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CONSTRUCTION OF EQUIVALENCE MAPS IN PSEUDO-HERMITIAN GEOMETRY VIA LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

We discuss an equivalence problem of pseudo-Hermitian structures on 3-dimensional manifolds, and develop a method of constructing equivalence maps by using systems of linear partial differential equations. It is proved that a pseudo-Hermitian structure is transformed to a standard model of pseudo-Hermitian structure constructed on the Heisenberg group if and only if it has the vanishing pseudo-Hermitian torsion and the pseudo-Hermitian curvature. A system of linear partial differential equations whose coefficients are associated with a given pseudo-Hermitian structure is introduced, and plays a central role in this paper. The system is integrable if and only if the pseudo-Hermitian structure has vanishing torsion and curvature. The equivalence map is constructed by using a normal basis of the solution space of the system.

1. Introduction

In a series of papers [2], [3], [4], and [5], Cartan elaborated the equivalence problem of geometric structures. He gave an algorithm of solving the problem by using absorptions of torsions, prolongations, group reductions and fixing connections, etc. Necessary conditions are given by invariants for an existence of a local equivalence. But Cartan's method does not lead directly to a concrete construction of equivalence. A method of a quadrature should be contrived depending on each geometric structure.

In classical complex function theory, a conformal equivalence is given by the projectification of solutions of the linear second order differential equation using the Schwarzian derivative as the coefficients. For higher dimensional conformal geometry and the contact projective geometry, Schwarzian derivatives and systems of linear differential equations are settled by several authors, so that the

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projectification of solutions gives an equivalence of the structure (see Gunning [8], Matsumoto et al. [9], Yoshida [15], Ozawa et al. [10], and Sato et al. [12], [13]).

In this paper, we study the pseudo-Hermitian geometry under equivalence via contact diffeomorphism. Since the problem is local, we consider the structure on an open set U in the 3-dimensional Heisenberg group H (see §2.1 for basics on Heisenberg group). We take the standard left invariant frame v_i $(i = 1, 2, 3)$ and coframe α_i $(i = 1, 2, 3)$. We fix the contact structure $\ker \alpha_3$. For a complex valued function σ on U, we denote Z_σ and $\bar{Z}_\sigma$ the complex vector fields defined by $Z_\sigma = v_1 + \sigma v_2$ and $\bar{Z}_\sigma = v_1 + \bar{\sigma} v_2$, respectively. If the imaginary part of a complex valued function σ on U does not vanish, then σ induces a decomposition of the complexified contact plane field $\mathbf{C} \otimes \ker \alpha_3$;

$$\mathbf{C} \otimes \ker \alpha_3 = \mathbf{C} Z_\sigma \oplus \mathbf{C} \bar{Z}_\sigma.$$

(See §2.2.) Thus we have a correspondence between a function σ and a CR-structure which has the underlying contact structure equal to $\ker \alpha_3$. The CR structure (U, σ, η) together with a fixed contact form $e^\eta \alpha_3$ gives a pseudo-Hermitian structure. The unique canonical linear connection ∇ on a pseudo-Hermitian structure is defined by Tanaka and Webster so that the contact form and the CR structure are parallel with respect to ∇ (see Tanaka [14], Webster [16] and Dragomir et al. [7, Theorem 1.3])

Let T be the Reeb vector field of the contact form $e^\eta \alpha_3$ (see equation (2)). The pseudo-Hermitian torsion τ and the Tanaka-Webster curvature κ (or p-H torsion and T-W curvature for short, respectively) are calculated in Lemmas 2 and 3 as

$$\tau = \frac{-1}{\sigma - \bar{\sigma}} (T(\bar{\sigma}) - \bar{\sigma} \bar{Z}_\sigma(v_2(e^{-\eta})) - \bar{Z}_\sigma(v_1(e^{-\eta})))$$

$$\kappa = -\bar{Z}_\sigma(s) + Z_\sigma(m) + m(s - \bar{m}) - \sqrt{-1} hp.$$

$$\text{where } m = -\frac{Z_\sigma(\bar{\sigma}) - \bar{Z}_\sigma(\sigma)}{\sigma - \bar{\sigma}} - \bar{Z}_\sigma(\eta),\ s = v_2(\sigma) + 2Z_\sigma(\eta),$$

$$h = \sqrt{-1}(\sigma - \bar{\sigma}) e^\eta, \quad \text{and} \quad p = \frac{\bar{\sigma} Z_\sigma(v_2(e^{-\eta})) + Z_\sigma(v_1(e^{-\eta})) - T(\sigma)}{\sigma - \bar{\sigma}}$$

(For a more explicit formula, see Proposition 1.)

A Sasakian structure is defined by specializing a metric structure on a pseudo-Hermitian structure. We use the terminology in a slightly different context where (U, σ, η) is a Sasakian structure if the torsion τ of the Tanaka-Webster connection vanishes. If σ is equal to the constant function $\sigma_0 = -\sqrt{-1}$ on H, then $(H, \sigma_0, \eta = 0)$ is a Sasakian structure with a vanishing T-W curvature. We regard $(H, \sigma_0, 0)$ as the standard flat Sasakian structure.

