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SURFACES WITH INFLECTION POINTS IN EUCLIDEAN 4-SPACE

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Abstract

For a surface in the Euclidean 4-space, we prove a reduction theorem for the codimension of a surface all whose points are inflection points.

1. Introduction

The curvature ellipse is much interested in the study of a surface M in the Euclidean 4-space $\mathbf{R}^4$ (cf. [8, 9, 4]). At a point $p \in M$, the *curvature ellipse* $\mathscr{E}_p$ is defined by the image $\{\Pi(v,v) \in T_p^\perp M \mid v \in T_pM, |v| = 1\}$, in the normal space $T_p^\perp M$, of the unit circle in the tangent plane T_pM under the second fundamental form Π. If the curvature ellipse $\mathscr{E}_p$ degenerates to a segment contained in a straight line passing through $\mathbf{0}_p$ of $T_p^\perp M$, we say that p is an *inflection point*. A sufficient and necessary condition for p being an inflection point is that there exists a unit normal vector $\nu_p \in T_p^\perp M$ such that the ν-component $\langle \Pi, \nu \rangle$ of Π at p vanishes. In particular, if M lies an affine 3-space in $\mathbf{R}^4$, then all points are inflection points. On the other hand, the converse does not hold (e.g. Example 5.2, (ii)). Lane [7] proved that if the surface is exclusively made of inflection points, then it is locally either a developable surface or lies in a 3-space (cf. Little [8]). In this paper, we present the following reduction theorem.

Theorem 1. *Let X be a conformal immersion from a connected Riemann surface S into $\mathbf{R}^4$. Assume that the Gauss curvature K does not vanish anywhere. If all points of S are inflection points, then the surface $X(S)$ lies in an affine 3-space in $\mathbf{R}^4$.*

In order to prove this theorem, we introduce a new complex-valued local invariant Λ in Section 2. For the *resultant* Δ_p of X at $p \in S$ and the normal

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curvature $K_N(p)$, $\Lambda(p)$ satisfies that

$$4\Delta_p = (K_N(p))^2 - 4|\Lambda(p)|^2.$$

The local invariant Δ_p was considered in [8, 9, 4], in order to study of the curvature ellipse $\mathscr{E}_p$. The sign of Δ_p determines the position of $\mathscr{E}_p$ in $T_p^{\perp}M$, that is, whether the origin $\mathbf{0}_p$ of $T_p^{\perp}M$ lies inside of $\mathscr{E}_p$ or outside of $\mathscr{E}_p$ or on $\mathscr{E}_p$ (Lemma 2, cf. [8, Section 2]). However, the relation between Δ_p and curvatures of M is not clear since Δ_p is a polynomial of degree 4 with respect to the components of second fundamental form Π. On the other hand, the invariant $\Lambda(p)$ is a quadratic polynomial with respect to the components of the mean curvature vector and the Hopf differential. Hence, the criterion on the position of $\mathscr{E}_p$ in $T_p^{\perp}M$ is explicitly expressed in terms of cuvatures of $X(S)$.

In Section 2, we recall the definition of curvature ellipses $\mathscr{E}_p$ and the invariant Δ_p. Then we introduce the invariant Λ. Moreover, we give another simple proof of the above fact (i.e., Lemma 2) by using Λ and K_N. In Section 3, we represent Λ in terms of the Gauss maps. In Section 4, we prove Theorem 1. In Section 5, we give some examples of surfaces in $\mathbf{R}^4$.

2. Curvature ellipses

We prepare the terminologies following [8] (see also [4]).

Let S be a connected Riemann surface and $X : S \to \mathbf{R}^4$ a conformal immersion. From now on, we identify locally S with $X(S)$ ($\subset \mathbf{R}^4$) via the immersion X. Let $\{e_1, e_2, e_3, e_4\}$ denote an orthonormal frame on an open neighborhood of S, chosen e_1 and e_2 are tangent vectors to S with the frame $\{e_1, e_2\}$ agreeing with the orientation of T_pS, and chosen so that e_3 and e_4 are normal to the surface with the frame $\{e_1, e_2, e_3, e_4\}$ agreeing with a fixed orientation of $\mathbf{R}^4$. As usual, define the dual forms $\omega_A = dX \cdot e_A$ and the connection forms $\omega_A^B = de_A \cdot e_B$. The indices A, B run from 1 to 4. Then we have the structure equations:

$$\omega_A^B = -\omega_B^A, \quad d\omega_A = \sum_{B=1}^{4} \omega_A^B \wedge \omega_B, \quad d\omega_A^B = \sum_{C=1}^{4} \omega_A^C \wedge \omega_C^B.$$

Since $\omega_3 = \omega_4 = 0$ on S, by the Cartan Lemma, we obtain the functions h_{ij}^{α} such that $\omega_i^{\alpha} = \sum_{j=1}^{2} h_{ij}^{\alpha}\omega_j$. The indices i, j run from 1 to 2, and α, β run from 3 to 4. We have the symmetry $h_{ij}^{\alpha} = h_{ji}^{\alpha}$. The second fundamental form Π of the surface is

$$\Pi = (d^2X \cdot e_3)e_3 + (d^2X \cdot e_4)e_4 = \sum_{\alpha=3}^{4} \sum_{i,j=1}^{2} h_{ij}^{\alpha}\omega_i\omega_j e_\alpha.$$

