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KODAI MATH. J.
37 (2014), 383–395

THE BEST CONSTANT OF THREE KINDS OF THE DISCRETE SOBOLEV INEQUALITIES ON THE COMPLETE GRAPH

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Abstract

We introduce a discrete Laplacian $\boldsymbol{A}$ on the complete graph with N vertices, that is, K_N. We obtain the best constants of three kinds of discrete Sobolev inequalities on K_N. The background of the first inequality is the discrete heat operator $(d/dt + \boldsymbol{A} + a_0\boldsymbol{I}) \cdots (d/dt + \boldsymbol{A} + a_{M-1}\boldsymbol{I})$ with positive distinct characteristic roots $a_0, \dots, a_{M-1}$. The second one is the difference operator $(\boldsymbol{A} + a_0\boldsymbol{I}) \cdots (\boldsymbol{A} + a_{M-1}\boldsymbol{I})$ and the third one is the discrete polyharmonic operator $\boldsymbol{A}^M$. Here $\boldsymbol{A}$ is an $N \times N$ real symmetric positive-semidefinite matrix whose eigenvector corresponding to zero eigenvalue is $\mathbf{1} = {}^t(1, 1, \dots, 1)$. A discrete heat kernel, a Green's matrix and a pseudo Green's matrix are obtained by means of $\boldsymbol{A}$.

1. Introduction

For any fixed $M = 1, 2, 3, \dots,$ we put $a = (a_0, \dots, a_{M-1})$ and assume $0 < a_0 < a_1 < \cdots < a_{M-1}$. We introduce the characteristic polynomial

$$P(z) = \prod_{j=0}^{M-1} (z + a_j)$$

and the function

$$e(t) = \sum_{j=0}^{M-1} b_j e^{-a_j t}, \quad b_j = \frac{1}{P'(-a_j)} = \frac{1}{\prod_{k=0, k \neq j}^{M-1} (-a_j + a_k)}. \tag{1.1}$$

The coefficients b_j appear in the partial fraction expansion

$$\frac{1}{P(z)} = \sum_{j=0}^{M-1} b_j \frac{1}{z + a_j}. \tag{1.2}$$

2010 *Mathematics Subject Classification*. Primary 46E39, Secondary 35K08.

Key words and phrases. discrete Sobolev inequality, discrete Laplacian, discrete heat kernel, Green's matrix, reproducing relation.

Received July 5, 2013; revised October 29, 2013.

We assume $N = 2, 3, 4, \ldots$. We set the indices of vertices as Figure 1 and introduce the discrete Laplacian $\boldsymbol{A}$ as

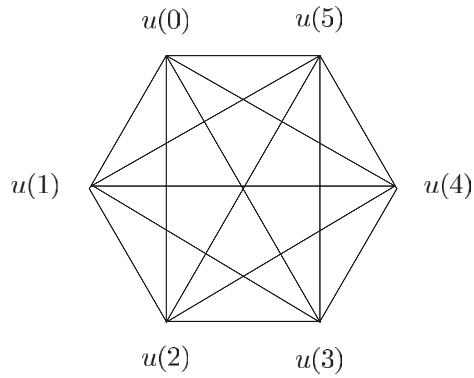

FIGURE 1. Complete graph K_6.

$$\boldsymbol{A} = \boldsymbol{A}(N) = (a(N; i, j)) \quad (0 \le i, j \le N-1),$$
$$a(N; i, j) = \begin{cases} N-1 & (i = j), \\ -1 & (i \ne j). \end{cases}$$

Here $\boldsymbol{A}$ is an $N \times N$ real symmetric positive-semidefinite matrix which has an eigenvalue 0 and whose eigenvector is $\mathbf{1} = {}^t(1, 1, \ldots, 1)$. We introduce the constants C_0, $C_0(a)$ and $C_1(a)$ as

$$C_0 = \frac{N-1}{N^{M+1}}, \quad C_0(a) = \sum_{j=0}^{M-1} b_j \frac{a_j + 1}{a_j(a_j + N)},$$
$$C_1(a) = \sum_{i,j=0}^{M-1} b_i b_j \frac{a_i + a_j + 2}{(a_i + a_j)(a_i + a_j + 2N)}.$$