Let σ_j be complex valued functions, and let η_j be real valued functions on U_j, respectively $(j = 1, 2)$. Then we have two pseudo-Hermitian structures $(U_j, \sigma_j, e^{\eta_j} \alpha_3)$. If a differentiable map $\varphi : U_1 \to U_2$ satisfies

$$e^{\eta_1}\alpha_3 = \varphi^*(e^{\eta_2}\alpha_3)$$
$$\varphi_*(\mathbf{C}(v_1 + \sigma_1 v_2)) = \mathbf{C}(v_1 + \sigma_2 v_2),$$

then φ is called a pseudo-Hermitian map.

Let (U, σ, η) be a pseudo-Hermitian structure. It is well known that *there exists a pseudo-Hermitian map* $\varphi : (U, \sigma, \eta) \to (H, -\sqrt{-1}, 0)$ *if and only if the p-H torsion and the T-W curvature of* (U, σ, η) *vanish* (see Blair et al. [1] and Cho et al. [6].) We will construct the pseudo-Hermitian map from (U, σ, η) to $(H, -\sqrt{-1}, 0)$ by using a system of linear partial differential equations. For this purpose, given a pseudo-Hermitian structure (U, σ, η), consider the following system of linear partial differential equations:

$$\text{(F)} \qquad \begin{cases} 0 = \bar{Z}_\sigma(f) \\ 0 = Z_\sigma^2(f) - (v_2(\sigma) + 2Z_\sigma(\eta))Z_\sigma(f), \end{cases}$$

where f is a complex valued unknown function on U. We prove the following result in Section 4.

Theorem 1. *If the pseudo-Hermitian structure* (U, σ, η) *satisfies* $\tau = \kappa = 0$, *then the system of equations* (F) *is integrable.*

In Section 5, we define a normality condition for a basis of the solution space of (F) and prove the following:

Theorem 2. *Suppose that a pseudo-Hermitian structure* (U, σ, η) *on an open set* U *in the Heisenberg group* H *satisfies* $\tau = \kappa = 0$. *Then there exists a normal basis* $\{f_1, f_2, f_3\}$ *of the solution space of* (F) *such that* f_1 *is a constant function equal to* $\frac{1}{2}$. *For such solutions* f_2 *and* f_3, *the map* $\Phi := (T(f_3)^{-1/2} f_2, T(f_3)^{-1}\Re(f_3)) : (U, \sigma, \eta) \to (H, -\sqrt{-1}, 0)$ *is a pseudo-Hermitian map, where* T *is the Reeb vector field of the contact form* $e^\eta \alpha_3$, *and* $\Re(f_3)$ *is the real part of* f_3.

2. CR and pseudo-Hermitian structures

We introduce a left invariant contact structure on the 3-dimensional Heisenberg group, by using a standard frame $\{v_1, v_2, v_3\}$ and coframe $\{\alpha_1, \alpha_2, \alpha_3\}$. And then we summarize basic facts on CR and pseudo-Hermitian structures.

2.1. Heisenberg group and frames. The set $H = \mathbf{C} \oplus \mathbf{R}$ endowed with the product structure

$$(z, t) \cdot (w, s) = (z + w, t + s - 2\Im(z\bar{w})) \quad \text{for } (z, t), (w, s) \in \mathbf{C} \oplus \mathbf{R}$$

is called the *Heisenberg group*. By identifying H with $\mathbf{R}^3$ and using coordinates $(x, y, t) = (z, t)$, we introduce the vector fields

$$\text{(1)} \qquad v_1 = \frac{1}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \quad v_2 = \frac{1}{2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \quad v_3 = \frac{\partial}{\partial t}$$

on H. They form a left invariant frame, and satisfy the so-called *Heisenberg's relation*;

$$[v_2, v_1] = v_3, \quad [v_3, v_1] = [v_3, v_2] = 0.$$

In the following, we call $\{v_1, v_2, v_3\}$ the *Heisenberg frame*. The real 1-forms

$$\alpha_1 = 2\,dx, \quad \alpha_2 = 2\,dy, \quad \alpha_3 = dt + 2(x\,dy - y\,dx) = dt + \sqrt{-1}(z\,d\bar{z} - \bar{z}\,dz)$$

on H are left invariant. They form the dual frame to the Heisenberg frame (i.e. $\alpha_i(v_j) = \delta_{ij}$ for $i, j = 1, 2, 3$), and satisfy the relations

$$d\alpha_3 = \alpha_1 \wedge \alpha_2, \quad d\alpha_1 = d\alpha_2 = 0$$

We call $\{\alpha_1, \alpha_2, \alpha_3\}$ the *Heisenberg coframe*.

The plane field spanned by v_1 and v_2 defines a *contact structure* on H, which we denote by $\mathscr{D}$. Throughout the paper, we fix the contact structure $\mathscr{D}$ on H. For any real valued function η, a 1-form $e^{\eta}\alpha_3$ is a contact form of $\mathscr{D}$.

The *Reeb vector field* (or the *characteristic vector field*) T of a contact form α is the uniquely determined vector field satisfying the equations

$$\iota_T\alpha = 1, \quad \iota_T(d\alpha) = 0,$$

where ι is the interior product. The Reeb vector field T of the contact form $e^{\eta}\alpha_3$ is given by

$$T = e^{-\eta}v_2(\eta)v_1 - e^{-\eta}v_1(\eta)v_2 + e^{-\eta}v_3. \tag{2}$$

Especially, v_3 is the Reeb vector field of the contact form α_3.