The Gauss curvature K is defined by the formula

$$d\omega_1^2 = -K\omega_1 \wedge \omega_2.$$

The normal curvature K_N is also defined by the formula

$$d\omega_3^4 = -K_N \omega_1 \wedge \omega_2.$$

Both the Gauss curvature K and the normal curvature K_N are described in terms of the components h_{ij}^α:

$$K = h_{11}^3 h_{22}^3 - (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2,$$
$$K_N = (h_{11}^3 - h_{22}^3) h_{12}^4 - (h_{11}^4 - h_{22}^4) h_{12}^3.$$

For a given point $p \in S$, consider the unit circle S_p^1 in T_pS parametrized by the angle θ. We call the following map $\boldsymbol{\eta}$ from S_p^1 to the normal space $T_p^\perp S$ the *normal curvature vector*. Denote by γ_θ the unit-speed curve on S satisfying $\gamma_\theta(0) = p$ and $\gamma_\theta'(0) = \xi_\theta = \cos\theta e_1 + \sin\theta e_2$, and define $\boldsymbol{\eta}(\theta) = \boldsymbol{\eta}(\xi_\theta)$ by the normal part of $\gamma_\theta''(0)$. Then we obtain that

$$\begin{aligned}
\boldsymbol{\eta}(\theta) &= \sum_{\alpha=3}^{4} \sum_{i,j=1}^{2} h_{ij}^\alpha \omega_i(\xi_\theta)\omega_j(\xi_\theta) e_\alpha \\
&= (e_3 \quad e_4) \begin{pmatrix} h_{11}^3 \cos^2\theta + 2h_{12}^3 \cos\theta \sin\theta + h_{22}^3 \sin^2\theta \\ h_{11}^4 \cos^2\theta + 2h_{12}^4 \cos\theta \sin\theta + h_{22}^4 \sin^2\theta \end{pmatrix} \\
&= (e_3 \quad e_4) \begin{pmatrix} \frac{1}{2}(h_{11}^3 + h_{22}^3) + \frac{1}{2}(h_{11}^3 - h_{22}^3)\cos 2\theta + h_{12}^3 \sin 2\theta \\ \frac{1}{2}(h_{11}^4 + h_{22}^4) + \frac{1}{2}(h_{11}^4 - h_{22}^4)\cos 2\theta + h_{12}^4 \sin 2\theta \end{pmatrix}.
\end{aligned}$$

Recall that the mean curvature vector $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = \frac{1}{2}(h_{11}^3 + h_{22}^3)e_3 + \frac{1}{2}(h_{11}^4 + h_{22}^4)e_4.$$

Then we have

$$\boldsymbol{\eta}(\theta) - \boldsymbol{H} = (e_3 \quad e_4)\mathscr{H}\begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}, \quad \text{where } \mathscr{H} = \begin{pmatrix} \frac{1}{2}(h_{11}^3 - h_{22}^3) & h_{12}^3 \\ \frac{1}{2}(h_{11}^4 - h_{22}^4) & h_{12}^4 \end{pmatrix}.$$

The normal curvature K_N coincides with $2\det(\mathscr{H})$. When K_N is not zero at p, the locus $\mathscr{E}_p$ of $\boldsymbol{\eta}(\theta)$ is an ellipse centered at $\boldsymbol{H}$ in $T_p^\perp S$. So we call the locus $\mathscr{E}_p$ the *curvature ellipse* at p. When K_N is zero at p, the curvature ellipse $\mathscr{E}_p$ is a segment.

At a point p in S, if the origin $\mathbf{0}_p$ of $T_p^\perp S$ lies outside the curvature ellipse $\mathscr{E}_p$, then the point p is said to be *hyperbolic*. The point p of S is said to be *elliptic* if $\mathbf{0}_p$ lies inside $\mathscr{E}_p$, and the point p of S is said to be *parabolic* if $\mathbf{0}_p$ lies on $\mathscr{E}_p$. (In [11], the hyperbolic points are said to be *convex* and the elliptic points are said to be *aconvex*.) When $\mathscr{E}_p$ degenerates to a segment contained in a straight line passing through $\mathbf{0}_p$, the point p of M is said to be an *inflection point*. At an inflection point p in S, $K_N = 0$ at p. Moreover, we can choose a unit normal vector $\tilde{e}_3 \in T_p^{\perp} S$ such that the components of the second fundamental form with respect to $\tilde{e}_3$ are zero, that is,

$$d^2 X \cdot \tilde{e}_3 = \sum_{i,j=1}^{2} \tilde{h}_{ij}^3 \omega_i \omega_j = \mathbf{0}.$$

The last condition is a necessary and sufficient condition for that p is an inflection point.