For any $\boldsymbol{u} = {}^t(u(0), u(1), \ldots, u(N-1)) \in \mathbf{C}^N$ and $\boldsymbol{u}(t) = {}^t(u(0, t), u(1, t), \ldots, u(N-1, t)) \in \mathbf{C}^N$ on the complete graph K_N, we define three kinds of the Sobolev energy:

$$E(\boldsymbol{u}) = \boldsymbol{u}^* \boldsymbol{A}^M \boldsymbol{u}, \quad E(a; \boldsymbol{u}) = \boldsymbol{u}^* \prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I}) \boldsymbol{u},$$
$$F(a; \boldsymbol{u}(t)) = \int_{-\infty}^{\infty} \left\| \prod_{j=0}^{M-1} \left(\frac{d}{dt} + \boldsymbol{A} + a_j \boldsymbol{I} \right) \boldsymbol{u}(t) \right\|^2 dt,$$

where $\|\boldsymbol{u}(t)\|^2 = (\boldsymbol{u}(t))^* \boldsymbol{u}(t)$. We introduce the following three matrices

(1.3) Discrete heat kernel: $\boldsymbol{H}(t) = \exp(-t\boldsymbol{A})$,

$$\text{(1.4)} \qquad \text{Green's matrix:} \quad \boldsymbol{G}(a) = \left(\prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I}) \right)^{-1} = \int_0^\infty e(t)\boldsymbol{H}(t)\,dt,$$

$$\text{(1.5)} \qquad \text{Pseudo Green's matrix:} \quad \boldsymbol{G}_* = \lim_{a \to +0} \left(\boldsymbol{G}(a) - \prod_{j=0}^{M-1} a_j^{-1} \boldsymbol{E}_0 \right),$$

where $\boldsymbol{E}_0 = N^{-1}\boldsymbol{1}\,{}^t\boldsymbol{1}$ is a projection matrix to the eigenspace associated with the eigenvalue 0 of $\boldsymbol{A}$. In this paper, we obtain the best constants of three kinds of discrete Sobolev inequalities on K_N as the following theorems.

Theorem 1.1. *For any $\boldsymbol{u} \in \mathbf{C}^N$ with ${}^t\boldsymbol{1}\boldsymbol{u} = 0$, there exists a positive constant C which is independent of $\boldsymbol{u}$, such that the discrete Sobolev inequality*

$$\text{(1.6)} \qquad \left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le CE(\boldsymbol{u})$$

holds. Among such C, the best constant is C_0. If we replace C by C_0 in (1.6), *the equality holds iff $\boldsymbol{u}$ is parallel to any column of $\boldsymbol{G}_*$.*

Theorem 1.2. *For any $\boldsymbol{u} \in \mathbf{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}$, such that the discrete Sobolev inequality*

$$\text{(1.7)} \qquad \left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le CE(a; \boldsymbol{u})$$

holds. Among such C, the best constant is $C_0(a)$. If we replace C by $C_0(a)$ in (1.7), *the equality holds iff $\boldsymbol{u}$ is parallel to any column of $\boldsymbol{G}(a)$.*

Theorem 1.3. *For any $\boldsymbol{u}(t) \in \mathbf{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}(t)$, such that the discrete Sobolev-type inequality*

$$\text{(1.8)} \qquad \left(\sup_{0 \le j \le N-1,\, -\infty < s < \infty} |u(j,s)| \right)^2 \le CF(a; \boldsymbol{u}(t))$$

holds. Among such C, the best constant is $C_1(a)$. If we replace C by $C_1(a)$ in (1.8), *the equality holds iff $\boldsymbol{u}(t)$ is parallel to any column of*

$$\text{(1.9)} \qquad \int_{|t|}^\infty \frac{1}{2} e\left(\frac{t+\sigma}{2}\right) e\left(\frac{t-\sigma}{2}\right) \boldsymbol{H}(\sigma)\,d\sigma \quad (-\infty < t < \infty).$$

Research on discrete Sobolev inequalities was performed in [1] on graphs, in our previous papers [3, 5] on periodic one-dimensional lattices and in [2, 4] on regular polyhedra.

2. Difference equations

First, we explain three difference equations concerning the discrete heat kernel (1.3), the Green's matrix (1.4) and the pseudo Green's matrix (1.5).

PROPOSITION 2.1. *For any* $\boldsymbol{f}(t) \in \mathbf{C}^N$, *the discrete heat equation*

$$\prod_{j=0}^{M-1}\left(\frac{d}{dt} + \boldsymbol{A} + a_j \boldsymbol{I}\right)\boldsymbol{u} = \boldsymbol{f}(t) \quad (-\infty < t < \infty) \tag{2.1}$$

has the unique solution given by

$$\boldsymbol{u}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{f}(s)\, ds \quad (-\infty < t < \infty), \tag{2.2}$$

$$\boldsymbol{H}_*(t) = Y(t)e(t)\boldsymbol{H}(t) \quad (-\infty < t < \infty), \tag{2.3}$$

where $Y(t) = 1$ $(0 \le t < \infty)$, 0 $(-\infty < t < 0)$ *is the Heaviside step function and* $e(t)$ *is defined in* (1.1).