2.2. CR structures based on $\mathscr{D}$. A *CR structure* based on the given contact structure $(H, \mathscr{D})$ is a direct sum decomposition of the complexification of $\mathscr{D}$ into 1-dimensional subspaces;

$$\mathbf{C} \otimes \mathscr{D} = \mathscr{D}_{(1,0)} \oplus \mathscr{D}_{(0,1)}.$$

Let f and g be complex valued functions on H. The line fields $\mathscr{D}_{(1,0)} = \mathbf{C}(fv_1 + gv_2)$ and $\mathscr{D}_{(0,1)} = \mathbf{C}(\bar{f}v_1 + \bar{g}v_2)$ intersect only at the zero section if and only if the imaginary part of $f\bar{g}$ doesn't vanish. If it is the case, f itself doesn't vanish, and the vector field $v_1 + (g/f)v_2$ gives the same decomposition of $\mathbf{C} \otimes \mathscr{D}$. Therefore a CR structure based on $\mathscr{D}$ uniquely corresponds to a complex valued function σ $(= g/f)$ with non-vanishing imaginary part $\Im(\sigma)$. Putting $Z_\sigma = v_1 + \sigma v_2$, we have the following one-to-one correspondence:

$$\{\text{CR structures based on } (H, \mathscr{D})\} \leftrightarrow \{\sigma : H \to \mathbf{C}; \Im(\sigma) \neq 0\}$$

$$\mathbf{C}Z_\sigma \oplus \mathbf{C}\bar{Z}_\sigma \leftrightarrow \sigma,$$

where $\bar{Z}_\sigma = Z_{\bar{\sigma}} = v_1 + \bar{\sigma}v_2$. For an open set $U \subset H$ and a smooth function $\sigma : U \to \mathbf{C}$ with non-vanishing imaginary part, we denote by (U, σ) the CR structure based on the contact structure $\mathscr{D}$ on U defined by the complex vector field $Z_\sigma = v_1 + \sigma v_2$.

By definition, a complex valued function f is called a *CR function* with respect to the CR structure defined by σ if f satisfies $\bar{Z}_\sigma(f) = 0$. Let U_i be open subsets of H and $\sigma_i : U_i \to \mathbf{C}$ be smooth functions with non-vanishing imaginary part for $i = 1, 2$. A contact map is necessarily a local diffeomorphism. If the differential $\varphi_* : TU_1 \to TU_2$ of a contact map $\varphi : U_1 \to U_2$ preserves the decompositions $\mathbf{C}Z_{\sigma_i} \oplus \mathbf{C}\bar{Z}_{\sigma_i}$, then φ is called a *CR map*.

For the CR-structure defined by σ, there exists a unique complex structure J_σ along $\mathscr{D}$ whose complexification has Z_σ and $\bar{Z}_\sigma$ as the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenvectors, respectively. If we write $\sigma = a + \sqrt{-1}b$, then the explicit formula of J_σ is given by

$$J_\sigma(pv_1 + qv_2) = \left(-\frac{a}{b}p + \frac{1}{b}q\right)v_1 + \left(-\frac{a^2+b^2}{b}p + \frac{a}{b}q\right)v_2.$$

If $\sigma \equiv -\sqrt{-1}$, then the complex structure $J_{-\sqrt{-1}}$ maps v_1 and v_2 to

$$J_{-\sqrt{-1}}(v_1) = v_2 \quad \text{and} \quad J_{-\sqrt{-1}}(v_2) = -v_1,$$

respectively. We will regard the CR structure $(H, -\sqrt{-1})$ as a *standard model* of CR structure.

By using the complex structure J_σ on the contact plane field $\mathscr{D}$, the notions of CR functions and maps are explained as follows. A complex valued function f is a CR function if and only if

$$\sqrt{-1}v(f) = J_\sigma(v)(f)$$

holds for all contact elements $v \in \mathscr{D}$. Suppose two complex valued functions σ_i with $i = 1, 2$ define CR structures on open sets $U_i \subset H$, respectively. Let $\varphi : U_1 \to U_2$ be a contact map with respect to the contact plane field $\mathscr{D}$. Then φ is a CR map if and only if

$$J_{\sigma_2}(\varphi_*(v)) = \varphi_*(J_{\sigma_1}(v))$$

holds for all contact elements $v \in \mathscr{D}$.

Originally CR structure was introduced as an abstraction of real hypersurfaces in complex spaces. Here we briefly explain it for a 3-dimensional real hypersurface M in the 2-dimensional complex space $\mathbf{C}^2$. At each point p of M, the intersection

$$\mathscr{D}_p = T_pM \cap \sqrt{-1}T_pM$$

is necessarily a 2-dimensional real subspace in T_pM, and induces a contact structure $\mathscr{D} = \bigcup_{p \in M} \mathscr{D}_p$ on M. Since $\mathscr{D}_p$ is closed under the multiplication $J = \times\sqrt{-1}$, J defines a complex structure on each plane $\mathscr{D}_p$, and thus a CR structure on M. We call it *the natural CR structure* on the real hypersurface M. Let f and g be CR functions on an open set $U \subset H$ with respect to a CR structure defined by σ. Suppose the image M of the map $\varphi = (f, g) : U \to \mathbf{C}^2$ is an embedded real hypersurface. Then φ is a CR map from (U, σ) to M with the natural CR structure.