The *resultant* Δ_p of X at p is defined by

$$\Delta_p = \frac{1}{4} \begin{vmatrix} h_{11}^3 & 2h_{12}^3 & h_{22}^3 & 0 \\ h_{11}^4 & 2h_{12}^4 & h_{22}^4 & 0 \\ 0 & h_{11}^3 & 2h_{12}^3 & h_{22}^3 \\ 0 & h_{11}^4 & 2h_{12}^4 & h_{22}^4 \end{vmatrix},$$

which is the resultant of the two polynomials $h_{11}^3 x^2 + 2h_{12}^3 xy + h_{22}^3 y^2$ and $h_{11}^4 x^2 + 2h_{12}^4 xy + h_{22}^4 y^2$. By the resultant Δ_p, we can distinguish the position of $\mathscr{E}_p$ in $T_p^{\perp} M$ as follows:

Lemma 2 ([8], [9]). *At a point p of S, assume that $K_N \neq 0$.*
(i) *p is a hyperbolic point if and only if $\Delta_p < 0$.*
(ii) *p is a parabolic point if and only if $\Delta_p = 0$.*
(iii) *p is an elliptic point if and only if $\Delta_p > 0$.*

Set $h^\alpha = \frac{1}{2}(h_{11}^\alpha + h_{22}^\alpha)$ and $\varphi^\alpha = \frac{1}{2}(h_{11}^\alpha - h_{22}^\alpha) - \boldsymbol{i} h_{12}^\alpha$ $(\alpha = 3, 4)$, where $\boldsymbol{i}$ denotes the imaginary unit. Then we have $\boldsymbol{H} = h^3 e_3 + h^4 e_4$, $K = (h^3)^2 + (h^4)^2 - |\varphi^3|^2 - |\varphi^4|^2$ and $K_N = 2\,\mathrm{Im}(\varphi^3 \overline{\varphi^4})$. Moreover, we set

$$\Lambda = -h^3 \varphi^4 + h^4 \varphi^3.$$

Then, we have the following lemma by a straightforward computation.

Lemma 3. *At a point p of S,*

$$4\Delta_p = (K_N(p))^2 - 4|\Lambda(p)|^2. \tag{1}$$

Remark 4. We can write

$$d(e_1 - \boldsymbol{i} e_2) \cdot (e_3 + \boldsymbol{i} e_4) \wedge d(e_1 - \boldsymbol{i} e_2) \cdot (e_3 - \boldsymbol{i} e_4) = -2\boldsymbol{i} \Lambda \phi \wedge \bar{\phi},$$

where $\phi = \omega_1 + \boldsymbol{i} \omega_2$.

The normal curvature vector $\boldsymbol{\eta}(\theta)$ at $p \in S$ is given by

$$\boldsymbol{\eta}(\theta) = (e_3 \quad e_4) \begin{pmatrix} h^3 + \mathrm{Re}(\varphi^3 e^{\boldsymbol{i}2\theta}) \\ h^4 + \mathrm{Re}(\varphi^4 e^{\boldsymbol{i}2\theta}) \end{pmatrix}, \tag{2}$$

and

$$\frac{d\boldsymbol{\eta}}{d\theta} = (e_3 \quad e_4)\begin{pmatrix} -2\ \mathrm{Im}(\varphi^3 e^{i2\theta}) \\ -2\ \mathrm{Im}(\varphi^4 e^{i2\theta}) \end{pmatrix}.$$

Proof of Lemma 2. We give here a different proof from that in [8, Section 2]. When $\mathbf{0}_p$ lies outside the curvature ellipse $\mathscr{E}_p$, there exist $\theta_1, \theta_2 \in [0, \pi)$ $(\theta_1 \neq \theta_2)$ such that the tangent vectors $\frac{d\boldsymbol{\eta}}{d\theta}(\theta_i)$ of $\mathscr{E}_p$ is a scalar multiplication of the position vectors $\boldsymbol{\eta}(\theta_i)$ $(i = 1, 2)$. This implies that the following equation for θ must have two distinct solutions:

$$0 = \det\begin{pmatrix} \boldsymbol{\eta}(\theta) & -\frac{1}{2}\frac{d\boldsymbol{\eta}}{d\theta} \end{pmatrix} = \begin{vmatrix} h^3 + \mathrm{Re}(\varphi^3 e^{i2\theta}) & \mathrm{Im}(\varphi^3 e^{i2\theta}) \\ h^4 + \mathrm{Re}(\varphi^4 e^{i2\theta}) & \mathrm{Im}(\varphi^4 e^{i2\theta}) \end{vmatrix}. \tag{3}$$

This equation implies that $h^3 + \varphi^3 e^{i2\theta}$ and $h^4 + \varphi^4 e^{i2\theta}$ lie on the same line through the origin in the complex plane. We then obtain

$$\begin{aligned} 0 &= \mathrm{Im}\{(h^3 + \varphi^3 e^{i2\theta})\overline{(h^4 + \varphi^4 e^{i2\theta})}\} \\ &= \mathrm{Im}\{(-h^3\varphi^4 + h^4\varphi^3)e^{i2\theta}\} + \frac{1}{2}K_N. \end{aligned} \tag{4}$$

Then, we have $|\Lambda| = |-h^3\varphi^4 + h^4\varphi^3| > \frac{1}{2}|K_N|$.

When $\mathbf{0}_p \in \mathscr{E}_p$, there exists only one $\theta \in \mathbf{R}/\pi\mathbf{Z}$ satisfying (4). Then, we have $|\Lambda| = |-h^3\varphi^4 + h^4\varphi^3| = \frac{1}{2}|K_N|$.