Proof of Proposition 2.1. By the Fourier transform

$$\boldsymbol{u}(t) \hat{\to} \hat{\boldsymbol{u}}(\omega) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t}\boldsymbol{u}(t)\, dt,$$

(2.1) is transformed into

$$\prod_{j=0}^{M-1}(\sqrt{-1}\omega + \boldsymbol{A} + a_j\boldsymbol{I})\hat{\boldsymbol{u}}(\omega) = \hat{\boldsymbol{f}}(\omega) \quad (-\infty < \omega < \infty).$$

Solving this and remarking the formula (1.2), we have $\hat{\boldsymbol{u}}(\omega) = \hat{\boldsymbol{H}}_*(\omega)\hat{\boldsymbol{f}}(\omega)$, where

$$\begin{aligned}\hat{\boldsymbol{H}}_*(\omega) &= \left(\prod_{j=0}^{M-1}(\sqrt{-1}\omega\boldsymbol{I} + \boldsymbol{A} + a_j\boldsymbol{I})\right)^{-1} \\ &= \sum_{j=0}^{M-1} b_j(\sqrt{-1}\omega\boldsymbol{I} + \boldsymbol{A} + a_j\boldsymbol{I})^{-1} = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t} Y(t)e(t)\boldsymbol{H}(t)\, dt.\end{aligned}$$

From the inverse Fourier transform, we have (2.2) and (2.3). It should be noted that $\boldsymbol{H}_*(t)$ satisfies the relations:

$$\left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{H}_* = \boldsymbol{O},$$

$$\boldsymbol{H}_*(t-s)|_{s=t-0} - \boldsymbol{H}_*(t-s)|_{s=t+0} = \boldsymbol{I} \quad (-\infty < t < \infty),$$

where $\boldsymbol{O}$ is zero matrix. This completes the proof of Proposition 2.1. ■

Let us put ω as $\omega = \exp(\sqrt{-1}2\pi/N)$ and $\hat{a}(k) = 0$ $(k = 0)$, N $(1 \le k \le N-1)$ be the eigenvalues of $\boldsymbol{A}$. Then, the Jordan canonical form of $\boldsymbol{A}$ is given by

$$\hat{\boldsymbol{A}} = \mathrm{diag}\{\hat{a}(0), \hat{a}(1), \dots, \hat{a}(N-1)\} = \mathrm{diag}\{0, N, \dots, N\}.$$

Further, $\boldsymbol{\varphi}_k$ $(0 \le k \le N-1)$ denote the eigenvectors of $\boldsymbol{A}$ as follows:

$$\boldsymbol{\varphi}_k = \frac{1}{\sqrt{N}}\,{}^t(1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k}) \in \mathbf{C}^N \quad (0 \le k \le N-1).$$

These eigenvectors are chosen to satisfy the relation $\boldsymbol{\varphi}_k^* \boldsymbol{\varphi}_l = \delta(k-l)$, where $\delta(k) = 1$ $(k = 0)$, 0 $(k \ne 0)$. We introduce a unitary $N \times N$ matrix $\boldsymbol{W} = (\boldsymbol{\varphi}_0, \dots, \boldsymbol{\varphi}_{N-1})$ and orthogonal projection matrices $\boldsymbol{E}_k = \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^*$ $(0 \le k \le N-1)$. It is easy to see that the relations,

$$\boldsymbol{W}^*\boldsymbol{W} = \boldsymbol{W}\boldsymbol{W}^* = \boldsymbol{I}, \quad \boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k, \quad \boldsymbol{E}_k^* = \boldsymbol{E}_k,$$

hold. Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{I}$ and $\boldsymbol{A}$ as

$$\boldsymbol{I} = \boldsymbol{W}\boldsymbol{W}^* = \sum_{k=0}^{N-1} \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{N-1} \boldsymbol{E}_k, \tag{2.4}$$

$$\begin{aligned}\boldsymbol{A} = \boldsymbol{W}\hat{\boldsymbol{A}}\boldsymbol{W}^* &= \sum_{k=0}^{N-1} \hat{a}(k)\boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^* = \sum_{k=0}^{N-1} \hat{a}(k)\boldsymbol{E}_k = N\sum_{k=1}^{N-1} \boldsymbol{E}_k \\ &= N(\boldsymbol{I} - \boldsymbol{E}_0).\end{aligned}$$