2.3. Pseudo-Hermitian structure. A *pseudo-Hermitian structure* $(U, \sigma, f\alpha_3)$ is a CR structure together with a fixed contact form $f\alpha_3$. For the sake of simplicity, we consider only the case where

$$\Im(\sigma) < 0 \quad \text{and} \quad f > 0. \tag{3}$$

Let $(U_i, \sigma_i, e^{\eta_i}\alpha_3)$ be pseudo-Hermitian structures $(i = 1, 2)$. A CR map $\varphi : (U_1, \sigma_1, e^{\eta_1}\alpha_3) \to (U_2, \sigma_2, e^{\eta_2}\alpha_3)$ is called a *pseudo-Hermitian map* if it satisfies

$$\varphi^*(e^{\eta_2}\alpha_3) = e^{\eta_1}\alpha_3.$$

Let $\sigma : U \to \mathbf{C}$ be a complex valued function with negative imaginary part, and $\eta : U \to \mathbf{R}$ be a real valued function. We use the abbreviation

$$(U, \sigma, \eta)$$

for a pseudo-Hermitian structure on an open set $U \subset H$ based on the contact structure $\mathscr{D}$ which consists of a decomposition

$$\mathbf{C} \otimes \mathscr{D} = \mathbf{C}(v_1 + \sigma v_2) \oplus \mathbf{C}(v_1 + \bar{\sigma} v_2)$$

and a prescribed contact form $e^{\eta}\alpha_3$.

For a given pseudo-Hermitian structure (U, σ, η), we use the frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ formed by vector fields

$$\begin{aligned} Z_\sigma &= v_1 + \sigma v_2, \quad \bar{Z}_\sigma = v_1 + \bar{\sigma} v_2, \\ T &= e^{-\eta} v_2(\eta) v_1 - e^{-\eta} v_1(\eta) v_2 + e^{-\eta} v_3 \end{aligned} \tag{4}$$

of the complexified tangent bundle $\mathbf{C} \otimes TU$. Here T is the Reeb vector field of the contact form $e^{\eta}\alpha_3$ (see (2)).

The *Levi form* L of a pseudo-Hermitian structure (U, σ, η) is the restriction to $\mathbf{C} \otimes \mathscr{D}$ of the Hermitian form

$$L(V, W) = -\sqrt{-1}\, d(e^{\eta}\alpha_3)(V, \bar{W}).$$

The real coefficient h of the Levi form L with respect to the frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ is given by

$$h := L(Z_\sigma, Z_\sigma) = \sqrt{-1}(\sigma - \bar{\sigma}) e^{\eta}. \tag{5}$$

By the hypothesis (3), the function h is positive valued.

The dual frame $\{\theta^1, \theta^{\bar{1}}, \theta\}$ of the frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ is given by the 1-forms

$$\begin{aligned} \theta^1 &= \frac{1}{\sigma - \bar{\sigma}}(-\bar{\sigma}\alpha_1 + \alpha_2 + \bar{Z}_\sigma(\eta)\alpha_3), \\ \theta^{\bar{1}} &= \frac{1}{\sigma - \bar{\sigma}}(\sigma\alpha_1 - \alpha_2 - Z_\sigma(\eta)\alpha_3), \quad \theta = e^{\eta}\alpha_3. \end{aligned} \tag{6}$$

The frames $\{Z_\sigma, \bar{Z}_\sigma, T\}$ and $\{\theta^1, \theta^{\bar{1}}, \theta\}$ will be referred to as the *canonical frame* and the *canonical coframe* of the pseudo-Hermitian structure (U, σ, η). Let

m be the function defined by

$$m = \frac{\bar{Z}_\sigma(\sigma) - Z_\sigma(\bar{\sigma})}{\sigma - \bar{\sigma}} - \bar{Z}_\sigma(\eta). \tag{7}$$

Then the Lie brackets of vector fields in the canonical frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ are equal to

$$\begin{cases} [Z_\sigma, \bar{Z}_\sigma] = -mZ_\sigma + \bar{m}\bar{Z}_\sigma - \sqrt{-1}hT \\ [Z_\sigma, T] = pZ_\sigma - q\bar{Z}_\sigma, \\ [\bar{Z}_\sigma, T] = -\bar{q}Z_\sigma + \bar{p}\bar{Z}_\sigma \end{cases} \tag{8}$$

where p and q are the functions defined by

$$\begin{cases} p = (\bar{\sigma}Z_\sigma(v_2(e^{-\eta})) + Z_\sigma(v_1(e^{-\eta})) - T(\sigma))/(\sigma - \bar{\sigma}), \\ q = (\sigma Z_\sigma(v_2(e^{-\eta})) + Z_\sigma(v_1(e^{-\eta})) - T(\sigma))/(\sigma - \bar{\sigma}). \end{cases} \tag{9}$$

Since $\{Z_\sigma, \bar{Z}_\sigma, T\}$ is dual to $\{\theta^1, \theta^{\bar{1}}, \theta\}$, and satisfy (8), the exterior derivatives of the 1-forms in the canonical coframe $\{\theta^1, \theta^{\bar{1}}, \theta\}$ are equal to

$$\begin{cases} d\theta = \sqrt{-1}h\theta^1 \wedge \theta^{\bar{1}} \\ d\theta^1 = -p\theta^1 \wedge \theta + m\theta^1 \wedge \theta^{\bar{1}} - \bar{q}\theta \wedge \theta^{\bar{1}} \\ d\theta^{\bar{1}} = q\theta^1 \wedge \theta - \bar{m}\theta^1 \wedge \theta^{\bar{1}} + \bar{p}\theta \wedge \theta^{\bar{1}}. \end{cases} \tag{10}$$

3. Tanaka-Webster connection

Let (U, σ, η) be a pseudo-Hermitian structure on an open set $U \subset H$. In this section, we give explicit formulas of the pseudo-Hermitian torsion τ (or p-H torsion for short) and the curvature κ of Tanaka-Webster connection (or T-W curvature for short) in terms of the functions σ and η.

