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BANACH SPACES OF BOUNDED DIRICHLET FINITE HARMONIC FUNCTIONS ON RIEMANN SURFACES

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Abstract

The Banach space of bounded Dirichlet finite harmonic functions on an open Riemann surface will be seen to be reflexive and also separable if and only if the underlying Riemann surface does not carry any unbounded Dirichlet finite harmonic function.

1. Introduction

There are many properties commonly considered for general Banach spaces such as separability, reflexivity, uniform convexity, and many others. It is not only interesting in its own right but also important and quite useful to know whether a given special space enjoys these properties or not. Having the intention to apply to the harmonic classification theory of Riemann surfaces we have considered the separability and the reflexivity for the spaces $HB(R)$ and $HD(R)$ (cf. [11], [12]) explained below.

We denote by $H(R)$ the linear space of real valued harmonic functions u on an open (i.e., noncompact) Riemann surface R ([1]). Two major important linear subspaces of $H(R)$ repeatedly considered thus far in the classification theory of Riemann surfaces are $HB(R)$ and $HD(R)$. The former space $HB(R)$ consists of *bounded* harmonic functions u on R and forms a Banach space under the supremum norm $\|u\|_{HB}$, i.e.,

$$\|u\|_{HB} := \sup_{z \in R} |u(z)|. \tag{1.1}$$

The letter B in $HB(R)$ is thus used to suggest the initial of boundedness. This space is important in view of the normal family argument for harmonic functions. The latter space $HD(R)$ is the family of *Dirichlet finite* harmonic functions u

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on R, where a $u \in H(R)$ is said to be Dirichlet finite if the Dirichlet integral $D(u;R)$ of u taken over R is finite, i.e.,

$$(1.2)\qquad D(u;R) := \int_R du \wedge *du = \int_R |\nabla u(z)|^2\, dxdy < +\infty \quad (z = x + iy).$$

Then $HD(R)$ forms a Banach space, and actually a Hilbert space equipped with the inner product $(u,v)_{HD}$ for u and v in $HD(R)$ given by

$$(1.3)\qquad (u,v)_{HD} := u(a)v(a) + D(u,v;R),$$

where $D(u,v;R)$ is the mutual Dirichlet integral of u and v in $HD(R)$ defined by

$$(1.4)\qquad D(u,v;R) := \int_R du \wedge *dv = \int_R \nabla u(z)\cdot\nabla v(z)\, dxdy \quad (z = x + iy)$$

so that $D(u;R) = D(u,u;R)$ and a is an arbitrarily chosen and then fixed reference point in R. The choice of a is not essential in the sense that the change of a only produces the homeomorphically linear isomorphic Hilbert space. Of course, the norm $\|u\|_{HD}$ of u is given by $\|u\|_{HD} = \sqrt{(u,u)_{HD}}$. The letter D in $HD(R)$ is used as in the case of $HB(R)$ to suggest the initial D of Dirichlet finiteness. This space is important in connection with the so called Dirichlet principle in the harmonic function thory.

For these two spaces $HB(R)$ and $HD(R)$ we considered the separability and the reflexivity as Banach spaces and we obtained the result as indicated in the following table ([11], [12]).

Table 1.5

space	$HB(R)$ (dim $< \infty$)	$HB(R)$ (dim $= \infty$)	$HD(R)$ (dim $\leqq \infty$)
reflexivity	yes	no	yes
separability	yes	no	yes

Here, for example, $HB(R)$ (dim $< \infty$) means that $\dim HB(R) < \infty$, where $\dim X$ for a linear space X is the linear dimension of X. Thus $HD(R)$ (dim $\leqq \infty$) means that $\dim HD(R) \leqq \infty$ or equivalently that unconditional for the dimension of $HD(R)$. Recall that any general finite n dimensional ($n \in \mathbf{N}$: the set of positive integers) Banach space is homeomorphically linear isomorphic to the n dimensional Euclidean space $\mathbf{R}^n$ so that it is always separable and also reflexive. One of our motivations of deriving the above table was to give a short, simple, and easy proof to the useful Masaoka theorem [6] that the identity $HB(R) = HD(R)$ as sets holds if and only if $\dim HB(R) = \dim HD(R) < \infty$. The essential part of the proof of this result is the implication of the latter assertion $\dim HB(R) = \dim HD(R) < \infty$ from the former condition of set identity $HB(R) = HD(R)$. By the open mapping principle, $HB(R) = HD(R)$ assures that $HB(R)$ and $HD(R)$ are homeomorphically linear isomorphic as Banach spaces and thus either parts of reflexivity or separability of the table 1.5 shows that $\dim HB(R) = \dim HD(R) < \infty$, as required. Our proof is thus ultrasimple. And the more, as an effect of our proof in [11] and [12], what we have really proven above is the following generlization of the Masaoka result.

The following 3 *conditions are equivalent by pairs*: (i) *the Banach spaces* $HB(R)$ *and* $HD(R)$ *are isomorphic as topological linear spaces*; (ii) $HB(R) = HD(R)$ *as sets*; (iii) $\dim HB(R) = \dim HD(R) < \infty$.

The significance of this generalization reveals itself in the fact that the method of the original proof in [6] or that of relatively simplified proof of it in [8] cannot at all take care of the above generalization.

In treating spaces $HB(R)$ and $HD(R)$ it is not only useful and convenient but also important to consider the third Banach space

$$HBD(R) := HB(R) \cap HD(R), \tag{1.6}$$

which forms a Banach space under the combined norm

$$\|u\|_{HBD} := \sup_{z \in R} |u(z)| + \sqrt{D(u; R)} \tag{1.7}$$

for $u \in HBD(R)$. Since, in general, we always have two inclusion relations $HBD(R) \subset HB(R)$ and $HBD(R) \subset HD(R)$, the condition $HB(R) = HD(R)$ is identical with two inverse inclusions $HBD(R) = HB(R)$ and $HBD(R) = HD(R)$. In this sense the above Masaoka theorem falls in the category of the inverse inclusion problem in the classification theory of Riemann surfaces.

The purpose of the present paper is to discuss the reflexivity and the separability of the Banach space $HBD(R)$. Since the parents $HB(R)$ and $HD(R)$ of their child $HBD(R)$ have entirely opposite characters with respect to both of reflexivity and separability, the question is which endowments of his (or her) parents the child $HBD(R)$ inherits more. Even for the simplest Riemann surface $\mathbf{D}$, the unit disc, it seems to be considerably hard to tell wether $HBD(\mathbf{D})$ is reflexive or not and also separable or not. The possibility of representing the Banach aspace $HBD(\mathbf{D})$ isometrically and linear isomorphically as the Banach space of Borel functions f on $\partial\mathbf{D}$ with finite norm $\|f\|$, the supremum and Douglas combined norm, given by

$$\|f\| := \operatorname{ess.sup}_{\partial\mathbf{D}} |f| + \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^2}\, dsdt} < \infty$$

does not seem to be too much helpful. We will establish a complete solution to the question mentioned above for general Riemann surfaces R (cf. Theorem 2.8 in the next section 2). Based upon this general solution it turns out that the following table 1.8 is obtained for the Riemann surface $\mathbf{D}$ (cf. Section 6).

Table 1.8

space	$HB(\mathbf{D})$	$HD(\mathbf{D})$	$HBD(\mathbf{D})$
reflexivity	no	yes	no
separability	no	yes	no

2. Prediction based upon examples

There is an open Riemann surface $\mathbf{W}_B$ with the following two properties ([9]):

(2.1) $$HBD(\mathbf{W}_B) = HB(\mathbf{W}_B);$$

(2.2) $$\dim HBD(\mathbf{W}_B) = \infty.$$

From (2.1), (2.2) and the table 1.5 it trivially follows the validity of the following table.

Table 2.3

space	$HB(\mathbf{W}_B)$	$HD(\mathbf{W}_B)$	$HBD(\mathbf{W}_B)$
reflexivity	no	yes	no
separability	no	yes	no

In contrast with the above $\mathbf{W}_B$ we have also constructed an open Riemann surface $\mathbf{W}_D$ with the following two properties ([13]):

(2.4) $$HBD(\mathbf{W}_D) = HD(\mathbf{W}_D);$$

(2.5) $$\dim HBD(\mathbf{W}_D) = \infty.$$

By the above conditions (2.4), (2.5) and Table 1.5 we can off hand give the following table.

