lambda_{1-n}^n \|u\|^2 - a_3 \sum_{i=1}^{k} \left(\sqrt{u_{i+1}^2 + u_i^2}\right)^\alpha - a_4 k \\
&\geq \frac{\gamma}{2}\lambda_{1-n}^n \|u\|^2 - a_3 \left\{ \left[\sum_{i=1}^{k} \left(\sqrt{u_{i+1}^2 + u_i^2}\right)^\alpha \right]^{1/\alpha} \right\}^\alpha - a_4 k \\
&\geq \frac{\gamma}{2}\lambda_{1-n}^n \|u\|^2 - a_3 c_2^\alpha \left\{ \left[\sum_{i=1}^{k} (u_{i+1}^2 + u_i^2) \right]^{1/2} \right\}^\alpha - a_4 k \\
&\geq \frac{\gamma}{2}\lambda_{1-n}^n \|u\|^2 - 2^\alpha a_3 c_2^\alpha \|u\|^\alpha - a_4 k \\
&\to +\infty \quad \text{as } \|u\| \to +\infty
\end{aligned}$$

By the continuity of J, we know from the above inequality that there exist lower bounds of values of the functional. And this means that J attains its minimal value at some point which is just the critical point of J with the finite norm. $\square$

3.4. Proof of Theorem 1.4

Proof. Assume, for the sake of contradiction, that the BVP (1.1) with (1.2) has a nontrivial solution. Then J has a nonzero critical point $u^\star$. Since

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}) - f(i, u_{i+1}, u_i, u_{i-1}),$$

we get

$$\begin{aligned}
\sum_{i=1}^{k} f(i, u_{i+1}^\star, u_i^\star, u_{i-1}^\star) u_i^\star &= \sum_{i=1}^{k} [(-1)^n \Delta^n(\gamma_{i-n+1}\Delta^n u_{i-n}^\star)] u_i^\star \\
&= \sum_{i=1-n}^{k} \gamma_{i+1}(\Delta^n u_i^\star)^2 \leq 0.
\end{aligned} \tag{3.1}$$

On the other hand, it follows from (F_6) that

$$\sum_{i=1}^{k} f(i, u_{i+1}^{\star}, u_i^{\star}, u_{i-1}^{\star}) u_i^{\star} > 0. \tag{3.2}$$

This contradicts (3.1) and hence the proof is complete. □

4. Examples

As an application of Theorems 1.2, 1.3 and 1.4, we give three examples to illustrate our main results.

Example 4.1. For $i \in \mathbf{Z}(1,k)$, assume that

$$\Delta^3(\gamma_{i-2}\Delta^3 u_{i-3}) = -\beta u_i[\varphi(i)(u_{i+1}^2 + u_i^2)^{\beta/2-1} + \varphi(i-1)(u_i^2 + u_{i-1}^2)^{\beta/2-1}], \tag{4.1}$$

with boundary value conditions

$$u_{-2} = u_{-1} = u_0 = 0, \quad u_{k+1} = u_{k+2} = u_{k+3} = 0, \tag{4.2}$$

where γ_i is real valued for each $i \in \mathbf{Z}(-1, k+1)$ and $\gamma_i > 0$, $\beta > 2$, φ is continuously differentiable and $\varphi(i) > 0$, $i \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$.

We have

$$f(i, v_1, v_2, v_3) = \beta v_2[\varphi(i)(v_1^2 + v_2^2)^{\beta/2-1} + \varphi(i-1)(v_2^2 + v_3^2)^{\beta/2-1}]$$

and

$$F(i, v_1, v_2) = \varphi(i)(v_1^2 + v_2^2)^{\beta/2}.$$

It is easy to verify all the assumptions of Theorem 1.2 are satisfied and then the BVP (4.1) with (4.2) possesses at least two nontrivial solutions.

Example 4.2. For $i \in \mathbf{Z}(1,k)$, assume that

$$\Delta^5(9^{i-4}\Delta^5 u_{i-5}) = -\alpha u_i[\psi(i)(u_{i+1}^2 + u_i^2)^{\alpha/2-1} + \psi(i-1)(u_i^2 + u_{n-1}^2)^{\alpha/2-1}], \tag{4.3}$$

with boundary value conditions

$$u_{-4} = u_{-3} = u_{-2} = u_{-1} = u_0 = 0, \quad u_{k+1} = u_{k+2} = u_{k+3} = u_{k+4} = u_{k+5} = 0, \tag{4.4}$$

where $1 < \alpha < 2$, ψ is continuously differentiable and $\psi(i) > 0$, $i \in \mathbf{Z}(1,k)$ with $\psi(0) = 0$.