Proposition 1. *The p-H torsion τ and the T-W curvature κ are explicitly given by*

$$\begin{aligned} \tau = \frac{-e^{-\eta}}{\sigma - \bar{\sigma}} & (v_2(\eta)v_1(\bar{\sigma}) - v_1(\eta)v_2(\bar{\sigma}) - (v_1(\eta) + \bar{\sigma}v_2(\eta))^2 \\ & + v_1v_1(\eta) + \bar{\sigma}^2 v_2v_2(\eta) + v_3(\bar{\sigma}) + \bar{\sigma}v_4(\eta)) \end{aligned}$$

$$\begin{aligned} \kappa = & -2(\bar{\sigma}v_2(\sigma) - \sigma v_2(\bar{\sigma}))^2/(\sigma - \bar{\sigma})^2 \\ & + (4(\sigma v_2(\bar{\sigma}) - \bar{\sigma}v_2(\sigma))v_1(\eta) + 4(\sigma^2 v_2(\bar{\sigma}) - \bar{\sigma}^2 v_2(\sigma))v_2(\eta) \\ & \quad - v_1(\bar{\sigma})v_2(\sigma) + v_1(\sigma)v_2(\bar{\sigma}) - v_1v_1(\sigma - \bar{\sigma}) \\ & \quad + \sigma^2 v_2v_2(\bar{\sigma}) - \bar{\sigma}^2 v_2v_2(\sigma) - \bar{\sigma}v_4(\sigma) + \sigma v_4(\bar{\sigma}))/(\sigma - \bar{\sigma}) \\ & - (2(v_1v_1 + \sigma\bar{\sigma}v_2v_2)(\eta) + (\sigma + \bar{\sigma})v_4(\eta) \\ & \quad + v_1(\eta)^2 + \sigma\bar{\sigma}v_2(\eta)^2 + (\sigma + \bar{\sigma})v_1(\eta)v_2(\eta) + v_1(\sigma + \bar{\sigma})v_2(\eta)) \end{aligned}$$

where v_1, v_2, and v_3 are the left invariant vector fields on the Heisenberg group H given by (1), *and v_4 denotes the second order operator $v_1 v_2 + v_2 v_1$;*

$$v_4 = \frac{1}{2}\frac{\partial^2}{\partial x \partial y} - x\frac{\partial^2}{\partial x \partial t} + y\frac{\partial^2}{\partial y \partial t} - 2xy\frac{\partial^2}{\partial t^2}.$$

The proposition will be proved in successive subsections.

3.1. Tanaka-Webster connection. There exists a unique 1-form ω such that

$$d\theta^1 = \theta^1 \wedge \omega + \tau\theta \wedge \theta^{\bar{1}}, \quad \omega + \bar{\omega} = h^{-1}\, dh, \tag{11}$$

where h is the real coefficient of the Levi form (see (5)). By using ω, we define a connection ∇ on the complexified tangent bundle $\mathbf{C} \otimes TU$ by

$$\nabla Z_\sigma = \omega \otimes Z_\sigma, \quad \nabla \bar{Z}_\sigma = \bar{\omega} \otimes \bar{Z}_\sigma, \quad \nabla T = 0.$$

The connection ∇ is called the *Tanaka-Webster connection* (or *T-W connection* for short) of the pseudo-Hermitian structure (U, σ, η), and the 1-form ω is called the connection form of ∇ with respect to the canonical frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$. The coefficient τ is called the *pseudo-Hermitian torsion* of the connection ∇. See Tanaka [14], Webster [16], and Dragomir et al. [7], for details of Tanaka-Webster connection.

Let m and p be the functions defined in (7) and (9), respectively. Define a function s by

$$s = v_2(\sigma) + 2Z_\sigma(\eta). \tag{12}$$

Lemma 1. *The connection form of T-W connection of the pseudo-Hermitian structure (U, σ, η) is given by*

$$\omega := s\theta^1 - p\theta + m\theta^{\bar{1}}.$$

Proof. We will show that ω satisfies (11). For the first equation, we have

$$\begin{aligned} \theta^1 \wedge \omega + \tau\theta \wedge \theta^{\bar{1}} &= \theta^1 \wedge (s\theta^1 - p\theta + m\theta^{\bar{1}}) - \bar{q}\theta \wedge \theta^{\bar{1}} \\ &= -p\theta^1 \wedge \theta + m\theta^1 \wedge \theta^{\bar{1}} - \bar{q}\theta \wedge \theta^{\bar{1}}. \end{aligned}$$

In view of (10), we see the first equation holds. For the second, we use the following expression:

$$h^{-1}\, dh = h^{-1}Z_\sigma(h)\theta^1 + h^{-1}\bar{Z}_\sigma(h)\theta^{\bar{1}} + h^{-1}T(h)\theta.$$

It is easy to verify that

$$s + \bar{m} = Z_\sigma(\eta) + \frac{Z_\sigma(\sigma - \bar{\sigma})}{\sigma - \bar{\sigma}} = h^{-1}Z_\sigma(h). \tag{13}$$