Table 2.6

space	$HB(\mathbf{W}_D)$	$HD(\mathbf{W}_D)$	$HBD(\mathbf{W}_D)$
reflexivity	no	yes	yes
separability	no	yes	yes

Constructing $\mathbf{W}_D$ was really a painstaking task but that of $\mathbf{W}_B$ was relatively easy. Anyway, then, what is the essential distinction between $\mathbf{W}_B$ and $\mathbf{W}_D$? Clearly (2.4) is equivalent to saying that $\mathbf{W}_D$ does not admit any unbounded Dirichlet finite harmonic function on $\mathbf{W}_D$. On the other hand (2.1) means that

$HB(\mathbf{W}_B) \subset HD(\mathbf{W}_D)$. If the inclusion is not proper so that $HB(\mathbf{W}_B) = HD(\mathbf{W}_B)$, then by the Masaoka theorem stated and proved in Section 1 we must conclude

$$\dim HB(\mathbf{W}_B) = \dim HD(\mathbf{W}_B) < \infty,$$

which yields $\dim HBD(\mathbf{W}_B) < \infty$. This contradicts (2.2). Hence we can say that $\mathbf{W}_B$ admits an unbounded Dirichlet finite harmonic function on it. Let us consider the condition

$$HD(R) = HBD(R) \tag{2.7}$$

for general open Riemann surfaces R. Then $\mathbf{W}_B$ (resp. $\mathbf{W}_D$) falls in the class of open Riemann surfaces R for which (2.7) is invalid (resp. valid). Based upon this observation accompanied with Tables 2.3 and 2.6 it may not be too bold to conjecture that $HBD(R)$ is (resp. is not) reflexive and also separable if the condition (2.7) is (resp. is not) satisfied by R. Hereafter we proceed along this line until we fortunately come to the stage that we can say the above conjecture is certainly the case. For the sake of convenience for reference and also to make the relevant situation impressive, we wish to propose in this occasion to call Riemann surfaces R satisfying the condition (2.7) *HD-singular*, i.e., a Riemann surface R is HD-singular if there is no unbounded Dirichlet finite harmonic function on R. After all, we will prove the following result as the main assertion of this paper.

Theorem 2.8 (The main theorem). *The following three conditions are equivalent by pairs*:

(a) *the Banach space $HBD(R)$ is reflexive*;
(b) *the Banach space $HBD(R)$ is separable*;
(c) *the base Riemann surface R is HD-singular*: $HD(R) = HBD(R)$.

In the next section 3 we will consider capacities $\mathrm{cap}(K)$ for compact subsets K of the Royden harmonic boundary δ of the Riemann surface R. In terms of capacities we consider one more condition on R

(d) $\inf_{\zeta\in\delta} \mathrm{cap}(\{\zeta\}) > 0$

in addition to those (a), (b), and (c) above. In the later sections 4 and 5 we will prove Theorem 4.1 maintaining the equivalence of (a), (c), and (d) above by pairs and then Theorem 5.1 asserting the equivalence of (b), (c), and (d) above by pairs, from which we can derive four conditions (a), (b), and (c) in the above theorem and (d) just added above are altogether equivalent by pairs. The proof of the above main theorem of this paper will be complete in this fashion.

3. The capacity on the Royden harmonic boundary

First we recall the definition of the capacity $\mathrm{cap}(K)$ for compact subsets K of the Royden harmonic boundary δ of the Riemann surface R in question and certain related properties of them. We call a function f belonging to $L^{1,2}(R) \cap C(R)$ a *Royden function* on R, where $L^{1,2}(R)$ is the Dirichlet space (cf. [4]) on R which consists of local Sobolev functions $f \in W^{1,2}_{\mathrm{loc}}(R)$ on R with $D(f;R) < \infty$. The Royden compactification R^* of R is the smallest compactification of R such that every Royden function on R can be uniquely extended to R^* so as to be a $[-\infty, +\infty]$-valued continuous function on R^* which is unique up to homeomorphisms; we denote by

$$\gamma = \gamma R := R^* \backslash R$$

the Royden boundary of R and by

$$\delta = \delta R$$

the *Royden harmonic boundary* of R which is the totality of regular points in γ with respect to the harmonic Dirichlet problem in the sense of the Perron-Wiener-Brelot method; δ is a compact subset of R^* and $\delta = \emptyset$ if and only if R is parabolic, $R \in \mathcal{O}_G$ in notation, characterized by the nonexistence of Green functions on R (cf. e.g. [2], [14], [5], etc.). Hereafter, unless the contrary is explicitly stated, we assume that R is *hyperbolic*, i.e., $R \notin \mathcal{O}_G$.

An *end* W of R is a subregion of R such that $W = R \backslash \overline{R}_0$ with R_0 a regular subregion of R, i.e., R_0 is a relatively compact subregion of R whose relative boundary ∂R_0 consists of a finite number of mutually disjoint analytic Jordan curves so that $\partial W = \partial R_0$ as sets and thus W is surrounded by the relative boundary ∂W and the Royden boundary γ of R. The *relative class* $HD(W; \partial W)$ of the absolute class $HD(R)$ is given by

$$HD(W; \partial W) := \{u \in HD(W) \cap C(R) : u | R \backslash W = 0\}, \tag{3.1}$$

which forms a Banach space and actually a Hilbert space equipped with the norm $\sqrt{D(u;R)}$. The canonical isomorphism τ of the relative class $HD(W; \partial W)$ to the absolute class $HD(R)$ is given by

$$(\tau u)|\delta = u|\delta \tag{3.2}$$

for every $u \in HD(W; \partial W)$. It is easily seen (cf. e.g. [1]) that τ is bijective from $HD(W; \partial W)$ to $HD(R)$; τ and τ^{-1} are linear isomorphisms between $HD(W; \partial W)$ and $HD(R)$; τ and τ^{-1} are order preserving, i.e., $\tau u \geqq 0$ on R if and only if $u \geqq 0$ on R; τ and τ^{-1} are homeomorphisms between $HD(W; \partial W)$ and $HD(R)$, i.e., there exists a constant $C \in [1, \infty)$ such that

$$C^{-2} D(u;R) \leqq (\tau u)(a)^2 + D(\tau u; R) \leqq C^2 D(u;R) \tag{3.3}$$

for every $u \in HD(W; \partial W)$. In short, Banach spaces $HD(W; \partial W)$ and $HD(R)$ are homeomorphically and linearly isomorphic by the canonical isomorphism τ. As the relative class corresponding to the absolute class $HBD(R)$ we also consider the class

$$HBD(W; \partial W) := \{u \in HBD(W) \cap C(R) : u | R \backslash W = 0\},$$

which also forms a Banach space equipped with the norm (1.7). As above we also conclude that Banach spaces $HBD(W; \partial W)$ and $HBD(R)$ are homeomorphically linear isomorphic by the canonical isomorphism τ in (3.2).

For a compact subset $K \subset \delta$ the *capacity* cap(K) of K relative to an end W of R is given by

$$\text{cap}(K) := \inf_f D(f : R), \tag{3.4}$$

where f runs over every Royden function f on R such that $f|K \geqq 1$ and $f|R\backslash W \leqq 0$. Properties cap$(K) = 0$ or cap$(K) > 0$ for any compact set $K \subset \delta$ do not depend upon the choice of W (cf. e.g. [10], [5]). It is easily seen (cf. e.g. [10]) that there is a Cauchy sequence $(v_n)_{n\in\mathbf{N}}$ in $HD(W; \partial W)^+$ such that $0 \leqq v_n|R \leqq 1$, $v_n|K = 1$, and

$$\lim_{n\to\infty} D(v_n; R) = \text{cap}(K). \tag{3.5}$$

Using this simple observation we can show the following fact.

Proposition 3.6. *A compact subset $K \subset \delta$ is of vanishing capacity, i.e., cap$(K) = 0$, if and only if there exists an $h \in HD(R)$ such that $h|K = +\infty$.*

Before giving a proof to the above assertion we add the following remark. The spaces $L^{1,2}(R) \cap C(R)$ and $HD(R)$ (and also $HBD(R)$) form vector lattices (i.e., Riesz spaces) with respect to the usual function ordering. We denote by

$$f \cup g = \sup(f, g) \quad \text{and} \quad f \cap g = \inf(f, g)$$

the lattice operations in $L^{1,2} \cap C(R)$ and by

$$u \vee v = \sup(u, v) \quad \text{and} \quad u \wedge v = \inf(u, v)$$

in $HD(R)$ (and also in $HBD(R)$) so that e.g. $(f \cup g)(z) = \max(f(z), g(z))$ for every $z \in R$ but $u \vee v$ is the least harmonic majorant of u and v and thus $(u \cup v)(z) \leqq (u \vee v)(z)$ for every u and v in $HD(R)$ (and also in $HBD(R)$) and for every $z \in R$ but $(u \vee v)(\zeta) = (u \cup v)(\zeta)$ for every $\zeta \in \delta$. We know (cf. e.g. [2], [4]) that

$$D(f \cup g; R) + D(f \cap g; R) = D(f; R) + D(g; R) \tag{3.7}$$

for every f and g in $L^{1,2} \cap C(R)$. From the above relation (3.7), by using the Dirichlet principle, it follows that

$$D(u \vee v; R) + D(u \wedge v; R) \leqq D(u; R) + D(v; R) \tag{3.8}$$

for every u and v in $HD(R)$ (and also in $HBD(R)$). Based upon (3.8) we can replace $h \in HD(R)$ in the above proposition by $h \in HD(R)^+$ since we only have to take $h \vee 0$ because $h \vee 0|\delta = \max(h, 0)$ on δ. We can also replace $h \in HD(R)$ in the above proposition by $h \in HD(W : \partial W)^+$ in view of (3.2) and the fact $D(\tau u; R) \leqq D(u; R)$ for every u in $HD(W; \partial W)$.