Using $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we have

$$\boldsymbol{A}^M = N^M \sum_{k=1}^{N-1} \boldsymbol{E}_k = N^M(\boldsymbol{I} - \boldsymbol{E}_0), \tag{2.5}$$

$$\prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I}) = \prod_{j=0}^{M-1} \sum_{k=0}^{N-1} (\hat{a}(k) + a_j)\boldsymbol{E}_k = \sum_{k=0}^{N-1} P(\hat{a}(k))\boldsymbol{E}_k. \tag{2.6}$$

Proposition 2.2. *For any $\boldsymbol{f} \in \mathbf{C}^N$, the difference equation*

$$\prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I})\boldsymbol{u} = \boldsymbol{f}$$

has the unique solution given by $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{f}$, where $\boldsymbol{G} = \boldsymbol{G}(a)$ is the Green's matrix expressed as

$$\boldsymbol{G} = \sum_{j=0}^{M-1} b_j \left[\frac{1}{a_j + N}\boldsymbol{I} + \left(\frac{1}{a_j} - \frac{1}{a_j + N}\right)\boldsymbol{E}_0\right]. \tag{2.7}$$

Proof of Proposition 2.2. From (2.4) and (2.6), we have

$$\sum_{k=0}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = \prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I})\boldsymbol{u} = \sum_{k=0}^{N-1} P(\hat{a}(k))\boldsymbol{E}_k \boldsymbol{u}.$$

Multiplying $\boldsymbol{E}_l$ from the left to both sides of the above relation and using the relation $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l \boldsymbol{u} = (P(\hat{a}(l)))^{-1} \boldsymbol{E}_l \boldsymbol{f}$. Then, we see that

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{N-1} \boldsymbol{E}_l \boldsymbol{u} = \sum_{l=0}^{N-1} \frac{1}{P(\hat{a}(l))} \boldsymbol{E}_l \boldsymbol{f} = \boldsymbol{G}\boldsymbol{f},$$

$$\boldsymbol{G} = \sum_{l=0}^{N-1} \frac{1}{P(\hat{a}(l))} \boldsymbol{E}_l = \frac{1}{P(0)} \boldsymbol{E}_0 + \frac{1}{P(N)} \sum_{l=1}^{N-1} \boldsymbol{E}_l$$
$$= \frac{1}{P(N)} \boldsymbol{I} + \left(\frac{1}{P(0)} - \frac{1}{P(N)} \right) \boldsymbol{E}_0.$$

Using (1.2), we have (2.7). This completes the proof of Proposition 2.2. ∎

Proposition 2.3. *For any* $\boldsymbol{f} \in \mathbf{C}^N$ *with the solvability condition* ${}^t\mathbf{1}\boldsymbol{f} = 0$, *the difference equation* $\boldsymbol{A}^M \boldsymbol{u} = \boldsymbol{f}$ *with the orthogonality condition* ${}^t\mathbf{1}\boldsymbol{u} = 0$ *has the unique solution given by* $\boldsymbol{u} = \boldsymbol{G}_* \boldsymbol{f}$, *where* $\boldsymbol{G}_*$ *is the pseudo Green's matrix expressed as*

$$\boldsymbol{G}_* = \frac{1}{N^M} (\boldsymbol{I} - \boldsymbol{E}_0) = \frac{1}{N^M} \sum_{k=1}^{N-1} \boldsymbol{E}_k. \tag{2.8}$$

$\boldsymbol{G}_*$ *satisfies* $\boldsymbol{A}^M \boldsymbol{G}_* = \boldsymbol{G}_* \boldsymbol{A}^M = \boldsymbol{I} - \boldsymbol{E}_0$, $\boldsymbol{G}_* \boldsymbol{E}_0 = \boldsymbol{E}_0 \boldsymbol{G}_* = \boldsymbol{O}$.

Proof of Proposition 2.3. From (2.4), (2.5) and $\boldsymbol{E}_0 \boldsymbol{f} = N^{-1}\mathbf{1}{}^t\mathbf{1}\boldsymbol{f} = \mathbf{0}$, where $\mathbf{0}$ is zero vector, we have

$$\sum_{k=1}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \sum_{k=0}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = \boldsymbol{A}^M \boldsymbol{u} = N^M \sum_{k=1}^{N-1} \boldsymbol{E}_k \boldsymbol{u}.$$

Multiplying $\boldsymbol{E}_l$ from the left to both sides of the above relation and using the relation $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l \boldsymbol{u} = N^{-M} \boldsymbol{E}_l \boldsymbol{f}$ $(1 \le l \le N-1)$. Then, using $\boldsymbol{E}_0 \boldsymbol{u} = N^{-1}\mathbf{1}{}^t\mathbf{1}\boldsymbol{u} = \mathbf{0}$, we see that