By equation (9), it holds that

$$
\begin{aligned}
p+\bar{p} &= \frac{1}{\sigma-\bar{\sigma}}(\bar{\sigma}Z_\sigma(v_2(e^{-\eta}))-\sigma\bar{Z}_\sigma(v_2(e^{-\eta})) \\
&\qquad + Z_\sigma(v_1(e^{-\eta}))-\bar{Z}_\sigma(v_1(e^{-\eta}))-T(\sigma-\bar{\sigma})) \\
&= \frac{1}{\sigma-\bar{\sigma}}(\bar{\sigma}e^{-\eta}(v_1(\eta)v_2(\eta)-v_1v_2(\eta))+\bar{\sigma}\sigma e^{-\eta}(v_2(\eta)v_2(\eta)-v_2v_2(\eta)) \\
&\qquad -\sigma e^{-\eta}(v_1(\eta)v_2(\eta)-v_1v_2(\eta))-\bar{\sigma}\sigma e^{-\eta}(v_2(\eta)v_2(\eta)-v_2v_2(\eta)) \\
&\qquad + e^{-\eta}(v_1(\eta)^2-v_1v_1(\eta))+\sigma e^{-\eta}(v_2(\eta)v_1(\eta)-v_2v_1(\eta)) \\
&\qquad - e^{-\eta}(v_1(\eta)^2-v_1v_1(\eta))-\bar{\sigma}e^{-\eta}(v_2(\eta)v_1(\eta)-v_2v_1(\eta)))-\frac{T(\sigma-\bar{\sigma})}{\sigma-\bar{\sigma}} \\
&= -e^{-\eta}v_3(\eta)-\frac{T(\sigma-\bar{\sigma})}{\sigma-\bar{\sigma}}.
\end{aligned}
$$

Since equation (2) implies $T(\eta)=e^{-\eta}v_3(\eta)$, we have

$$p+\bar{p}=-T(\eta)-\frac{T(\sigma-\bar{\sigma})}{\sigma-\bar{\sigma}}=-h^{-1}T(h). \tag{14}$$

From (13) and (14), it follows that

$$\omega+\bar{\omega}=h^{-1}\,dh.$$

This completes the proof. □

3.2. Pseudo-Hermitian torsion. Let $\{Z_\sigma,\bar{Z}_\sigma,T\}$ and $\{\theta^1,\theta^{\bar{1}},\theta\}$ be the canonical frame and coframe of (U,σ,η) defined in (4) and (6), respectively.

LEMMA 2. *The p-H torsion τ of the pseudo-Hermitian structure (U,σ,η) is equal to $-\bar{q}$, that is,*

$$\tau=\frac{-1}{\sigma-\bar{\sigma}}(T(\bar{\sigma})-\bar{\sigma}\bar{Z}_\sigma(v_2(e^{-\eta}))-\bar{Z}_\sigma(v_1(e^{-\eta}))). \tag{15}$$

Proof. Let Tor be the torsion of the T-W connection;

$$\mathrm{Tor}(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y].$$

Then the p-H torsion τ is defined by the following equation:

$$\mathrm{Tor}(T,Z_\sigma)=\bar{\tau}\bar{Z}_\sigma.$$

From equations (8) and (10), we have

$$
\begin{aligned}
\mathrm{Tor}(T,Z_\sigma) &= \nabla_T Z-\nabla_{Z_\sigma}T-[T,Z_\sigma] \\
&= \omega(T)Z_\sigma+(pZ_\sigma-q\bar{Z}_\sigma) \\
&= -q\bar{Z}_\sigma.
\end{aligned}
$$

Therefore the p-H torsion τ is equal to $-\bar{q}$. □

3.3. Tanaka-Webster curvature. By definition, the *T-W curvature* κ is the coefficient of $\theta^1 \wedge \theta^{\bar{1}}$-component in the exterior derivative $d\omega$ of the connection form ω.

LEMMA 3. *The T-W curvature κ of a pseudo-Hermitian structure (U, σ, η) is given by*

$$\kappa = -\bar{Z}_\sigma(s) + Z_\sigma(m) + m(s - \bar{m}) - \sqrt{-1}hp. \tag{16}$$

Proof. The exterior derivative $d\omega$ of the connection form ω obtained in Lemma 1 is equal to $d\omega := ds \wedge \theta^1 - dp \wedge \theta + dm \wedge \theta^{\bar{1}}$, where $ds = Z_\sigma(s)\theta^1 + \bar{Z}_\sigma(s)\theta^{\bar{1}} + T(s)\theta$, and dp and dm are similar. Therefore the exterior derivative $d\omega$ is equal to

$$\begin{aligned} d\omega = &-(T(s) + ps + Z_\sigma(p))\theta^1 \wedge \theta \\ &+ (-\bar{Z}_\sigma(s) + Z_\sigma(m) + m(s - \bar{m}) - \sqrt{-1}hp)\theta^1 \wedge \theta^{\bar{1}} \\ &+ (\bar{Z}_\sigma(p) + T(m) + \bar{p}m)\theta \wedge \theta^{\bar{1}} \end{aligned}$$

Thus we find the T-W curvature is as given in (16). □

Proof. (Proposition 1) By a simple calculation, we verify that τ and κ obtained in Lemmas 2 and 3 is equal to the formula in the proposition. □