Proof of Proposition 3.6. Suppose first that cap$(K) = 0$ for a given compact subset $K \subset \delta$. Then we can find a Cauchy sequence $(u_n)_{n\in\mathbf{N}}$ in $HD(W; \partial W)^+$

such that $u_n|K = 1$ $(n \in \mathbf{N})$ and $\lim_{n\to\infty} D(u_n; R) = \mathrm{cap}(K) = 0$. By choosing a subsequence, if necessary, we can assume that $D(u_n; R) \leqq 1/4^n$ $(n \in \mathbf{N})$ so that $\sum_{n\in\mathbf{N}} u_n$ is convergent in $HD(W; \partial W)$. Then $u := \sum_{n\in\mathbf{N}} u_n$ is locally uniformly convergent on R and $u \in HD(W; \partial W)^+$. For any $\zeta \in K$ we have

$$u(\zeta) \geqq \sum_{n=1}^{m} u_n(\zeta) = m$$

for every $m \in \mathbf{N}$ and therefore $u(\zeta) = +\infty$, i.e., $u|K = +\infty$. Thus $h := \tau u \in HD(R)^+$ is the required function.

Conversely suppose the existence of an $h \in HD(R)$ such that $h|K = +\infty$. Let $u := \tau^{-1}h$, which belongs to $HD(W; \partial W)$ and $u|K = h|K = +\infty$. Then $u \cap n$ is a Royden function on R with $u \cap n = 0$ on $R \backslash W$ and $D(u \cap n; R) \leqq D(u : R)$. Let v_n be the harmonic part of the (relative) Royden decomposition of $u \cap n$ with respect to W (cf. [14]) so that $v_n \in HD(W; \partial W)$, $v_n|\delta = (u \cap n)|\delta = \min(u|\delta, n)$ and in particular $v_n|K = n$, and $D(v_n; R) \leqq D(u \cap n; R) \leqq D(u; R)$. Finally let $u_n := v_n/n$. Then $u_n \in HD(W; \partial W)$, $u_n|K = 1$, and $D(u_n; R) \leqq n^{-2}D(u; R)$. Thus u_n is one of the competing functions determining $\mathrm{cap}(K)$ so that we can conclude that

$$\mathrm{cap}(K) \leqq D(u_n; R) \leqq D(u; R)/n^2$$

for every $n \in \mathbf{N}$ and a fortiori $\mathrm{cap}(K) = 0$ as required. □

As a consequence of Proposition 3.6 we state the following characterization of (2.7), i.e. the HD-singularity of R, in terms of the capacity on the harmonic boundary $\delta = \delta R$ of R.

Proposition 3.9. *The following four conditions are equivalent by pairs*:
(a) $HD(R) = HBD(R)$, *i.e.*, R *is* HD*-singular*;
(b) $HD(W; \partial W) = HBD(W; \partial W)$;
(c) $\mathrm{cap}(\{\zeta\}) > 0$ *for every* ζ *in* δ;
(d) $\inf_{\zeta\in\delta} \mathrm{cap}(\{\zeta\}) > 0$.

Proof. The canonical isomorphism τ of $HD(W; \partial W)$ onto $HD(R)$ sends its subspace $HBD(W; \partial W)$ onto $HBD(R)$ and therefore the conditions (a) and (b) are seen to be equivalent to each other by observing one more fact, the Royden-Virtanen theorem that $HBD(R)$ (resp. $HBD(W; \partial W)$) is dense in the Hilbert space $HD(R)$ (resp. $HD(W; \partial W)$). Proposition 3.6 assures the equivalence of the conditions (a) and (c). It is trivial that (d) implies (c) and hence the proof of the whole theorem will be over if we show that (c) implies (d). Contrariwise we assume that $\inf_{\zeta\in\delta} \mathrm{cap}(\{\zeta\}) = 0$ although the condition (c) is supposed to be valid. Then we can find $\zeta_n \in \delta$ for each $n \in \mathbf{N}$ such that

$$0 < \mathrm{cap}(\{\zeta_n\}) < (1/2) \cdot (256)^{-n}.$$

As a competing function to determine $\mathrm{cap}(\{\zeta_n\})$ we can find (cf. [10]) a $u_n \in HD(W;\partial W)^+$ such that $0 \leqq u_n \leqq 1$ on R, $u_n(\zeta_n) = 1$, and $D(u_n;R) < (256)^{-n}$. Observe that

$$D(4^n u_n : R) = (16)^n D(u_n;R) \leqq (16)^n \cdot (256)^{-n} = (16)^{-n}$$

and clearly $4^n u_n(\zeta_n) = 4^n$. We can define

$$u := \sum_{n \in \mathbf{N}} 4^n u_n$$

on R because the estimate

$$\sqrt{D(u;R)} \leqq \sum_{n \in \mathbf{N}} \sqrt{D(4^n u_n;R)} \leqq \sum_{n \in \mathbf{N}} 4^{-n} = 1/3 < \infty$$

assures that the above series defining u converges locally uniformly on R so that $u \in HD(W;\partial W)^+$. Observe that

$$u(\zeta_n) \geqq 4^n u_n(\zeta_n) = 4^n$$

for every $n \in \mathbf{N}$ and thus $\sup_R u = \max_\delta u = +\infty$. Then there exists a $\zeta \in \delta$ such that $u(\zeta) = +\infty$ so that, again by Proposition 3.6, we must conclude that $\mathrm{cap}(\{\zeta\}) = 0$, contradicting (c). □

4. Reflexivity

In this section we will prove that the Banach space $HBD(R)$ is reflexive if and only if the base Riemann surface R is HD-singular, i.e., there is no unbounded Dirichlet finite harmonic function on R. We start by recalling the reflexivity of Banach spaces. Let X be a Banach space and we denote by X^* the dual space of X, i.e., the space of bounded linear functional x^* on X and the value of x^* at $x \in X$ is denoted by $\langle x, x^* \rangle$. An $x \in X$ can be viewed as a functional $\hat{x}$ on X^* given by $\langle x^*, \hat{x} \rangle := \langle x, x^* \rangle$ for every $x^* \in X^*$. We denote by $\hat{X} := \{\hat{x} : x \in X\} \subset X^{**}$ and the isometric linear mapping $x \mapsto \hat{x}$ of X into X^{**} is referred to as the natural embedding of X into X^{**}, i.e., by identifying $\hat{X}$ with X we consider $X \subset X^{**}$. When $X = X^{**}$, we say that the Banach space X is *reflexive*. A typical situation occurs when X is a Hilbert space: a Hilbert space X is always reflexive in view of its Riesz self duality: $X = X^*$. We will use the following characterization of the reflexivity of X (see [3, p. 425]): X is reflexive if and only if the closed unit ball in X is weakly compact. The main assertion of this section is the following result.

THEOREM 4.1. *The following three conditions are equivalent by pairs*:
(a) *the Banach space* $HBD(R)$ *is reflexive*;
(b) *the base Riemann surface* R *is* HD*-singular, i.e.*, $HD(R) = HBD(R)$;
(c) *the Royden harmonic boundary* δ *of* R *satisfies* $\inf_{\zeta \in \delta} \mathrm{cap}(\{\zeta\}) > 0$.

Proof. The equivalence of (b) and (c) is established in Proposition 3.9. Suppose the validity of (b): $HD(R) = HBD(R)$. Since the identity mapping $\imath : HBD(R) \to HD(R)$ given by $\imath(u) = u$ for every $u \in HBD(R)$ is clealy a linear isomorphism of $HBD(R)$ onto $HD(R)$. Since

$$\|\imath(u)\|_{HD} := \sqrt{\imath(u)(a)^2 + D(\imath(u); R)} = \sqrt{u(a)^2 + D(u; R)}$$
$$\leqq \sup_R |u| + \sqrt{D(u; R)} =: \|u\|_{HBD}$$

for every $u \in HBD(R)$, $\imath$ is a continuous bijective mapping of $HBD(R)$ to $HD(R)$ so that the Banach open mapping principle assures that $\imath^{-1}$ is continuous. Hence the Banach space $HBD(R)$ is homeomorphically linear isomorphic with the Hilbert space $HD(R)$, which is reflexive and a fortiori $HBD(R)$ is reflexive: (b) implies (a). Conversely, supposing (a) is valid the proof will be complete if we show the validity of (b), or equivalently, that of (c). We will show this by contradiction so that we assume the existence of a point $\xi \in \delta$ with $\mathrm{cap}(\{\xi\}) = 0$ in spite of that $HBD(R)$ is reflexive, i.e., the closed unit ball $HBD(R)_1$ of $HBD(R)$ is weakly compact in the Banach space $HBD(R)$.