When $\mathbf{0}_p$ lies inside $\mathscr{E}_p$, there exists no solution of the above equation (4). Then, we have $|\Lambda| = |-h^3\varphi^4 + h^4\varphi^3| < \frac{1}{2}|K_N|$. □

LEMMA 5. *At a point p of S, assume that $K_N = 0$.*

(I) *The curvature ellipse $\mathscr{E}_p$ consists of only one point if and only if $\varphi^3 = \varphi^4 = 0$ at p. In this case, the origin $\mathbf{0}_p$ of T_pS lies on $\mathscr{E}_p$ if and only if $\boldsymbol{H} = \mathbf{0}$ at p.*

(II) *The curvature ellipse $\mathscr{E}_p$ is a segment* (*which is not only one point*) *if and only if $\varphi^3 \neq 0$ or $\varphi^4 \neq 0$ at p.*

 (i) *The origin $\mathbf{0}_p$ of T_pS lies on the segment as the curvature ellipse $\mathscr{E}_p$ if and only if $\Lambda = 0$, $|h^3| \leqq |\varphi^3|$ and $|h^4| \leqq |\varphi^4|$ at p.*

 (ii) *The origin $\mathbf{0}_p$ of T_pS lies at the end points of the segment as the curvature ellipse $\mathscr{E}_p$ if and only if $\Lambda = 0$, $|h^3| = |\varphi^3|$ and $|h^4| = |\varphi^4|$ at p.*

Proof. (I) It follows from the equation (2).

(II) When $\mathbf{0}_p$ lies on the segment $\mathscr{E}_p$, there exists $\theta \in \mathbf{R}/\pi\mathbf{Z}$ such that $\boldsymbol{\eta}(\theta) = \mathbf{0}$, and hence $\mathrm{Re}(\varphi^\alpha e^{i2\theta}) = -h^\alpha$ $(\alpha = 3, 4)$. This implies that $|\varphi^\alpha| \geqq |h^\alpha|$, and hence

$$
\begin{aligned}
0 = \frac{1}{2}K_N &= \operatorname{Im}(\varphi^3 e^{i2\theta}\overline{\varphi^4 e^{i2\theta}}) \\
&= -\operatorname{Re}(\varphi^3 e^{i2\theta}) \operatorname{Im}(\varphi^4 e^{i2\theta}) + \operatorname{Im}(\varphi^3 e^{i2\theta}) \operatorname{Re}(\varphi^4 e^{i2\theta}) \\
&= h^3 \operatorname{Im}(\varphi^4 e^{i2\theta}) - h^4 \operatorname{Im}(\varphi^3 e^{i2\theta}) \\
&= -\operatorname{Im}\{(-h^3\varphi^4 + h^4\varphi^3)e^{i2\theta}\}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
(-h^3\varphi^4 + h^4\varphi^3)e^{i2\theta} &= \operatorname{Re}\{(-h^3\varphi^4 + h^4\varphi^3)e^{i2\theta}\} \\
&= -h^3 \operatorname{Re}(\varphi^4 e^{i2\theta}) + h^4 \operatorname{Re}(\varphi^3 e^{i2\theta}) = 0.
\end{aligned}
$$

Then, we obtain that $\Lambda = -h^3\varphi^4 + h^4\varphi^3 = 0$.

Conversely, assume that $\Lambda = 0$, $|h^3| \leqq |\varphi^3|$ and $|h^4| \leqq |\varphi^4|$ at p. There exists $\theta_\alpha \in \mathbf{R}/\pi\mathbf{Z}$ satisfying $\operatorname{Re}(\varphi^\alpha e^{i2\theta_\alpha}) = -h^\alpha$. The equation $-h^3\varphi^4 + h^4\varphi^3 = 0$ implies the existence of $w \in \mathbf{C}$ satisfying $\begin{pmatrix} h^3 \\ h^4 \end{pmatrix} = w \begin{pmatrix} \varphi^3 \\ \varphi^4 \end{pmatrix}$, and hence $\arg \varphi^3 = \arg \varphi^4 (=: \theta_0)$. Hence, we have $|\varphi^\alpha| \cos(\theta_0 + 2\theta_\alpha) = |w|\,|\varphi^\alpha|$, and then $\theta_3 = \theta_4 (=: \theta)$. This gives that $\boldsymbol{\eta}(\theta) = \mathbf{0}$. Now we can conclude the assersion (i).

When $\mathbf{0}_p$ lies at the end points of the segment $\mathscr{E}_p$, there exists $\theta \in \mathbf{R}/\pi\mathbf{Z}$ such that $\boldsymbol{\eta}(\theta) = \mathbf{0}$ and $\dfrac{d\boldsymbol{\eta}}{d\theta} = \mathbf{0}$. Hence, we have $\varphi^\alpha e^{i2\theta} = -h^\alpha$, and then $\Lambda = -h^3\varphi^4 + h^4\varphi^3 = 0$ and $|\varphi^\alpha| = |h^\alpha|$.