We have

$$\gamma_i = 9^i, \quad f(i, v_1, v_2, v_3) = \alpha v_2[\psi(i)(v_1^2 + v_2^2)^{\alpha/2-1} + \psi(i-1)(v_2^2 + v_3^2)^{\alpha/2-1}]$$

and

$$F(i, v_1, v_2) = \psi(i)(v_1^2 + v_2^2)^{\alpha/2}.$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied and then the BVP (4.3) with (4.4) possesses at least one solution.

Example 4.3. For $i \in \mathbf{Z}(1,k)$, assume that

$$-\Delta^8 u_{i-4} = \frac{12}{7} u_i[(u_{i+1}^2 + u_i^2)^{-1/7} + (u_i^2 + u_{i-1}^2)^{-1/7}], \tag{4.5}$$

with boundary value conditions

$$u_{-3} = u_{-2} = u_{-1} = u_0 = 0, \quad u_{k+1} = u_{k+2} = u_{k+3} = u_{k+4} = 0. \tag{4.6}$$

We have

$$\gamma_i \equiv -1, \quad f(i, v_1, v_2, v_3) = \frac{12}{7} v_2[(v_1^2 + v_2^2)^{-1/7} + (v_2^2 + v_3^2)^{-1/7}]$$

and

$$F(i, v_1, v_2) = (v_1^2 + v_2^2)^{6/7}.$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied and then the BVP (4.5) with (4.6) has no nontrivial solutions.

References

[1] R. P. Agarwal, Difference equations and inequalities, Theory, methods and applications **228**, Marcel Dekker, New York, 2000.

[2] C. D. Ahlbrant, Dominant and recessive solutions of symmetric three term recurrences, J. Differential Equations **107** (1994), 238–258.

[3] C. D. Ahlbrant and A. C. Peterson, The (n,n)-disconjugacy of a $2n$th-order linear difference equation, Comput. Math. Appl. **28**(1–3) (1994), 1–9.

[4] D. Anderson, A $2n$th-order linear difference equation, Commun. Appl. Anal. **2** (1998), 521–529.

[5] D. Arcoya, Positive solutions for semilinear Dirichlet problems in an annulus, J. Differential Equations **94** (1991), 217–227.

[6] Z. Bai and H. Wang, On positive solutions of some nonlinear fourth-order beam equations, J. Math. Anal. Appl. **270** (2002), 357–368.

[7] X. C. Cai and J. S. Yu, Existence of periodic solutions for a $2n$th-order nonlinear difference equation, J. Math. Anal. Appl. **329** (2007), 870–878.

[8] M. Cecchi, M. Marini and G. Villari, On the monotonicity property for a certain class of second order differential equations, J. Differential Equations **82** (1989), 15–27.

[9] S. Z. Chen, Disconjugacy, disfocality, and oscillation of second order difference equations, J. Differential Equations **107** (1994), 383–394.

[10] P. Chen, Existence of homoclinic orbits in discrete Hamilton systems without Palais-Smale condition, J. Difference Equ. Appl. **19** (2013), 1781–1794.

[11] P. Chen and H. Fang, Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations, Adv. Difference Equ. **2007** (2007), 1–9.

[12] P. Chen and X. H. Tang, Existence and multiplicity of homoclinic orbits for $2n$th-order nonlinear difference equations containing both many advances and retardations, J. Math. Anal. Appl. **381** (2011), 485–505.

[13] P. Chen and X. H. Tang, Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation, Appl. Math. Comput. **217** (2011), 4408–4415.

[14] P. Chen and X. H. Tang, Existence of homoclinic solutions for the second-order discrete p-Laplacian systems, Taiwanese J. Math. **15** (2011), 2123–2143.

[15] P. Chen and X. H. Tang, New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, Math. Comput. Modelling **55** (2012), 723–739.

[16] P. Chen and X. H. Tang, Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems, Bull. Belg. Math. Soc. **20** (2013), 193–212.