For any open neighborhood V of ξ and for any positive number ε, we maintain the existence of $u \in HBD(R)$ with the following 4 properties: $0 \leqq u \leqq 1$ on R; $u(\xi) = 1$; $u|\delta\setminus V = 0$; $D(u; R) < \varepsilon$. In fact, we first take a $v \in HBD(R)$ such that $0 \leqq v \leqq 1$ on R, $v(\xi) = 1$, and $v|\delta\setminus V = 0$. Since $\mathrm{cap}(\{\xi\}) = 0$, there exists a competing Cauchy sequence $(w_n)_{n\in\mathbf{N}} \subset HD(W; \partial W)$ for $\mathrm{cap}(\{\xi\})$ converging to zero, i.e., $\lim_{n\to\infty} D(w_n; R) = 0$. We can assume that $0 \leqq w_n \leqq 1$ on R and $w_n(\xi) = 1$. Then the sequence $(w_n)_{n\in\mathbf{N}}$ also converges to zero locally uniformly on R. Put $f_n := w_n v$ for each $n \in \mathbf{N}$. We see that $0 \leqq f_n \leqq 1$ on R, $f_n(\xi) = w_n(\xi)v(\xi) = 1$, $f_n|\delta\setminus V = w_n \cdot (v|\delta\setminus V) = 0$, and

$$|\nabla f_n| \leqq w_n|\nabla v| + v|\nabla w_n| \leqq w_n|\nabla v| + |\nabla w_n|$$

on R. Since $w_n^2 \leqq w_n$ on R, we see that

$$D(f_n; R) \leqq 2\int_R w_n|\nabla v(z)|^2\, dxdy + 2D(w_n; R) \quad (z = x + iy) \tag{4.2}$$

for every $n \in \mathbf{N}$. Since $\int_R 1 \cdot |\nabla v(z)|^2\, dxdy = D(v; R) < +\infty$, $0 \leqq w_n(z) \leqq 1$ on R, and $w_n(z) \to 0$ $(n \to \infty)$ locally uniformly on R, the Lebesgue convergence theorem assures that the first term of the right hand side of (4.2) tends to zero as $n \to \infty$. This with $\lim_{n\to\infty} D(w_n; R) = 0$ implies that

$$\lim_{n\to\infty} D(f_n; R) = 0. \tag{4.3}$$

We denote by u_n the harmonic part and by g_n the potential part of the Royden decomposition (cf. e.g. [14]) of f_n for each $n \in \mathbf{N}$ so that $f_n = u_n + g_n$ on R and $g_n|\delta = 0$ or $f_n|\delta = u_n|\delta$. We can thus find an $n \in \mathbf{N}$ such that $D(u_n; R) < \varepsilon$. Then $u := u_n$ is a required one.

We denote by A the totality of open neighborhoods α of $\xi \in \delta$. We make A an ordered set by giving an order on A by $\alpha_1 \preceq \alpha_2$ if $\alpha_1 \supset \alpha_2$. For each $\alpha \in A$

we let $2u_\alpha$ be the function u constructed in the foregoing paragraph for $V = \alpha$ and $\varepsilon = 1$ so that $u_\alpha \in HBD(R)$, $0 \leqq u_\alpha \leqq 1/2$ on R, $u_\alpha(\xi) = 1/2$, $u_\alpha|\delta\backslash\alpha = 0$, and $D(u_\alpha; R) < 1/4$. In particular, $\|u_\alpha\|_{HBD} \leqq 1$, i.e., the directed net $(u_\alpha)_{\alpha\in A}$ is contained in the closed unit ball $HBD(R)_1$ of $HBD(R)$, which is weakly compact as a consequence of (a): the Banach space $HBD(R)$ is reflexive. Therefore the directed net $(u_\alpha)_{\alpha\in A}$ contains a weakly convergent subnet $(u_\beta)_{\beta\in B}$, where B is a cofinal subnet of A so that B forms a base of neighborhood system of ξ consisting of certain open neighborhoods β of ξ. We denote by $\check{\zeta}$ the Dirac measure on R^* having its support at $\zeta \in \delta$ so that, as is easily seen, $\check{\zeta} \in HBD(R)^*$. Let $h \in HBD(R)$ be the weak limit of $(u_\beta)_{\beta\in B}$. Then

$$h(\zeta) = \langle h, \check{\zeta} \rangle = \lim_\beta \langle u_\beta, \check{\zeta} \rangle = \lim_\beta u_\beta(\zeta)$$

for every $\zeta \in \delta$, i.e., $(u_\beta)_{\beta\in B}$ converges to h pointwise on δ. If $\zeta \in \delta\backslash\{\xi\}$, then $\zeta \notin \beta$ for every $\beta \in B$ with $\beta \succeq \beta_0$ for some $\beta_0 \in B$. Hence $h(\zeta) = 0$ since $u_\beta|\delta\backslash\beta = 0$ $(\beta \succeq \beta_0)$. Needless to say, $u_\beta(\xi) = 1/2$ for every $\beta \in B$ implies that $h(\xi) = 1/2$. In view of $h \in HBD(R)\,|\,\delta \subset C(\delta)$ with $h|\delta\backslash\{\xi\} = 0$ and $h(\xi) = 1/2$, we conclude that ξ is an isolated point in δ. Hence $\mathrm{hm}(\{\xi\})$, the harmonic measure of $\{\xi\}$, must be strictly positive. Since there exists a constant $\tau \in (0, +\infty)$ depending upon R and W such that

$$\mathrm{hm}(K)^2 \leqq \tau \cdot \mathrm{cap}(K)$$

for every compact subset $K \subset \delta$ (cf. [10]), we must conclude that

$$0 < \mathrm{hm}(\{\xi\})^2 \leqq \tau \cdot \mathrm{cap}(\{\xi\}),$$

which contradicts the starting assumption of $\mathrm{cap}(\{\xi\}) = 0$ in the present part of proving the implication (b), or equivalently (c), from (a), by contradiction. □

5. Separability

In this section we will prove that the Banach space $HBD(R)$ is separable if and only if the base Riemann surface R is HD-singular, i.e., there is no unbounded Dirichlet finite harmonic function on R so that $HD(R) = HBD(R)$. We have seen in the foregoing section 4 as Theorem 4.1 that $HBD(R)$ is reflexive if and only if R is HD-singular. Therefore the result in this section assures that the reflexivity and the separability are equivalent in the case of the Banach space $HBD(R)$. Thus the proof of Theorem 2.8, the main assertion of this paper, will also be completed in this section. Namely we will prove the following result as the main assertion of this section.

Theorem 5.1. *The following three conditions are equivalent by pairs*:
(a) *the Banach space* $HBD(R)$ *is separable*;
(b) *the base Riemann surface* R *is* HD-*singular, i.e.*, $HD(R) = HBD(R)$;
(c) *the Royden harmonic boundary* δ *of* R *satisfies* $\inf_{\zeta\in\delta} \mathrm{cap}(\{\zeta\}) > 0$.

Proof. The equivalence of (b) and (c) is established in Proposition 3.9. The condition (b) assures, as we saw in the proof of Theorem 4.1, that Banach spaces $HD(R)$ and $HBD(R)$ are homeomorphically linear isomorphic. Since $HD(R)$ is separable (see [12]), we can conclude that $HBD(R)$ is separable, i.e., (b) implies (a). To complete the proof we thus have to show that (a) implies (b), or equivalently, the negation of (b), i.e., $HD(R)\setminus HBD(R) \neq \emptyset$, implies the negation of (a), i.e., $HBD(R)$ is not separable. Thus we assume the existence of an $h \in HD(R)$ with $\sup_R |h| = +\infty$. Since $HD(R)$ forms a vector lattice, we can assume the existence of an $h \in HD(R)^+$ with $\sup_R h = +\infty$. We say $(\varepsilon, \alpha) \in (0, +\infty) \times (1, +\infty)$ is an *admissible couple*. When an admissible couple (ε, α) is given, a pair $(a, b) \in \mathbf{R} \times \mathbf{R}$ will be referred to as an (ε, α)-*pair* if it satisfies the following three conditions:

$$\alpha < a < 3a < b; \tag{5.2}$$

$$\frac{a+b}{2} \in h(\delta); \tag{5.3}$$

$$D(h; \{z \in R : h(z) > a\}) < \varepsilon, \tag{5.4}$$

where $h(\delta)$ is the range set $\{h(\zeta) : \zeta \in \delta\}$. We first show the existence of an (ε, α)-pair (a, b) for any given admissible couple (ε, α). Let $V_a := \{z \in R : h(z) > a\}$ for any $a \in (\alpha, +\infty)$. In view of $V_a \downarrow \emptyset$ as $a \uparrow +\infty$, we see that $D(h, V_a) \downarrow 0$ as $a \uparrow +\infty$. This shows the existence of an $a \in (\alpha, +\infty)$ such that the condition (5.4) is satisfied. Fixing the $a \in (\alpha, +\infty)$ just found, we put $K_a := \{\zeta \in \delta : h(\zeta) \geqq 3a\}$. Since $h \in HD(R)^+ \setminus HBD(R)$, K_a is the closure of the nonempty open subset $\{\zeta \in \delta : h(\zeta) > 3a\}$ of δ and hence $\mathrm{cap}(K_a) > 0$. Thus we see the existence of an $s \in K_a$ such that $3a \leqq h(s) < +\infty$. Otherwise $K_a \subset \{\zeta \in \delta : h(\zeta) = +\infty\}$ and hence $\mathrm{cap}(K_a) = 0$ by Proposition 3.6, contradicting $\mathrm{cap}(K_a) > 0$. Let

$$b := 2h(s) - a > 2 \cdot 3a - 3a = 3a,$$

which satisfies (5.2), and moreover $(a + b)/2 = h(s) \in h(\delta)$, i.e., (5.3) is fulfilled. We have thus established the existence of an (ε, α)-pair (a, b).