Conversely, assume that $\Lambda = 0$ and $|\varphi^\alpha| = |h^\alpha|$ at p. Then we can get $\theta \in \mathbf{R}/\pi\mathbf{Z}$ satisfying $\boldsymbol{\eta}(\theta) = \dfrac{d\boldsymbol{\eta}}{d\theta} = \mathbf{0}$ similarly to the above. □

Little [8] has also proved the following equivalent condition on inflection points. In the following theorem,

$$
\mathscr{S} = \begin{pmatrix} \begin{vmatrix} h^3_{11} & h^3_{12} \\ h^4_{11} & h^4_{12} \end{vmatrix} & \dfrac{1}{2}\begin{vmatrix} h^3_{11} & h^3_{22} \\ h^4_{11} & h^4_{22} \end{vmatrix} \\ \dfrac{1}{2}\begin{vmatrix} h^3_{11} & h^3_{22} \\ h^4_{11} & h^4_{22} \end{vmatrix} & \begin{vmatrix} h^3_{12} & h^3_{22} \\ h^4_{12} & h^4_{22} \end{vmatrix} \end{pmatrix}.
$$

We remark that $\Delta = \det \mathscr{S}$ and $K_N = \operatorname{trace} \mathscr{S}$.

Theorem ([8, Theorem 1.2]). *Let $p \in S$. The following three conditions are equivalent.*

(a) *p is an inflection point,*
(b) *$\mathscr{S} = 0$ at p,*
(c) *$\Delta_p = 0$ and $K_N(p) = 0$.*

Here, we give briefly another proof of the equivalence of (a) and (c). First, note that p is an inflection point if and only if the equation (3) (and hence (4)) holds at p for any θ. Then, the equation (4) with $K_N = 0$ implies that $\Lambda = 0$, and hence that $\Delta = 0$.

Moreover, we can get the following characterization in terms of Λ.

LEMMA 6. *A point p in S is an inflection point if and only if $\Lambda = 0$ and $K_N = 0$ at p. When $\boldsymbol{H}_p \neq \boldsymbol{0}$ especially, p is an inflection point if and only if $\Lambda = 0$ at p.*

Proof. Set $\varphi = \varphi^3 e_3 + \varphi^4 e_4$ and $h_{ij} = h_{ij}^3 e_3 + h_{ij}^4 e_4$. Then, we obtain that $\varphi \wedge \boldsymbol{H} = \Lambda e_3 \wedge e_4 = \dfrac{1}{2} h_{11} \wedge h_{22} - \dfrac{\boldsymbol{i}}{2} h_{12} \wedge (h_{11} + h_{22})$. Accordingly, the condition that $\Lambda = 0$ is equivalent to that $h_{11} \wedge h_{22} = h_{12} \wedge (h_{11} + h_{22}) = 0$. On the other hand, the condition that $K_N = 0$ is equivalent to $h_{12} \wedge (h_{11} - h_{22}) = 0$. Since the condition that $\mathscr{S} = 0$ is equivalent to $h_{11} \wedge h_{22} = h_{12} \wedge h_{11} = h_{12} \wedge h_{22} = 0$, we obtain the first assertion.

When $\boldsymbol{H} \neq \boldsymbol{0}$, $h_{12} \wedge (h_{11} + h_{22}) = 0$ implies that there exists a real number a satisfying $h_{12} = a(h_{11} + h_{22})$. Then $h_{12} \wedge h_{11} = a h_{22} \wedge h_{11} = h_{22} \wedge h_{12}$. Hence the conditions $\boldsymbol{H} \neq \boldsymbol{0}$ and $\Lambda = 0$ imply $h_{12} \wedge h_{11} = h_{12} \wedge h_{22} = 0$. Therefore, we obtain the second assertion. □

Remark 7. When $\boldsymbol{H}_p = \boldsymbol{0}$, it is clear that p is an inflection point if and only if $K_N = 0$ at p.

3. Gauss maps

Following Hoffman-Osserman [6], we will recall some terminologies.

Let S be a connected Riemann surface and $X : S \to \mathbf{R}^4$ a conformal immersion. If $z = \xi + \boldsymbol{i}\eta$ is a local conformal parameter on S, the (conjugate) Gauss map $\overline{G}$ of X is the map from S into the complex quadric Q_2 in the complex projective 3-space $\mathbf{C}P^3$ defined by

$$\overline{G}(z) = \left[\frac{\partial X}{\partial z}\right]. \tag{5}$$

Q_2 is biholomorphic to the product $S^2 \times S^2$ of the Riemann sphere $S^2 = \hat{\mathbf{C}}$. The identification $\hat{\mathbf{C}} \times \hat{\mathbf{C}} \cong Q_2$ is given by the map

$$\begin{aligned} &\varphi : \hat{\mathbf{C}} \times \hat{\mathbf{C}} \to Q_2 \subset \mathbf{C}P^3, \\ &(w_1, w_2) \mapsto (1 + w_1 w_2, \boldsymbol{i}(1 - w_1 w_2), w_1 - w_2, -\boldsymbol{i}(w_1 + w_2)). \end{aligned}$$

Set $f_k = \pi_k \circ \overline{G}$ $(k = 1, 2)$, where π_1 and π_2 are the projections from Q_2 on $S^2 = \hat{\mathbf{C}}$. Then, the Gauss map $\overline{G}(z)$ is expressed by the pair $(f_1(z), f_2(z))$ of the functions.