Let w and w' be the coefficients of $\theta^1 \wedge \theta$- and $\theta \wedge \theta^{\bar{1}}$-components in $d\omega$, respectively;

$$\begin{cases} w = -T(s) - ps - Z_\sigma(p), \\ w' = \bar{Z}_\sigma(p) + T(m) + \bar{p}m. \end{cases} \tag{17}$$

We will use the following lemma in Subsection 4.2:

LEMMA 4. *If the p-H torsion τ vanishes, then $w = w' = 0$.*

Proof. Suppose $\tau = 0$. Then we have $d\theta^1 = \theta^1 \wedge \omega$, and thus

$$0 = d^2\theta^1 = d(\theta^1 \wedge \omega) = -\theta^1 \wedge d\omega,$$

which implies $w' = 0$. Since $d\theta^{\bar{1}} = \theta^{\bar{1}} \wedge \bar{\omega} = \theta^{\bar{1}} \wedge (h^{-1}\,dh - \omega)$, we have

$$0 = d^2\theta^{\bar{1}} = d(\theta^{\bar{1}} \wedge \bar{\omega}) = -\theta^{\bar{1}} \wedge d\bar{\omega} = \theta^{\bar{1}} \wedge d\omega,$$

which implies $w = 0$. □

Remark 1. It is well known that there exists a unique contact Riemannian structure on a pseudo-Hermitian structure. If the pseudo-Hermitian torsion vanishes, then the resulting contact Riemannian structure is called a *Sasakian structure*. In this sense a pseudo-Hermitian structure with $\tau = \kappa = 0$ is nothing but a flat Sasakian structure (see for instance Dragomir et al. [7]).

By Proposition 1, the p-H torsion and the T-W curvature of the left invariant pseudo-Hermitian structure $(H, -\sqrt{-1}, 0)$ vanish, because $\sigma = -\sqrt{-1}$ and $\eta = 0$ are constant. We regard $(H, -\sqrt{-1}, 0)$ as the *standard model* of a pseudo-Hermitian structure and also a flat Sasakian structure.

4. Fundamental system of equations

Let (U, σ, η) be a pseudo-Hermitian structure on an open set $U \subset H$. Throughout this section, we fix the canonical frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ of (U, σ, η) defined in (4). The following system of equations (F) will be called the *fundamental system of equations* of the pseudo-Hermitian structure (U, σ, η):

$$\text{(F)} \qquad \begin{cases} 0 = \bar{Z}_\sigma(f) \\ 0 = Z_\sigma^2(f) - (v_2(\sigma) + 2Z_\sigma(\eta))Z_\sigma(f). \end{cases}$$

If the dimension of the solution space of the system of equations is maximal, then we say that it is *integrable*. As we will see shortly, the complex dimension of the solution space of (F) is at most 3. We will prove that, if the pseudo-Hermitian structure has the vanishing p-H torsion and the vanishing T-W curvature, the system (F) is integrable.

4.1. Matrix form. In order to investigate the integrability of (F) and to apply Proposition 2, we convert (F) into a matrix form. Let h, m, p, and s be the functions defined in (5), (7), (9), and (12), respectively.

Lemma 5. *Suppose (U, σ, η) is a pseudo-Hermitian structure with $\tau = \kappa = 0$. Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be matrices defined by*

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & \sqrt{-1}h \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let $\tilde{f}$ denote the column vector $(f\ Z_\sigma(f)\ T(f))^t$. Then the system of equations (F) *is equivalent to the following system of equations*:

$$\text{(18)} \qquad Z_\sigma(\tilde{f}) = \mathscr{A}\tilde{f}, \quad \bar{Z}_\sigma(\tilde{f}) = \mathscr{B}\tilde{f}, \quad T(\tilde{f}) = \mathscr{C}\tilde{f},$$

Proof. Let (U, σ, η) be a pseudo-Hermitian structure with p-H torsion $\tau = 0$ on an open set $U \subset H$. The condition $\tau = 0$ implies that the Lie bracket $[Z_\sigma, T]$ is equal to

$$[Z_\sigma, T] = pZ_\sigma$$

(see equation (8) and Lemma 2). Let f be a solution of (F). By differentiating f by $[Z_\sigma, \bar{Z}_\sigma]$ in (8), we have

$$\text{(19)} \qquad \bar{Z}_\sigma Z_\sigma(f) = mZ_\sigma(f) + \sqrt{-1}hT(f),$$

which we differentiate by Z_σ and $\bar{Z}_\sigma$ to obtain

$$Z_\sigma \bar{Z}_\sigma Z_\sigma(f) = (Z_\sigma(m) + ms)Z_\sigma(f) + \sqrt{-1}Z_\sigma(h)T(f) + \sqrt{-1}hZ_\sigma T(f).$$

We differentiate the second equation of (F) by $\bar{Z}_\sigma$ to obtain

$$\bar{Z}_\sigma Z_\sigma Z_\sigma(f) = (\bar{Z}_\sigma(s) + ms)Z_\sigma(f) + \sqrt{-1}hsT(f).$$

Subtracting the above two equations, we obtain

$$[Z_\sigma, \bar{Z}_\sigma]Z_\sigma(f) = (Z_\sigma(m) - \bar{Z}_\sigma(s))Z_\sigma(f) + \sqrt{-1}(Z_\sigma(h) - hs)T(f) + \sqrt{-1}hZ_\sigma T(f).$$