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{N-1} \boldsymbol{E}_l \boldsymbol{u} = \sum_{l=1}^{N-1} \boldsymbol{E}_l \boldsymbol{u} = \frac{1}{N^M} \sum_{l=1}^{N-1} \boldsymbol{E}_l \boldsymbol{f} = \boldsymbol{G}_* \boldsymbol{f}.$$

So we have (2.8). Moreover, using (2.8) and $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we have

$$\boldsymbol{A}^M \boldsymbol{G}_* = \sum_{k,l=1}^{N-1} N^M \boldsymbol{E}_k \frac{1}{N^M} \boldsymbol{E}_l = \sum_{k=1}^{N-1} \boldsymbol{E}_k = \boldsymbol{I} - \boldsymbol{E}_0,$$

$$\boldsymbol{G}_* \boldsymbol{E}_0 = \sum_{k=1}^{N-1} \frac{1}{N^M} \boldsymbol{E}_k \boldsymbol{E}_0 = \boldsymbol{O}.$$

From the above relations, we see that $\boldsymbol{G}_*$ is a Penrose-Moore generalized inverse matrix of $\boldsymbol{A}^M$. This completes the proof of Proposition 2.3. ■

Next, we compute $\boldsymbol{H}$, $\boldsymbol{G}$ and $\boldsymbol{G}_*$ which represent the best constants of Sobolev inequalities. We introduce the N-dimension vector

$$\boldsymbol{\delta}_j = {}^t(\quad \delta(i-j) \quad)_{0 \le i \le N-1}.$$

LEMMA 2.1. *For any fixed* j $(0 \le j \le N-1)$, *we have the following relations*:

$$(2.9) \qquad {}^t\boldsymbol{\delta}_j \boldsymbol{H}(t) \boldsymbol{\delta}_j = \frac{1}{N}(1 + (N-1)e^{-Nt}).$$

$$(2.10) \qquad \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2 \, dt = C_1(a).$$

$$(2.11) \qquad {}^t\boldsymbol{\delta}_j \boldsymbol{G} \boldsymbol{\delta}_j = C_0(a).$$

$$(2.12) \qquad {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = C_0.$$

Proof of Lemma 2.1. Applying $\boldsymbol{A}^j = N^j(\boldsymbol{I} - \boldsymbol{E}_0)$ $(j = 1, 2, 3, \ldots)$ in (2.5) to (1.3), we have

$$\boldsymbol{H}(t) = \exp(-t\boldsymbol{A}) = \sum_{j=0}^{\infty} \frac{1}{j!}(-t)^j \boldsymbol{A}^j = \boldsymbol{I} + \sum_{j=1}^{\infty} \frac{1}{j!}(-t)^j \boldsymbol{A}^j$$
$$= \boldsymbol{I} + \left(\sum_{j=1}^{\infty} \frac{1}{j!}(-tN)^j \right)(\boldsymbol{I} - \boldsymbol{E}_0) = \boldsymbol{I} + (-1 + e^{-Nt})(\boldsymbol{I} - \boldsymbol{E}_0).$$

From the relation above, we have

$$\boldsymbol{H}(t) = e^{-Nt}\boldsymbol{I} + (1 - e^{-Nt})\boldsymbol{E}_0$$

and (2.9). Noting ${}^t\boldsymbol{H}(t) = \boldsymbol{H}(t)$, $(\boldsymbol{H}(t))^2 = \boldsymbol{H}(2t)$ and (2.3), we have

$$\int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2 \, dt = \int_{-\infty}^{\infty} {}^t(\boldsymbol{H}_*(t)\boldsymbol{\delta}_j)(\boldsymbol{H}_*(t)\boldsymbol{\delta}_j) \, dt$$
$$= \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j (\boldsymbol{H}_*(t))^2 \boldsymbol{\delta}_j \, dt = \int_0^{\infty} e^2(t)\, {}^t\boldsymbol{\delta}_j \boldsymbol{H}(2t) \boldsymbol{\delta}_j \, dt$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{i,j=0}^{M-1} b_i b_j e^{-(a_i+a_j)t} \frac{1}{N}(1+(N-1)e^{-2Nt})\,dt \\
&= \frac{1}{N}\sum_{i,j=0}^{M-1} b_i b_j \int_0^\infty [e^{-(a_i+a_j)t} + (N-1)e^{-(a_i+a_j+2N)t}]\,dt \\
&= \frac{1}{N}\sum_{i,j=0}^{M-1} b_i b_j \left[\frac{1}{a_i+a_j} + (N-1)\frac{1}{a_i+a_j+2N}\right] = C_1(a).
\end{aligned}
$$