Set $\Phi = \varphi(f_1, f_2)$ and

$$A = (f_2 - \overline{f_1}, -\boldsymbol{i}(f_2 + \overline{f_1}), 1 + \overline{f_1} f_2, -\boldsymbol{i}(1 - \overline{f_1} f_2)).$$

We conclude that

$$e_1 = \sqrt{2}\frac{\operatorname{Re}\Phi}{\|\Phi\|}, \quad e_2 = \sqrt{2}\frac{\operatorname{Im}\Phi}{\|\Phi\|}, \quad e_3 = \sqrt{2}\frac{\operatorname{Re}\bar{A}}{\|A\|}, \quad e_4 = \sqrt{2}\frac{\operatorname{Im}\bar{A}}{\|A\|}$$

give an adapted local frame field on S [6, Proposition 4.4]. It follows from $\bar{\Phi} \cdot \bar{A} = \bar{\Phi} \cdot A = 0$ that

$$\begin{aligned}
&d(e_1 - \boldsymbol{i}e_2) \cdot (e_3 + \boldsymbol{i}e_4) \wedge d(e_1 - \boldsymbol{i}e_2) \cdot (e_3 - \boldsymbol{i}e_4) \\
&\quad = \frac{4}{\|\Phi\|^4}(d\bar{\Phi} \cdot \bar{A}) \wedge (d\bar{\Phi} \cdot A) \\
&\quad = \frac{4}{\|\Phi\|^4}\{(\bar{\Phi}_z \cdot \bar{A})(\bar{\Phi}_{\bar{z}} \cdot A) - (\bar{\Phi}_{\bar{z}} \cdot \bar{A})(\bar{\Phi}_z \cdot A)\}\, dz \wedge d\bar{z} \\
&\quad = \frac{1}{(1 + |f_1|^2)(1 + |f_2|^2)} \overline{(f_{1\bar{z}} f_{2z} - f_{1z} f_{2\bar{z}})}\, dz \wedge d\bar{z} \\
&\quad = \overline{(F_1 \hat{F}_2 - \hat{F}_1 F_2)}\, dz \wedge d\bar{z},
\end{aligned}$$

where

$$F_k = F(f_k) = \frac{(f_k)_{\bar{z}}}{1 + |f_k|^2}, \quad \text{and} \quad \hat{F}_k = \hat{F}(f_k) = \frac{(f_k)_z}{1 + |f_k|^2}.$$

Denote the induced metric on S by the form $ds^2 = \lambda^2 |dz|^2$. Then we obtain the following

LEMMA 8.

$$\Lambda = \frac{\boldsymbol{i}}{2\lambda^2} \overline{(F_1 \hat{F}_2 - \hat{F}_1 F_2)}.$$

Remark 9. The equation (1) combined with this lemma implies that

$$4\Delta = (K_N)^2 - \frac{1}{\lambda^4} |F_1 \hat{F}_2 - \hat{F}_1 F_2|^2.$$

J. Monterde has also proved this equation in [10].

On the other hand, in [6, Proposition 4.5], it is also proved that the square norm of the mean curvature vector $\boldsymbol{H}$, the Gauss curvature K and normal curvature K_N of X are given by

$$|\boldsymbol{H}|^2 = \frac{2}{\lambda^2}(|F_1|^2 + |F_2|^2), \tag{6}$$

$$K = J_1 + J_2, \tag{7}$$

$$K_N = J_1 - J_2. \tag{8}$$

Here, J_k $(k = 1, 2)$ is the Jacobian of the map f_k from $(S, \lambda^2|dz|^2)$ to the sphere (S^2, g_0) of radius $1/\sqrt{2}$:

$$J_k = \frac{2}{\lambda^2}(|F_k|^2 - |\hat{F}_k|^2).$$

4. Inflection points

In this section, we prove Theorem 1.

In order to prove that $X(S)$ in $\mathbf{R}^4$ lies in an affine 3-space, we recall the following theorem for degenerate Gauss maps by Hoffman and Osserman [6]. A surface M in $\mathbf{R}^4$ is said to have *degenerate Gauss map* if the image of M under the Gauss map (5) lies in a hyperplane of $\mathbf{C}P^3$, that is, there exists a non-zero complex vector $B = (b_1, b_2, b_3, b_4)$ such that

$$b_1\varphi_1(z) + b_2\varphi_2(z) + b_3\varphi_3(z) + b_4\varphi_4(z) \equiv 0, \tag{9}$$

where $(\varphi_1(z), \varphi_2(z), \varphi_3(z), \varphi_4(z)) = \Phi(z) = \varphi(f_1(z), f_2(z))$.

Theorem 10 ([6, Theorem 5.3]). *Let M be a surface in $\mathbf{R}^4$ with degenerate Gauss map, so that* (9) *holds for some vector B. M lies in some affine* 3*-space in $\mathbf{R}^4$ if and only if B can be chosen to be a real vector.*

Proof of Theorem 1. First, if $K(p) \neq 0$ $(p \in S)$, we show that the pullbacks of the metric g_0 on S^2 by f_1, f_2 induce a same metric g on an open neighborhood around p in S.