Suppose an (ε, α)-pair (a, b) is given. We call the compact subset

$$E = \{\zeta \in \delta : a \leqq h(\zeta) \leqq b\} \tag{5.5}$$

of δ the (a, b)-*set*. A function $e \in HBD(R)^+$ will be referred to as an (a, b)-*function* if it satisfies the following 4 conditions:

$$0 \leqq e|R \leqq 1; \tag{5.6}$$

$$e|\delta\setminus E = 0; \tag{5.7}$$

$$1/4 \in e(\delta); \tag{5.8}$$

$$D(e; R) < \varepsilon. \tag{5.9}$$

We now prove the existence of an (a,b)-function e for any given (ε,α)-pair (a,b). To begin with we consider the function

$$u := \frac{((h \cup a) \cap b - a) \cdot (b - (h \cup a) \cap b)}{(b-a)^2} \tag{5.10}$$

on R based upon h. On setting

$$f := ((h \cup a) \cap b - a)/(b-a) \quad \text{and} \quad g := (b - (h \cup a) \cap b)/(b-a)$$

on R, we see that $0 \leqq f \leqq 1$ and $0 \leqq g \leqq 1$ on R. Then $u = f \cdot g$ on R and

$$\begin{aligned} |\nabla u|^2 &= |\nabla (f \cdot g)|^2 \leqq (f|\nabla g| + g|\nabla f|)^2 \\ &\leqq (f+g)^2 |\nabla((h \cup a) \cap b)|^2/(b-a)^2 \leqq 4|\nabla((h \cup a) \cap b)|^2/(b-a)^2. \end{aligned}$$

In view of $b - a \geqq 3a - a \geqq 2$, we have $4/(b-a)^2 \leqq 1$ and a fortiori

$$|\nabla u|^2 \leqq |\nabla((h \cup a) \cap b)|^2.$$

Hence, by (5.4), we conclude that

$$D(u; R) \leqq D((h \cup a) \cap b; R) \leqq D(h \cup a; R) = D(h; V_a) < \varepsilon. \tag{5.11}$$

By the existence of an $s \in \delta$ with $h(s) = (a+b)/2 \in (a,b)$, we see

$$u(s) = \left(\frac{b-a}{2}\right)^2 \bigg/ (b-a)^2 = 1/4$$

so that we obtain

$$1/4 \in u(\delta). \tag{5.12}$$

Finally we let e be the harmonic part of the Royden decomposition of u on R so that $e \in HBD(R)$ and

$$e|\delta = u|\delta. \tag{5.13}$$

Since $0 \leqq u \leqq 1$ on R and then on R^*, we have $0 \leqq u \leqq 1$ on δ so that $0 \leqq e \leqq 1$ on δ. The maximum principle (cf. [14]) yields that $0 \leqq e \leqq 1$ on R and a fortiori $e \in HBD(R)^+$ and (5.6) is deduced. By the definition (5.10) of u we see $u|\delta \backslash E = 0$ and thus, by (5.13), we infer (5.7). Conditions (5.12) and (5.13) assures the validity of (5.8). The Dirichlet principle $D(e; R) \leqq D(u; R)$ and (5.11) conclude (5.9). We have thus established the existence of an (a,b)-function e for any (ε,α)-pair (a,b).

We choose an arbitrary but then fixed sequence $(\varepsilon_n)_{n \in \mathbf{N}}$ of strictly positive numbers ε_n such that

$$\sum_{n \in \mathbf{N}} \sqrt{\varepsilon_n} < +\infty. \tag{5.14}$$

We also take arbitrarily and then fix a number $b_0 \in (1, +\infty)$. Viewing (ε_1, b_0) as an admissible couple, we take an (ε_1, b_0)-pair (a_1, b_1). With respect to the new admissible couple (ε_2, b_1) we take an (ε_2, b_1)-couple (a_2, b_2). Next for the admissible couple (ε_3, b_2) we choose an (ε_3, b_2)-pair (a_3, b_3). Repeating this process, when an (ε_n, b_{n-1})-pair (a_n, b_n) is obtained, we produce an (ε_{n+1}, b_n)-pair (a_{n+1}, b_{n+1}). Inductively we have thus constructed two sequences $(a_n)_{n \in \mathbf{N}}$ and $(b_n)_{n \in \mathbf{N}}$ such that (a_n, b_n) is an (ε_n, b_{n-1})-pair for each $n \in \mathbf{N}$ so that

$$(5.15) \qquad b_0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < a_{n+1} < b_{n+1} < \cdots .$$

From (5.9) and (5.14) it follows that

$$D(h; \{z \in R : h(z) > a_n\}) < \varepsilon_n \to 0 \quad (n \to +\infty)$$

so that we can conclude

$$(5.16) \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty.$$

For each $n \in \mathbf{N}$ we denote by E_n the (a_n, b_n)-set (cf. (5.5)) and by e_n an (a_n, b_n)-function (cf. (5.6)–(5.9)) so that $e_n \in HBD(R)^+$, $0 \leqq e_n|R \leqq 1$, $e_n|\delta \backslash E_n = 0$, $1/4 \in e_n(\delta)$, and $D(e_n; R) < \varepsilon_n$. Let I be the open interval $(0,1) \subset \mathbf{R}$ and

$$(5.17) \qquad \lambda = 0.\lambda_1 \lambda_2 \cdots \lambda_n \cdots$$

be the *infinite* dyadic fractional expression of $\lambda \in I$ so that $\lambda_j \in \{0, 1\}$ for all $j \in \mathbf{N}$ and there are at least one $\lambda_j = 0$ and infinitely many $\lambda_j = 1$. For each $\lambda \in I$ with (5.17) we can define the function

$$(5.18) \qquad f_\lambda := \sum_{j \in \mathbf{N}} \lambda_j e_j$$

on R, which is in $HBD(R)^+$ and satisfies $0 \leqq f_\lambda \leqq 1$ on R. In fact, since

$$0 \leqq \sum_{j \leqq n} \lambda_j e_j \leqq \sum_{j \leqq n} e_j \leqq 1$$

on R for every $n \in \mathbf{N}$ and partial sums $\sum_{j \leqq n} \lambda_j e_j$ $(n \in \mathbf{N})$ form an increasing sequence $(\sum_{j \leqq n} \lambda_j e_j)_{n \in \mathbf{N}}$ in $H(R)$, the Harnack principle assures that the series $\sum_{j \in \mathbf{N}} \lambda_j e_j$ in (5.18) is locally uniformly convergent on R and defines a function f_λ in $H(R)$ with $0 \leqq f_\lambda \leqq 1$ on R. By virtue of (5.14), we infer that

$$\sqrt{D(f_\lambda; R)} \leqq \sum_{j \in \mathbf{N}} \sqrt{D(\lambda_j e_j; R)} \leqq \sum_{j \in \mathbf{N}} \sqrt{D(e_j; R)} \leqq \sum_{j \in \mathbf{N}} \sqrt{\varepsilon_j} < +\infty,$$

i.e., $f_\lambda \in HD(R)$ so that $f_\lambda \in HBD(R)^+$, as required. We denote by F the totality of such f_λ, i.e.,

$$F := \{f_\lambda : \lambda \in I\}.$$

Choose any $\mu \in I$ different from $\lambda \in I$ with (5.17) and let

$$\mu = 0.\mu_1 \mu_2 \cdots \mu_n \cdots$$

be the infinite dyadic fractional expression of μ. Then $\lambda \neq \mu$ is equivalent to the existence of a $j \in \mathbf{N}$ such that $|\lambda_j - \mu_j| = 1$. Then we see that

$$|f_\lambda - f_\mu| = \sum_{i \in \mathbf{N}} |\lambda_i - \mu_i| e_i \geqq |\lambda_j - \mu_j| e_j = e_j$$

on δ and the maximum principle with (5.8) yields

$$\sup_R |f_\lambda - f_\mu| = \sup_\delta |f_\lambda - f_\mu| \geqq \sup_\delta e_j \geqq 1/4.$$