So we have (2.10). Since the proofs of (2.11) and (2.12) are standard and easy, we omit it. This completes the proof of Lemma 2.1. ■

3. Reproducing relation

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^N$, we introduce the inner products

$$
\begin{aligned}
&(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{u}, \\
&\|\boldsymbol{u}\|^2 = (\boldsymbol{u}, \boldsymbol{u}), \\
&(\boldsymbol{u}, \boldsymbol{v})_H = \left(\prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I})\boldsymbol{u}, \boldsymbol{v}\right) = \boldsymbol{v}^* \prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I})\boldsymbol{u}, \\
&\|\boldsymbol{u}\|_H^2 = (\boldsymbol{u}, \boldsymbol{u})_H = E(a; \boldsymbol{u}).
\end{aligned}
$$

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}_0^N := \{\boldsymbol{x} \mid \boldsymbol{x} \in \mathbf{C}^N \ \textit{and} \ {}^t\mathbf{1}\boldsymbol{x} = 0\}$, we introduce the inner product

$$
\begin{aligned}
&(\boldsymbol{u}, \boldsymbol{v})_A = (\boldsymbol{A}^M \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{A}^M \boldsymbol{u}, \\
&\|\boldsymbol{u}\|_A^2 = (\boldsymbol{u}, \boldsymbol{u})_A = E(\boldsymbol{u}).
\end{aligned}
$$

First, we show the positive definiteness of $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_A$.

Lemma 3.1.
(1) *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^N$, $(\boldsymbol{u}, \boldsymbol{v})_H$ *is defined as an inner product.*
(2) *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}_0^N$, $(\boldsymbol{u}, \boldsymbol{v})_A$ *is defined as an inner product.*

Proof of Lemma 3.1. (1) is obvious since $(\boldsymbol{A} + a_0 \boldsymbol{I}) \cdots (\boldsymbol{A} + a_{M-1} \boldsymbol{I})$ is positive definite. We show only (2). For $\boldsymbol{u} \in \mathbf{C}_0^N$, we have

$$
\begin{aligned}
\boldsymbol{u} &= \boldsymbol{I}\boldsymbol{u} = \sum_{k=0}^{N-1} \boldsymbol{E}_k \boldsymbol{u} = \sum_{k=1}^{N-1} \boldsymbol{E}_k \boldsymbol{u}, \\
\|\boldsymbol{u}\|^2 &= \sum_{k,l=0}^{N-1} \boldsymbol{u}^* \boldsymbol{E}_l^* \boldsymbol{E}_k \boldsymbol{u} = \sum_{k=1}^{N-1} \|\boldsymbol{E}_k \boldsymbol{u}\|^2.
\end{aligned}
$$

From the relation $\boldsymbol{E}_k = \boldsymbol{E}_k \boldsymbol{E}_k = \boldsymbol{E}_k^* \boldsymbol{E}_k$, we have

$$\|\boldsymbol{u}\|_A^2 = \boldsymbol{u}^* \boldsymbol{A}^M \boldsymbol{u} = N^M \sum_{k=1}^{N-1} \boldsymbol{u}^* \boldsymbol{E}_k^* \boldsymbol{E}_k \boldsymbol{u} = N^M \sum_{k=1}^{N-1} \|\boldsymbol{E}_k \boldsymbol{u}\|^2 = N^M \|\boldsymbol{u}\|^2.$$

Since $N > 0$, we have $\|\boldsymbol{u}\|_A^2 \geq 0$, and $\|\boldsymbol{u}\|_A^2 = 0$ holds iff $\boldsymbol{u} = \boldsymbol{0}$. This completes the proof of Lemma 3.1. ∎

Next, we show that $\boldsymbol{G}$ and $\boldsymbol{G}_*$ are a reproducing matrix for the inner products $(\cdot\,,\cdot)_H$ and $(\cdot\,,\cdot)_A$, respectively.

Lemma 3.2. *For any* $\boldsymbol{u} \in \mathbf{C}_0^N$ *and fixed* j $(0 \leq j \leq N-1)$, *we have the following reproducing relations*:
(1) $u(j) = (\boldsymbol{u}, \boldsymbol{G}_* \boldsymbol{\delta}_j)_A$.
(2) $C_0 = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{\delta}_j = \|\boldsymbol{G}_* \boldsymbol{\delta}_j\|_A^2 = E(\boldsymbol{G}_* \boldsymbol{\delta}_j)$.