When $\boldsymbol{H} = \mathbf{0}$ at $p \in S$, from (6), we have $|F_1| = |F_2| = 0$, and hence $(f_1)_{\bar{z}} = (f_2)_{\bar{z}} = 0$ at p. It follows from $K \neq 0$ and (7) that we have either $|\hat{F}_1| \neq 0$ or $|\hat{F}_2| \neq 0$ at p. Since p is an inflection point, we have $K_N = 0$ and, from (8), $|\hat{F}_1| = |\hat{F}_2|$, that is,

$$\frac{|df_1|^2}{(1+|f_1|^2)^2} = \frac{|df_2|^2}{(1+|f_2|^2)^2} \neq 0 \quad \text{at } p.$$

Now we consider the point p at which $\boldsymbol{H} \neq \mathbf{0}$. Since p is an inflection point, it follows from Lemmas 6 and 8 that

$$F_1\hat{F}_2 - \hat{F}_1F_2 = 0. \tag{10}$$

Since $\boldsymbol{H} \neq \mathbf{0}$, the equation (6) implies that $(F_1, F_2) \neq \mathbf{0}$. Hence, the equation (10) implies that there exists a complex number α such that at p

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = \alpha \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{11}$$

Since $K \neq 0$, the equation (7) combined with (11) implies that $|\alpha| \neq 1$. Since $K_N = 0$, it follows from (8) and $|\alpha| \neq 1$ that

$$|F_1| = |F_2| \neq 0.$$

This implies that f_1, f_2 are local diffeomorphisms. Moreover, we obtain that, for $k = 1, 2$,

$$\frac{|df_k|^2}{(1+|f_k|^2)^2}$$
$$= |F_k|^2 (d\xi \quad d\eta) \begin{pmatrix} (1+\mathrm{Re}(\alpha))^2 + \mathrm{Im}(\alpha)^2 & -2\,\mathrm{Im}(\alpha) \\ -2\,\mathrm{Im}(\alpha) & (1-\mathrm{Re}(\alpha))^2 + \mathrm{Im}(\alpha)^2 \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix},$$

where $z = \xi + \boldsymbol{i}\eta$. Since $|\alpha| \neq 1$, these are nondegenerate.

Consequently, the pullbacks of the metric g_0 by f_1, f_2 induce a same metric g on an open neighborhood around p in S.

Second, we show that $\phi \circ f_1 = f_2$ on S for an orientation-preserving isometry $\phi \in \mathrm{Isom}_+(S^2, g_0)$. For any point $p \in S$, we can take $\phi_p \in \mathrm{Isom}_+(S^2, g_0)$ such that $(\phi_p \circ f_1)(p) = f_2(p)$ and $d(\phi_p \circ f_1)_p = (df_2)_p$. From the above argument, there exist open neighborhoods U_p in S and V_p in S^2 such that $\phi_p \circ f_1, f_2$ are isometric diffeomorphism from (U_p, g) onto (V_p, g_0). Hence $\phi_p \circ f_1 = f_2$ on U_p (e.g. [5, Lemma 11.2]).

For a fixed point $p_0 \in S$, set $W = \{p \in S \,|\, (\phi_{p_0} \circ f_1)(p) = f_2(p)\}$. Then, W is nonempty and obviously closed. Moreover, for any point $p \in W$, there exists a finite sequence of points $\{p_k \,|\, k = 0, \dots, n\}$ in S such that $p_n = p$ and $U_{p_{k-1}} \cap U_{p_k} \neq \emptyset$ $(k = 1, \dots, n)$. On $U_{p_{k-1}} \cap U_{p_k}$, we have $\phi_{p_{k-1}} \circ f_1 = f_2 = \phi_{p_k} \circ f_1$. Since $f_1(U_{p_{k-1}} \cap U_{p_k}) \subset (S^2, g_0)$ contains obviously at least three distinct points, then $\phi_{p_{k-1}} = \phi_{p_k}$. Then $\phi_p = \phi_{p_0}$, and hence $U_p \subset W$. This implies that W is open. Since S is connected, $W = S$, that is, $\phi_{p_0} \circ f_1 = f_2$ on S.

The isometry ϕ_{p_0} can be expressed by

$$\phi_{p_0}(w) = \frac{Qw - \bar{P}}{Pw + \bar{Q}} \quad \text{for } w \in \hat{\mathbf{C}} = S^2 \quad (P, Q \in \mathbf{C},\ |P|^2 + |Q|^2 = 1).$$

Set $B = (b_1, b_2, b_3, b_4) = (-\mathrm{Re}(P), \mathrm{Im}(P), \mathrm{Re}(Q), -\mathrm{Im}(Q))$. Since $f_2 = \phi_{p_0} \circ f_1 = (Qf_1 - \bar{P})/(Pf_1 + \bar{Q})$, the Gauss map $\bar{G} = \varphi(f_1, f_2)$ of $X(S)$ satisfies the linear equation

$$b_1(1 + f_1 f_2) + b_2 \boldsymbol{i}(1 - f_1 f_2) + b_3(f_1 - f_2) - b_4 \boldsymbol{i}(f_1 + f_2) = 0.$$

Hence, the image of $\bar{G}$ is contained in the hyperplane in $\mathbf{C}P^3$, which is defined by the real vector B. Theorem 1 can now follow from Theorem 10 by Hoffman and Osserman. We then conclude that $X(S)$ lies in an affine 3-space in $\mathbf{R}^4$. $\square$

Remark 11. If all points of S are inflection points, the dimension of the first normal space

$$N_1^X(p) = \mathrm{span}\{\Pi(v, w) \,|\, v, w \in T_p S\}$$

at any point $p \in S$ is less than 2. Assume that N_1^X forms a rank-1 vector subbundle of the normal bundle $T^\perp S$. It is well known that $X(S)$ lies in an affine 3-space in $\mathbf{R}^4$ if and only if N_1^X is parallel in the normal connection of X (see [2]). Moreover, from Theorem 1 in [3], if N_1^X is nonparallel, we have that $K \equiv 0$.