A fortiori we obtain that

$$\|f_\lambda - f_\mu\|_{HBD} \geqq 1/4 \quad (\lambda \neq \mu). \tag{5.19}$$

Finally we choose an arbitrary dense subset G of $HBD(R)$. Since the closure of G in $HBD(R)$ is $HBD(R)$, there is a $g \in G$ such that $\|g - f_\lambda\|_{HBD} < 1/16$ for any $f_\lambda \in F$, or rather for any $\lambda \in I$. We choose and then fix one such g and denote it by $g_\lambda \in G$ so that $\|g_\lambda - f_\lambda\|_{HBD} < 1/16$. For any $(\lambda, \mu) \in I \times I$ with $\lambda \neq \mu$, we see, by (5.19), that

$$\begin{aligned}\|g_\lambda - g_\mu\|_{HBD} &\geqq \|f_\lambda - f_\mu\|_{HBD} - \|g_\lambda - f_\lambda\|_{HBD} - \|g_\mu - f_\mu\|_{HBD} \\ &\geqq \frac{1}{4} - \frac{1}{16} - \frac{1}{16} = \frac{1}{8}.\end{aligned}$$

This assures that the mapping $\lambda \mapsto g_\lambda$ of I to G is injective and therefore the cardinal number $\#G$ of G is at least the cardinal number $\#I$ of I, which is the cardinal number $\aleph$ of continuum, i.e., $\#G \geqq \aleph$. Thus we have seen that any dense subset of $HBD(R)$ cannot be countable so that $HBD(R)$ is not separable, which was to be shown. □

6. Surfaces of almost finite genus

In the introduction we stated that $HBD(\mathbf{D})$ is neither reflexive nor separable (cf. Table 1.8). In view of Theorems 4.1 and 5.1 the above assertion is equivalent to the existence of a point of vanishing capacity in the Royden harmonic boundary $\delta\mathbf{D}$ of $\mathbf{D}$. Actually not only some single point but also every point in $\delta\mathbf{D}$ is of vanishing capacity. In this section this fact is shown for a certain class of Riemann surfaces including the unit disc $\mathbf{D}$. In this fashion the proof of Table 1.8 will also be complete in this last section 6.

A Riemann surface R is said to be *of almost finite genus* if there exists a finite or countably infinite sequence $(A_n)_n$ of relatively compact annuli A_n in R such that

(α) $\bar{A}_n \cap \bar{A}_m = \emptyset$ $(n \neq m)$;
(β) $R \setminus \bigcup_n \bar{A}_n$ is a planar subregion of R;
(γ) $\sum_n 1/\text{mod}\, A_n < +\infty$.

By an annulus A_n on a Riemann surface R we mean a subregion which is conformally equivalent to a doubly connected plane region so that, if both components of ∂A_n are nondegenerate continua, then it has the canonical conformal representation $\{1 < |z| < \mu_n\}$ and its modulus $\operatorname{mod} A_n = \log \mu_n$. The notion for Riemann surfaces to be of almost finite genus was first introduced in [7] as a generalization of surfaces to be of finite genus related to the classification theory of Riemann surfaces. Thus surfaces of finite genus including those of zero genus, of course, are of almost finite genus but our main concern related to this notion lies in the nontrivial case of infinite genus. In this section we prove the following result.

THEOREM 6.1. *Every point in the Royden harmonic boundary δ of any open Riemann surface R of almost finite genus is of vanishing capacity.*

It is known (cf. [14]) that every point in δ of R of almost finite genus is of vanishing harmonic measure. Since we have (cf. Section 4) the inequality $\operatorname{hm}(K)^2 \leqq \kappa \cdot \operatorname{cap}(K)$ for every compact subset K of δ, where $\operatorname{hm}(K)$ is the harmonic measure of K and κ is a constant independent of K, the above theorem 6.1 is a generalization of our former result just stated above.

Proof of Theorem 6.1. We assume that the sequence $(A_n)_n$ of annuli A_n in R satisfying (α), (β), and (γ) is infinite one so that $(A_n)_n = (A_n)_{n \in \mathbf{N}}$. Each argument in the proof we are going to develope in the case of infinite genus can be easily or rather trivially modified so as to be applicable to the case of finite genus. Thus the condition (γ) above takes the following form

$$\sum_{n \in \mathbf{N}} 1/\operatorname{mod} A_n < \infty. \tag{6.2}$$

On replacing A_n by a bit smaller annulus in A_n for each $n \in \mathbf{N}$ we may assume that each component of ∂A_n is an analytic Jordan curve. For each $n \in \mathbf{N}$ we take a nondividing analytic Jordan curve $\alpha_n \subset A_n$ such that α_n separates one of two disjoint components of ∂A_n from the other. We denote by A_{nj} $(j = 1, 2)$ two annuli which are two components of $A_n \backslash \alpha_n$. We could also have chosen α_n so as to make the following relations hold:

$$\operatorname{mod} A_{nj} = \frac{1}{2} \operatorname{mod} A_n \quad (j = 1, 2).$$

We then set

$$\alpha := \bigcup_{n \in \mathbf{N}} \alpha_n.$$

By virtue of the condition (β) we see that $R \backslash \alpha$ is a planar subregion of R, i.e., $R \backslash \alpha$ can be embedded in the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

In addition to the original sequence $(A_n)_{n\in\mathbf{N}}$ we further take two more sequences $(B_n)_{n\in\mathbf{N}}$ and $(C_n)_{n\in\mathbf{N}}$ of annuli B_n and C_n as follows. The curve α_n is contained in C_n and α_n separates one component of ∂C_n from the other; $\bar{C}_n \subset B_n$ and C_n separates one component of ∂B_n from the other; $\bar{B}_n \subset A_n$ and B_n separates one component of ∂A_n from the other. Thus we have

$$\alpha_n \subset C_n \subset \bar{C}_n \subset B_n \subset \bar{B}_n \subset A_n$$

for each $n \in \mathbf{N}$. We denote by $(A_n\backslash\bar{B}_n)_j$ the component of $A_n\backslash\bar{B}_n$ contained in A_{nj} $(j = 1,2)$ and similarly we denote by $(B_n\backslash\bar{C}_n)_j$ the component of $B_n\backslash\bar{C}_n$ contained in A_{nj} $(j = 1,2)$. We could also have chosen the above B_n and C_n so as to satisfy the following two relations:

$$\text{mod}(A_n\backslash\bar{B}_n)_j = \frac{1}{4}\,\text{mod}\, A_n \quad (j = 1,2) \tag{6.4}$$

and similarly

$$\text{mod}(B_n\backslash\bar{C}_n)_j = \frac{1}{8}\,\text{mod}\, A_n \quad (j = 1,2). \tag{6.5}$$

For each $n \in \mathbf{N}$ we take a function $\varphi_n \in C(R)$ such that $\varphi_n|R\backslash A_n = 0$, $\varphi_n|\bar{B}_n = 1$, and $\varphi_n \in H(A_n\backslash\bar{B}_n)$. By the choice of B_n satisfying (6.4), we have

$$D(\varphi_n; R) = D(\varphi_n; A_n\backslash\bar{B}_n) = 16\pi/\text{mod}\, A_n$$

so that by (6.2) we have

$$D\left(\sum_{n\in\mathbf{N}} \varphi_n; R\right) = \sum_{n\in\mathbf{N}} D(\varphi_n; A_n\backslash\bar{B}_n) = 16\pi \sum_{n\in\mathbf{N}} 1/\text{mod}\, A_n < +\infty. \tag{6.6}$$

Thus $\varphi := \sum_{n\in\mathbf{N}} \varphi_n$ is a Royden function on R. Since $\sum_{n\leqq m} \varphi_n$ has a compact support for every $m \in \mathbf{N}$ and $\sum_{n\leqq m} \varphi_n \uparrow \varphi$ on R locally uniformly and

$$D\left(\varphi - \sum_{n\leqq m} \varphi_n; R\right) \downarrow 0$$

as $m \uparrow +\infty$, we conclude that φ is a Royden potential (i.e., Dirichlet potential) on R so that $\varphi|\delta = 0$ (cf. [14]). Clearly $\varphi|\bigcup_{n\in\mathbf{N}} \bar{B}_n = 1$ and a fortiori $\varphi|\overline{\bigcup_{n\in\mathbf{N}} \bar{B}_n} = 1$. Therefor we conclude that

$$\left(\overline{\bigcup_{n\in\mathbf{N}} \bar{B}_n}\right) \cap \delta = \emptyset \tag{6.7}$$

in R^*.