Proof of Lemma 3.2. Noting $\boldsymbol{G}_*^* = \boldsymbol{G}_*$, we have (1) as follows:

$$(\boldsymbol{u}, \boldsymbol{G}_* \boldsymbol{\delta}_j)_A = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_* \boldsymbol{A}^M \boldsymbol{u} = {}^t\boldsymbol{\delta}_j (\boldsymbol{I} - \boldsymbol{E}_0) \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} - \frac{1}{N} \mathbf{1}\,{}^t\mathbf{1} \boldsymbol{u} = u(j).$$

Putting $\boldsymbol{u} = \boldsymbol{G}_* \boldsymbol{\delta}_j$ in (1) and using (2.12), we obtain (2). ∎

Lemma 3.3. *For any* $\boldsymbol{u} \in \mathbf{C}^N$ *and fixed* j $(0 \leq j \leq N-1)$, *we have the following reproducing relations*:
(1) $u(j) = (\boldsymbol{u}, \boldsymbol{G} \boldsymbol{\delta}_j)_H$.
(2) $C_0(a) = {}^t\boldsymbol{\delta}_j \boldsymbol{G} \boldsymbol{\delta}_j = \|\boldsymbol{G} \boldsymbol{\delta}_j\|_H^2 = E(a; \boldsymbol{G} \boldsymbol{\delta}_j)$.

The proof of Lemma 3.3. Noting $\boldsymbol{G}^* = \boldsymbol{G}$, we have (1) as follows:

$$(\boldsymbol{u}, \boldsymbol{G} \boldsymbol{\delta}_j)_H = {}^t\boldsymbol{\delta}_j \boldsymbol{G} \prod_{j=0}^{M-1} (\boldsymbol{A} + a_j \boldsymbol{I}) \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{I} \boldsymbol{u} = u(j).$$

Putting $\boldsymbol{u} = \boldsymbol{G} \boldsymbol{\delta}_j$ in (1) and using (2.11), we obtain (2). ∎

4. Proof of theorems

This section is devoted to the proof of main theorems.

Proof of Theorem 1.1. Applying the Schwarz inequality to Lemma 3.2 (1) and using Lemma 3.2 (2), we have

$$|u(j)|^2 \leq \|\boldsymbol{u}\|_A^2 \|\boldsymbol{G}_* \boldsymbol{\delta}_j\|_A^2 = C_0 E(\boldsymbol{u}).$$

Taking the maximum with respect to j on both sides, we obtain the discrete Sobolev inequality

$$\left(\max_{0\le j\le N-1}|u(j)|\right)^2 \le C_0 E(\boldsymbol{u}). \tag{4.1}$$

For any fixed number j_0 $(0\le j_0\le N-1)$, if we take $\boldsymbol{u}=\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}$ in (4.1), then we have

$$\left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}|\right)^2 \le C_0 E(\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}) = (C_0)^2.$$

Combining this with the trivial inequality

$$(C_0)^2 = |{}^t\boldsymbol{\delta}_{j_0}\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}|^2 \le \left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}|\right)^2,$$

we have

$$\left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}|\right)^2 = C_0 E(\boldsymbol{G}_*\boldsymbol{\delta}_{j_0}).$$

This shows that C_0 is the best constant of (4.1) and the equality holds for any column of $\boldsymbol{G}_*$. This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Applying the Schwarz inequality to Lemma 3.3 (1) and using Lemma 3.3 (2), we have

$$|u(j)|^2 \le \|\boldsymbol{u}\|_H^2\|\boldsymbol{G}\boldsymbol{\delta}_j\|_H^2 = C_0(a)E(a;\boldsymbol{u}).$$

Taking the maximum with respect to j on both sides, we have the discrete Sobolev inequality

$$\left(\max_{0\le j\le N-1}|u(j)|\right)^2 \le C_0(a)E(a;\boldsymbol{u}). \tag{4.2}$$

For any fixed number j_0 $(0\le j_0\le N-1)$, if we take $\boldsymbol{u}=\boldsymbol{G}\boldsymbol{\delta}_{j_0}$ in (4.2), then we have

$$\left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_{j_0}|\right)^2 \le C_0(a)E(a;\boldsymbol{G}\boldsymbol{\delta}_{j_0}) = (C_0(a))^2.$$

Combining this with the trivial inequality

$$(C_0(a))^2 = |{}^t\boldsymbol{\delta}_{j_0}\boldsymbol{G}\boldsymbol{\delta}_{j_0}|^2 \le \left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_{j_0}|\right)^2,$$

we have

$$\left(\max_{0\le j\le N-1}|{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_{j_0}|\right)^2 = C_0(a)E(a;\boldsymbol{G}\boldsymbol{\delta}_{j_0}).$$