5. Examples

Example 5.1 (Whitney sphere). Let X be a conformal immersion from a Riemann sphere $\{(\cos u \cos v, \cos u \sin v, \sin u)\}$ into the complex 2-space $\mathbf{C}^2 \cong \mathbf{R}^4$ given by

$$\begin{aligned} X(u,v) &= (\alpha(u)e^{iv}, \beta(u)e^{iv}) \\ &= (\alpha(u)\cos v, \alpha(u)\sin v, \beta(u)\cos v, \beta(u)\sin v), \end{aligned}$$

where

$$\alpha(u) = \frac{\cos u}{1+\sin^2 u}, \quad \beta(u) = \frac{\cos u \sin u}{1+\sin^2 u}.$$

The plane curve $\gamma_a(u) = (\alpha(u), \beta(u))$ is the lemniscate of Bernoulli, and X gives the Whitney sphere. Following the computation in [1, Example 3.2], we have

$$\begin{aligned} & h^3 = 0, \quad h^4 = \frac{-2\sqrt{2}\cos u}{\sqrt{3-\cos 2u}}, \\ & \varphi^3 = -\boldsymbol{i}\frac{\sqrt{2}\cos u}{\sqrt{3-\cos 2u}}, \quad \varphi^4 = \frac{-\sqrt{2}\cos u}{\sqrt{3-\cos 2u}}, \\ & K_N = \frac{4\cos^2 u}{3-\cos 2u}, \quad \Lambda = \boldsymbol{i}\frac{4\cos^2 u}{3-\cos 2u}. \end{aligned}$$

Hence, X has only two infection points which are parabolic, and the other points are hyperbolic.

Example 5.2 (graphs in $\mathbf{R}^4$). For two functions $s(u,v)$ and $t(u,v)$, the graph surface $X(u,v)$ in $\mathbf{R}^4$ is given by

$$X(u,v) = (u, v, s(u,v), t(u,v)).$$

Set

$$\begin{aligned} & E = X_u \cdot X_u = 1 + (s_u)^2 + (t_u)^2, \quad F = X_u \cdot X_v = s_u s_v + t_u t_v, \\ & G = X_v \cdot X_v = 1 + (s_v)^2 + (t_v)^2, \quad g = EG - F^2, \\ & n_1 = (-s_u, -s_v, 1, 0), \quad n_2 = (-t_u, -t_v, 0, 1), \\ & E' = n_1 \cdot n_1 = 1 + (s_u)^2 + (s_v)^2, \quad F' = n_1 \cdot n_2 = s_u t_u + s_v t_v, \\ & G' = n_2 \cdot n_2 = 1 + (t_u)^2 + (t_v)^2, \quad g' = E'G' - (F')^2, \end{aligned}$$

$$e_1 = \frac{1}{\sqrt{E}} X_u, \quad e_2 = \sqrt{\frac{E}{g}} \left(X_v - \frac{F}{E} X_u \right),$$

$$e_3 = \frac{1}{\sqrt{E'}} n_1, \quad e_4 = \sqrt{\frac{E'}{g'}} \left(n_2 - \frac{F'}{E'} n_1 \right).$$

Using the orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we can compute the mean curvature vector $h^3 e_3 + h^4 e_4$ and K_N as follows:

$$h^\alpha = \frac{1}{E} X_{uu} \cdot e_\alpha + \frac{E}{g} \left(X_{vv} - 2\frac{F}{E} X_{uv} + \frac{F^2}{E^2} X_{uu} \right) \cdot e_\alpha \quad (\alpha = 3, 4)$$

$$K_N = \frac{1}{\sqrt{g'}(\sqrt{g})^3} \begin{vmatrix} E & F & G \\ s_{uu} & s_{uv} & s_{vv} \\ t_{uu} & t_{uv} & t_{vv} \end{vmatrix},$$

$$2\Lambda = \frac{\boldsymbol{i}}{\sqrt{g'}(\sqrt{g})^3} \begin{vmatrix} -E & \sqrt{g}\boldsymbol{i} - F & -\frac{1}{E}(\sqrt{g}\boldsymbol{i} - F)^2 \\ s_{uu} & s_{uv} & s_{vv} \\ t_{uu} & t_{uv} & t_{vv} \end{vmatrix}.$$

(i) For example, we set $s(u,v) = \dfrac{u^2}{2} + v$ and $t(u,v) = \dfrac{v^2}{2} + u$. Then, the graph surface has only hyperbolic points and no inflection point.

(ii) On the other hand, set $s = s(u)$ and $t = t(u)$. Then, we have that $K_N \equiv 0$ and $\Lambda \equiv 0$, and hence all points are inflection points. This graph is the product of a curve $(u, s(u), t(u))$ in $\mathbf{R}^3$ and a straight line in $\mathbf{R}^4$. Hence, the Gauss curvature K is obviously identically zero. Therefore, this implies that the assertion in Theorem 1 never hold without an assumption on the Gauss curvature K.

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