We repeat the same construction for $B_n\backslash\bar{C}_n$ as we have done above for $A_n\backslash\bar{B}_n$. For each $n \in \mathbf{N}$ we take a function $\psi_n \in C(R)$ such that $\psi_n|R\backslash B_n = 0$, $\psi_n|\bar{C}_n = 1$, and $\psi_n \in H(B_n\backslash\bar{C}_n)$. By the choice of (6.5) we have

$$D(\psi_n; R) = D(\psi_n; B_n\backslash\bar{C}_n) = 32\pi/\text{mod}\, A_n$$

so that by (6.2) we conclude that

$$D\left(\sum_{n\in\mathbf{N}}\psi_n; R\right) = \sum_{n\in\mathbf{N}} D(\psi_n; B_n\setminus\bar{C}_n) = 32\pi\sum_{n\in\mathbf{N}} 1/\mathrm{mod}\, A_n < +\infty. \tag{6.8}$$

Thus $\psi := \sum_{n\in\mathbf{N}}\psi_n$ is a Royden function on R. As in the case of φ, since $\sum_{n\leqq m}\psi_n$ has a compact support in R for every $m\in\mathbf{N}$ and $\sum_{n\leqq m}\psi_n\uparrow\psi$ locally uniformly on R and

$$D\left(\psi - \sum_{n\leqq m}\psi_n; R\right)\downarrow 0$$

as $m\uparrow +\infty$, we can conclude that ψ is a Royden potential on R so that $\psi|\delta = 0$. Clearly $\psi|\overline{\bigcup_{n\in\mathbf{N}}\bar{C}_n} = 1$. Let

$$\chi := 1 - \psi, \tag{6.9}$$

which will play very important role later based upon the properties

$$\chi|\alpha = \chi\left|\overline{\bigcup_{n\in\mathbf{N}}\bar{C}_n}\right. = 0$$

and

$$\chi|\delta = \chi|R\left\backslash\overline{\bigcup_{n\in\mathbf{N}}\bar{B}_n}\right. = \chi|R\left\backslash\overline{\bigcup_{n\in\mathbf{N}}\bar{A}_n}\right. = 1.$$

Let $(R\setminus\alpha)^*$ be the Royden compactification of $R\setminus\alpha$ as an abstract Riemann surface. The surface $R\setminus\alpha$ is a subregion of R^* and we denote by $\overline{R\setminus\alpha} = \overline{R\setminus\alpha}^{R^*}$ the closure of $R\setminus\alpha$ as a subset of R^*. On the other hand, since $R\setminus\alpha$ is planar, $R\setminus\alpha$ may be viewed as a subregion of the Riemann sphere $\hat{\mathbf{C}} := \mathbf{C}\cup\{\infty\}$ and we denote by $\overline{R\setminus\alpha} = \overline{R\setminus\alpha}^{\hat{\mathbf{C}}}$ the closure of $R\setminus\alpha$ as a subset of $\hat{\mathbf{C}}$. We need to know the relations between $(R\setminus\alpha)^*$ and $\overline{R\setminus\alpha}^{R^*}$ and also between $(R\setminus\alpha)^*$ and $\overline{R\setminus\alpha}^{\hat{\mathbf{C}}}$. We start from the former relation.

First we recall the following general observation (see [14, 5B–5E]). Let S be a subregion of any open Riemann surface R in general, where we do not exclude the case $S = R$. We denote by $\bar{S}$ (or $\bar{S}^{R^*}$ if we need to be more precise) the closure of S in R^* and by S^* the Royden compactification of S considered as an abstract Riemann surface. As for the relation between S^* and $\bar{S} = \bar{S}^{R^*}$ we know the unique existence of the *projection*

$$j = j(S^*, \bar{S}) : S^* \to \bar{S}$$

characterized by the following two conditions: $j : S^* \to \bar{S}$ is a surjective continuous mapping; j fixes S pointwise, i.e.,

$$j|S = id. \quad \text{(the identity mapping).} \tag{6.10}$$

We distinguish a part of the boundary $\overline{S}\backslash S$ as follows:

$$b_S := (\overline{S}\backslash\overline{\partial S}) \cap \gamma, \tag{6.11}$$

where ∂S is the relative boundary $(\overline{S}\backslash S) \cap R$ of S relative to R and $\gamma = \gamma R = R^*\backslash R$, the Royden boundary of R. Then we see that

$$S \cup j^{-1}(b_S) \approx S \cup b_S, \tag{6.12}$$

i.e., the projection $j : S \cup j^{-1}(b_S) \to S \cup b_S$ is a homeomorphism (i.e., bijective and bicontinuous mapping) so that $j : S \cup j^{-1}(b_S) \to S \cup b_S$ and $j^{-1} : S \cup b_S \to S \cup j^{-1}(b_S)$ are bijective and continuous.

We particularize the above general observation to our present situation of R of almost finite genus and its subregion $S := R\backslash\alpha$. Then clearly

$$\overline{S} = \overline{S}^{R^*} = \overline{R\backslash\alpha}^{R^*} = R^*$$

and we see the unique existence of the projection

$$j : (R\backslash\alpha)^* \to R^*,$$

which is a surjective continuous mapping with $j|R\backslash\alpha = id.$, the identity mapping. In this case we see that

$$b_S := (\overline{S}\backslash\overline{\partial S}) \cap \gamma = (R^*\backslash\overline{\alpha}) \cap \gamma = \gamma\backslash\overline{\alpha}$$

so that

$$j : (R\backslash\alpha) \cup j^{-1}(\gamma\backslash\overline{\alpha}) \to (R\backslash\alpha) \cup (\gamma\backslash\overline{\alpha}) = R^*\backslash\overline{\alpha}$$

is a homeomorphism with $j|R\backslash\alpha = id.$, or equivalently

$$j^{-1} : R^*\backslash\overline{\alpha} \to (R\backslash\alpha) \cup j^{-1}(\gamma\backslash\overline{\alpha}) \subset (R\backslash\alpha)^* \tag{6.13}$$

is a homeomorphism with $j^{-1}|R\backslash\alpha = id.$

Having finished the clarification as (6.13) of the relation between $(R\backslash\alpha)^*$ and $\overline{R\backslash\alpha}^{R^*}$, we turn next to the task of unravelling the relation between $(R\backslash\alpha)^*$ and $\overline{R\backslash\alpha}^{\hat{\mathbf{C}}}$ by bringing the fact that $R\backslash\alpha$ is a planar region into our consideration, i.e., $R\backslash\alpha \subset \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We use the proper coordinate z on $\hat{\mathbf{C}}$ so that $\hat{\mathbf{C}} = \{z : |z| \leqq +\infty\}$ and $\mathbf{C} = \{z : |z| < +\infty\}$. We denote by $\Delta(c,r)$ the disc in $\hat{\mathbf{C}}$ with radius $0 < r < \infty$ centered at $c \in \hat{\mathbf{C}}$ so that $\Delta(c,r) = \{z \in \mathbf{C} : |z - c| < r\}$ for $c \in \mathbf{C}$ and $\Delta(\infty, r) = \{z \in \hat{\mathbf{C}} : |z| > 1/r\}$. Hence $1/z$ is used as a local parameter at ∞. We set $\overline{\Delta}(c,r) := \overline{\Delta(c,r)}^{\hat{\mathbf{C}}}$. We denote by I the identity mapping of $\hat{\mathbf{C}}$, i.e., $I(z) = z$ for $z \in \hat{\mathbf{C}}$. We maintain that the identity mapping $I : R\backslash\alpha \to R\backslash\alpha$ is continued to a continuous surjective mapping

$$\overline{I} : (R\backslash\alpha)^* \to \overline{R\backslash\alpha}^{\hat{\mathbf{C}}}, \quad \overline{I}|R\backslash\alpha = I. \tag{6.14}$$

To see this fix an arbitrary point $a \in (R\backslash\alpha) \cap \mathbf{C}$ and suppose $\overline{\Delta}(a,\rho) \subset R\backslash\alpha$ $(0 < \rho < \infty)$. Let

$$T(z) := \frac{\rho}{z - a}$$

and $G := T(R\backslash\alpha)$, a subregion of $\hat{\mathbf{C}}$. By the conformal invariance of Royden compactifications, $T : R\backslash\alpha \to G$ can be continued to a homeomorphism $T^* : (R\backslash\alpha)^* \to G^*$. It is entirely clear that $T^{-1} : G \to R\backslash\alpha$ can be continued to a homeomorphism $\overline{T^{-1}} : \overline{G}^{\hat{\mathbf{C}}} \to \overline{R\backslash\alpha}^{\hat{\mathbf{C}}}$. We denote by $I_G : G \to G$ the identity mapping I. We show that I_G can be continued to a continuous surjective mapping $\overline{I_G} : G^* \to \overline{G}^{\hat{\mathbf{C}}}$. Observe that $\overline{\Delta}(\infty, 1) \subset G$ and therefore

$$D(I_G; G\backslash\overline{\Delta}(\infty, r)) = 2\,\mathrm{Area}(G\backslash\overline{\Delta}(\infty, r)) < +\infty$$

for every $0 < r < 1$. Hence I_G can be modefied on $\overline{\Delta}(\infty, 1)$ as a function I_{G1} such that $I_{G1}|G\backslash\overline{\Delta}(\infty, 1) = I$ and I_{G1} is a Royden function on G. Thus I_{G1} is continuous on G^* and a fortiori I_G is continuous on G^*. Hence we have seen that $I_G : G \to G$ can be continued to a continuous surjective mapping $\overline{I_G} : G^* \to \overline{G}^{\hat{\mathbf{C}}}$. Hence we can conlude that

$$I = T^{-1} \circ I_G \circ T : R\backslash\alpha \to R\backslash\alpha$$

can be continued to the continuous surjective mapping

$$\overline{I} = \overline{T^{-1}} \circ \overline{I_G} \circ T^* : (R\backslash\alpha)^* \to \overline{R\backslash\alpha}^{\hat{\mathbf{C}}}$$

(cf. Fig. 6.15). Thus (6.14) is established.