[17] P. Chen and X. H. Tang, Existence of homoclinic solutions for some second-order discrete Hamiltonian systems, J. Difference Equ. Appl. **19** (2013), 633–648.

[18] P. Eloe, A boundary value problem for a system of difference equations, Nonlinear Anal. **7** (1983), 813–820.

[19] H. Fang and D. P. Zhao, Existence of nontrivial homoclinic orbits for fourth-order difference equations, Appl. Math. Comput. **214** (2009), 163–170.

[20] C. J. Guo, D. O'regan and R. P. Agarwal, Existence of multiple periodic solutions for a class of first-order neutral differential equations, Appl. Anal. Discrete Math. **5** (2011), 147–158.

[21] C. J. Guo, D. O'regan, Y. T. Xu and R. P. Agarwal, Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods, Appl. Math. Inform. Mech. **4** (2012), 1–15.

[22] C. J. Guo and Y. T. Xu, Existence of periodic solutions for a class of second order differential equation with deviating argument, J. Appl. Math. Comput. **28** (2008), 425–433.

[23] Z. M. Guo and J. S. Yu, Existence of periodic and subharmonic solutions for second-order superlinear difference equations, Sci. China Math. **46** (2003), 506–515.

[24] Z. M. Guo and J. S. Yu, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, J. London Math. Soc. **68** (2003), 419–430.

[25] Z. M. Guo and J. S. Yu, Applications of critical point theory to difference equations, Fields Inst. Commun. **42** (2004), 187–200.

[26] J. K. Hale and J. Mawhin, Coincidence degree and periodic solutions of neutral equations, J. Differential Equations **15** (1974), 295–307.

[27] G. Han and Z. Xu, Multiple solutions of some nonlinear fourth-order beam equations, Nonlinear Anal. **68** (2008), 3646–3656.

[28] J. Hendrson and H. B. Thompson, Existence of multiple solutions for second-order discrete boundary value problems, Comput. Math. Appl. **43** (2002), 1239–1248.

[29] V. L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Mathematics and its applications **256**, Kluwer Academic Publishers, Dordrecht, 1993.

[30] Y. J. Liu and W. G. Ge, Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator, J. Math. Anal. Appl. **278** (2003), 551–561.

[31] J. Mawhin and M. Willem, Critical point theory and Hamiltonian systems **74**, Springer, New York, 1989.

[32] M. Migda, Existence of nonoscillatory solutions of some higher order difference equations, Appl. Math. E-notes **4** (2004), 33–39.

[33] T. Peil and A. Peterson, Asymptotic behavior of solutions of a two-term difference equation, Rocky Mountain J. Math. **24** (1994), 233–251.

[34] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS reg. conf. ser. math. **65**, Amer. Math. Soc., Providence, RI, New York, 1986.

[35] Y. Rodrgues, On nonlinear discrete boundary value problems, J. Math. Anal. Appl. **114** (1986), 398–408.

[36] A. N. Sharkovsky, Y. L. Maistrenko and E. Y. Romanenko, Difference equations and their applications **11**, Kluwer Academic Publishers, Dordrecht, 1993.
[37] H. P. Shi, W. P. Ling, Y. H. Long and H. Q. Zhang, Periodic and subharmonic solutions for second order nonlinear functional difference equations, Commun. Math. Anal. **5**(2) (2008), 50–59.
[38] D. Smets and M. Willem, Solitary waves with prescribed speed on infinite lattices, J. Funct. Anal. **149** (1997), 266–275.
[39] H. Y. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, J. Differential Equations **109** (1994), 1–7.
[40] Q. Yao, Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right, Nonlinear Anal. **69** (2008), 1570–1580.
[41] J. S. Yu and Z. M. Guo, Boundary value problems of discrete generalized Emden-Fowler equation, Sci. China Math. **49** (2006), 1303–1314.
[42] J. S. Yu and Z. M. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, J. Differential Equations **231** (2006), 18–31.
[43] Z. Zhou, J. S. Yu and Y. M. Chen, Periodic solutions of a $2n$th-order nonlinear difference equation, Sci. China Math. **53** (2010), 41–50.
[44] W. M. Zou and M. Schechter, Critical point theory and its applications, Springer, New York, 2006.

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