This shows that $C_0(a)$ is the best constant of (4.2) and the equality holds for any column of $\boldsymbol{G}$. This completes the proof of Theorem 1.2. ■

Proof of Theorem 1.3. Replacing t with s in (2.2), we have

$$\boldsymbol{u}(s) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(s-t)\boldsymbol{f}(t)\,dt,$$

or equivalently

$$u(j,s) = {}^t\boldsymbol{\delta}_j\boldsymbol{u}(s) = \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j\boldsymbol{H}_*(s-t)\boldsymbol{f}(t)\,dt = \int_{-\infty}^{\infty} {}^t(\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j)\boldsymbol{f}(t)\,dt. \tag{4.3}$$

Applying the Schwarz inequality to (4.3), we have

$$\begin{aligned}|u(j,s)|^2 &\le \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j\|^2\,dt \int_{-\infty}^{\infty} \|\boldsymbol{f}(t)\|^2\,dt \\ &= \int_{-\infty}^{\infty} \|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\|^2\,dt \int_{-\infty}^{\infty} \left\| \prod_{j=0}^{M-1}\left(\frac{d}{dt}+\boldsymbol{A}+a_j\boldsymbol{I}\right)\boldsymbol{u}(t)\right\|^2 dt \\ &= C_1(a)F(a;\boldsymbol{u}(t)),\end{aligned}$$

where we use (2.1) and (2.10). Taking the supremum with respect to j and s, we obtain the discrete Sobolev-type inequality

$$\left(\sup_{0\le j\le N-1,\,-\infty<s<\infty} |u(j,s)|\right)^2 \le C_1(a)F(a;\boldsymbol{u}(t)). \tag{4.4}$$

For any fixed number j_0 $(0 \le j_0 \le N-1)$, we introduce the vector $\boldsymbol{U}(t)$ defined by

$$\boldsymbol{U}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}\,ds, \tag{4.5}$$

$$U(j,t) = {}^t\boldsymbol{\delta}_j\boldsymbol{U}(t) = \int_{-\infty}^{\infty} {}^t\boldsymbol{\delta}_j\boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}\,ds.$$

Then we have

$$\begin{aligned}\left(\sup_{0\le j\le N-1,\,-\infty<s<\infty} |U(j,s)|\right)^2 &\le C_1(a)F(a;\boldsymbol{U}(t)) \\ &= C_1(a)\int_{-\infty}^{\infty} \left\| \prod_{j=0}^{M-1}\left(\frac{d}{dt}+\boldsymbol{A}+a_j\boldsymbol{I}\right)\boldsymbol{U}(t)\right\|^2 dt \\ &= C_1(a)\int_{-\infty}^{\infty} \|\boldsymbol{H}_*(-t)\boldsymbol{\delta}_{j_0}\|^2\,dt = (C_1(a))^2.\end{aligned}$$

Combining this with the trivial inequality

$$(C_1(a))^2 = |U(j_0, 0)|^2 \le \left(\sup_{0 \le j \le N-1, -\infty < s < \infty} |U(j,s)| \right)^2,$$

we have

$$\left(\sup_{0 \le j \le N-1, -\infty < s < \infty} |U(j,s)| \right)^2 = C_1(a) F(a; \boldsymbol{U}(t)).$$

This shows that $C_1(a)$ is the best constant of (4.4) and the equality holds for $\boldsymbol{u}(t) = \boldsymbol{U}(t)$. From (4.5), we have (1.9) as follows:

$$\begin{aligned}
\boldsymbol{U}(t) &= \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s) \boldsymbol{H}_*(-s) \boldsymbol{\delta}_{j_0} \, ds \\
&= \int_{-\infty}^{\infty} Y(t-s) e(t-s) \boldsymbol{H}(t-s) \, Y(-s) e(-s) \boldsymbol{H}(-s) \boldsymbol{\delta}_{j_0} \, ds \\
&= \int_{-\infty}^{0 \wedge t} e(t-s) e(-s) \boldsymbol{H}(t-2s) \boldsymbol{\delta}_{j_0} \, ds \\
&= \int_{|t|}^{\infty} \frac{1}{2} e\left(\frac{t+\sigma}{2}\right) e\left(\frac{t-\sigma}{2}\right) \boldsymbol{H}(\sigma) \boldsymbol{\delta}_{j_0} \, d\sigma.
\end{aligned}$$

This completes the proof of Theorem 1.3. ∎

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