Fig. 6.15

$$\begin{array}{ccc} (R\backslash\alpha)^* & \xrightarrow{T^*} & G^* \\ \overline{I}\downarrow & & \downarrow\overline{I_G} \\ \overline{R\backslash\alpha}^{\hat{\mathbf{C}}} & \xleftarrow[\overline{T^{-1}}]{} & \overline{G}^{\hat{\mathbf{C}}} \end{array}$$

Finally consider the mapping

$$J := \overline{I} \circ j^{-1} : R^*\backslash\overline{\alpha} \to \overline{R\backslash\alpha}^{\hat{\mathbf{C}}}. \tag{6.16}$$

In view of (6.13) and (6.14), we see that the mapping J in (6.16) is continuous, surjective, and $J|R\backslash\alpha = id$. Choose an arbitrary point $\zeta \in \delta = \delta R$. Our plan in the rest of this proof is to establish $\mathrm{cap}(\{\zeta\}) = 0$. Let $\xi := J(\zeta)$. Since $\{\zeta\}$ is not G_δ (cf. [14]) while every $\{z\}$ with $z \in R$ is G_δ, ξ cannot be in R because $J|R\backslash\alpha = id$. By virtue of (6.7) we have

$$\delta \subset R^* \Big\backslash \overline{\bigcup_{n\in\mathbf{N}} \overline{B}_n} \subset R^*\backslash\overline{\alpha}$$

and therefore $\xi \notin \overline{\alpha}$ either, or more accurately ξ does not belong to any component of the boundary $\overline{(R\backslash\alpha)}^{\hat{\mathbf{C}}}\backslash(R\backslash\alpha)$ of $R\backslash\alpha$ lying over α. We fix a disc $\Delta(a, \rho)$ with radius $0 < \rho < \infty$ centered at a point $a \in (R\backslash\alpha) \cap \hat{\mathbf{C}}$ such that $\overline{\Delta}(a, \rho) \subset R\backslash\alpha$. We choose two strictly decreasing zero sequences $(t_i)_{i\in\mathbf{N}}$ and $(\varepsilon_j)_{j\in\mathbf{N}}$ in the interval $(0, 1)$ such that $\overline{\Delta}(\xi, t_1) \cap \overline{\Delta}(a, \rho) = \emptyset$. Define the function $f_{ij} \in C(\hat{\mathbf{C}})$ such that $f_{ij}|\hat{\mathbf{C}}\backslash\overline{\Delta}(\xi, t_i) = 0$, $f_{ij}|\overline{\Delta}(\xi, t_i\varepsilon_j) = 1$, and $f_{ij} \in H(\Delta(\xi, t_i)\backslash\overline{\Delta}(\xi, t_i\varepsilon_j))$. Then

$$D(f_{ij};\hat{\mathbf{C}}) = D(f_{ij};\Delta(\xi,t_i)\backslash\bar{\Delta}(\xi,t_i\varepsilon_j))$$
$$= 2\pi/\mathrm{mod}(\Delta(\xi,t_i)\backslash\bar{\Delta}(\xi,t_i\varepsilon_j)) = 2\pi/\log(1/\varepsilon_j).$$

The function $f_{ij}\circ J$ may be viewed as a Royden function on $R\backslash\alpha$ because of (6.16) and $D(f_{ij};R\backslash\alpha) \leqq D(f_{ij};\hat{\mathbf{C}}) < +\infty$ but not on R since $f_{ij}\circ J$ may be discontinuous at α. The function χ in (6.9) vanishes in an open neighborhood of α and χ itself is a Royden function on R. Both of $f_{ij}\circ J$ and χ are bounded. Therefore, if we define

$$g_{ij} := \chi\cdot(f_{ij}\circ J)$$

on R, then g_{ij} is a Royden function on R. Clearly $g_{ij}(\zeta) = \chi(\zeta)\cdot f_{ij}(\xi) = 1\cdot 1 = 1$ and $g_{ij}\,|\,\bar{\Delta}(a,\rho) = \chi\cdot(f_{ij}\circ J\,|\,\bar{\Delta}(a,\rho)) = \chi\cdot 0 = 0$. Thus each g_{ij} is a competing function in the variation to determine $\mathrm{cap}(\{\zeta\})$ with respect to the end $R\backslash\bar{\Delta}(a,\rho)$, i.e.,

$$\mathrm{cap}(\{\zeta\}) \leqq D(g_{ij};R)$$

for every $(i,j)\in\mathbf{N}\times\mathbf{N}$.

We can find an increasing sequence $(k(i))_{i\in\mathbf{N}}\subset\mathbf{N}$ such that $k(i)\uparrow+\infty$ as $i\uparrow\infty$ and

$$\bar{\Delta}(\xi,t_i)\cap\left(\bigcup_{n<k(i)}(\bar{B}_n\backslash\alpha_n)\right) = \emptyset. \tag{6.17}$$

Aiming to estimate $D(g_{ij};R)$ we infer that

$$|\nabla g_{ij}|^2 = |\chi\nabla f_{ij} + f_{ij}\nabla\chi|^2 \leqq 2(\chi^2|\nabla f_{ij}|^2 + f_{ij}^2|\nabla\chi|^2)$$

on R and hence

$$D(g_{ij};R) \leqq 2\int_R \chi(z)^2|\nabla f_{ij}(z)|^2\,dxdy + 2\int_R f_{ij}(z)^2|\nabla\chi(z)|^2\,dxdy \quad (z = x+iy).$$

Observe that $0\leqq\chi\leqq 1$ and $0\leqq f_{ij}\leqq 1$ on R. Using $D(f_{ij};\hat{\mathbf{C}}) = 2\pi/\log(1/\varepsilon_j)$ we see that

$$\int_R \chi(z)^2|\nabla f_{ij}(z)|^2\,dxdy = \int_{R\backslash\alpha}\chi(z)^2|\nabla f_{ij}(z)|^2\,dxdy$$
$$\leqq \int_{\hat{\mathbf{C}}}|\nabla f_{ij}(z)|^2\,dxdy = 2\pi/\log(1/\varepsilon_j).$$

Clearly $\mathrm{supp}\, f_{ij}\subset\Delta(\xi,t_i)$ and $\mathrm{supp}|\nabla\chi|^2\subset\bigcup_{n\in\mathbf{N}}\bar{B}_n$ and thus (6.17) yields

$$\mathrm{supp}\, f_{ij}^2|\nabla\chi|^2 \subset \Delta(\xi,t_i)\cap\left(\bigcup_{n\in\mathbf{N}}\bar{B}_n\right)$$
$$= \Delta(\xi,t_i)\cap\left(\bigcup_{n\geqq k(i)}\bar{B}_n\right)\subset\bigcup_{n\geqq k(i)}\bar{B}_n,$$

where supp F for a function F indicates the support of F. Because of this and $|\nabla\chi|^2 = |\nabla\psi_n|^2$ on B_n we see that

$$\begin{aligned}\int_R f_{ij}(z)^2|\nabla\chi(z)|^2\,dxdy &= \int_{\bigcup_{n\geqq k(i)}\bar{B}_n} f_{ij}(z)^2|\nabla\chi(z)|^2\,dxdy \\ &\leqq \sum_{n\geqq k(i)}\int_{B_n}|\nabla\chi(z)|^2\,dxdy = \sum_{n\geqq k(i)}\int_{B_n}|\nabla\psi_n(z)|^2\,dxdy \\ &= \sum_{n\geqq k(i)} D(\psi_n; B_n) = \sum_{n\geqq k(i)} 32\pi/\mathrm{mod}\, A_n.\end{aligned}$$

After all we conclude that

$$D(g_{ij}; R) \leqq 4\pi/\log(1/\varepsilon_j) + 64\pi\sum_{n\geqq k(i)} 1/\mathrm{mod}\, A_n$$

so that

$$\mathrm{cap}(\{\zeta\}) \leqq 4\pi/\log(1/\varepsilon_j) + 64\pi\sum_{n\geqq k(i)} 1/\mathrm{mod}\, A_n.$$

By (6.2) and $\varepsilon_j \downarrow 0$ as $j \uparrow +\infty$, by letting $i \uparrow +\infty$ and $j \uparrow +\infty$ in the above inequality, we conclude that $\mathrm{cap}(\{\zeta\}) = 0$, which was to be shown. □

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