On the other hand, we have, by differentiating $Z_\sigma(f)$ by the vector field $[Z_\sigma, \bar{Z}_\sigma]$ in view of (8),

$$[Z_\sigma, \bar{Z}_\sigma]Z_\sigma(f) = -m(s - \bar{m})Z_\sigma(f) + \sqrt{-1}h\bar{m}T(f) - \sqrt{-1}hTZ_\sigma(f).$$

If $\kappa = 0$, the above two expressions of $[Z_\sigma, \bar{Z}_\sigma]Z_\sigma(f)$ yield

$$Z_\sigma T(f) + TZ_\sigma(f) = -pZ_\sigma(f), \tag{20}$$

where we used the identity $Z_\sigma(h) = h(s + \bar{m})$. By differentiating f by the vector field $[Z_\sigma, T]$, we have $Z_\sigma T(f) - TZ_\sigma(f) = pZ_\sigma(f)$. Thus we get

$$Z_\sigma T(f) = 0, \quad TZ_\sigma(f) = -pZ_\sigma(f). \tag{21}$$

The hypothesis $\tau = 0$ implies $[\bar{Z}_\sigma, T] = \bar{p}\bar{Z}_\sigma$, and thus, since f is a solution of (F), we have $[\bar{Z}_\sigma, T](f) = \bar{p}\bar{Z}_\sigma(f) = 0$ and $T\bar{Z}_\sigma(f) = 0$. Therefore we obtain

$$\bar{Z}_\sigma T(f) = 0. \tag{22}$$

By differentiating $T(f)$ by $[Z_\sigma, \bar{Z}_\sigma]$, we also find

$$T^2(f) = 0.$$

The equations so obtained as above

$$\bar{Z}_\sigma Z_\sigma(f) = mZ_\sigma(f) + \sqrt{-1}hT(f),$$
$$Z_\sigma T(f) = \bar{Z}_\sigma T(f) = TZ_\sigma(f) + pZ_\sigma(f) = T^2(f) = 0$$

imply that any solution f of (F) satisfies equations (18). The inverse implication is obvious. □

4.2. Integrability. Now we prove the following:

THEOREM 1. *If a pseudo-Hermitian structure* (U, σ, η) *has vanishing p-H torsion and vanishing T-W curvature, then the system of equations* (F) *for* (U, σ, η) *is integrable.*

Proof. From Proposition 2, it suffices to show the following three equalities

$$Z_\sigma(\mathscr{B}) - \bar{Z}_\sigma(\mathscr{A}) - [\mathscr{A},\mathscr{B}] = -m\mathscr{A} + \bar{m}\mathscr{B} - \sqrt{-1}h\mathscr{C} \tag{23}$$

$$Z_\sigma(\mathscr{C}) - T(\mathscr{A}) - [\mathscr{A},\mathscr{C}] = p\mathscr{A} \tag{24}$$

$$\bar{Z}_\sigma(\mathscr{C}) - T(\mathscr{B}) - [\mathscr{B},\mathscr{C}] = \bar{p}\mathscr{B}. \tag{25}$$

The subtraction of both sides of (23) has only two non-trivial entries. Thus to verify (23), it suffices to show

$$0 = Z_\sigma(m) - \bar{Z}_\sigma(s) + m(s-\bar{m}) - \sqrt{-1}hp \tag{26}$$

$$0 = Z_\sigma(h) - h(s+\bar{m}) \tag{27}$$

that are the $(2,2)$-component and the $(2,3)$-component. The equation (26) is equivalent to $\kappa = 0$, and (27) is an identity. Thus (23) is verified. For equation (24), the subtraction of both sides has only one non-trivial entry, and it suffices to show

$$0 = -Z_\sigma(p) - T(s) - ps \tag{28}$$

that are the $(2,2)$-component. The right hand side of (28) is equal to the coefficient w in (17), which vanishes, because we are supposing that the p-H torsion τ and the T-W curvature κ of (U,σ,η) are equal to 0 (see Lemma 4). Thus (24) is verified. Again the subtraction of both sides of (25) has two non-trivial entries. Here it remains to show

$$0 = \bar{Z}_\sigma(p) + T(m) + \bar{p}m \tag{29}$$

$$0 = -T(h) - h(p+\bar{p}) \tag{30}$$

that are the $(2,2)$-component and the $(2,3)$-component. The right hand side of (29) is equal to the coefficient w' in (17), which vanishes, because $\tau = 0$ and $\kappa = 0$, and (30) holds identically, as shown in (14). This completes the proof. □

5. Construction of pseudo-Hermitian map

We introduce a Hermitian inner product on the solution space of the fundamental system of equation (F) of a pseudo-Hermitian structure (U,σ,η). By using the Hermitian inner product, we will construct a pseudo-Hermitian map from (U,σ,η) into the standard pseudo-Hermitian structure $(U,-\sqrt{-1},0)$, provided the p-H torsion τ and the T-W curvature κ of the pseudo-Hermitian structure (U,σ,η) vanish. We continue using the canonical frame $(Z_\sigma,\bar{Z}_\sigma,T)$ of (U,σ,η) defined in (4).

5.1. Hermitian inner product on solution space. Define an inner product on the solution space of (F) by

$$\langle f,g\rangle = -\sqrt{-1}(fT(\bar{g}) - T(f)\bar{g}) + h^{-1}Z_\sigma(f)\bar{Z}_\sigma(\bar{g}). \tag{31}$$

LEMMA 6. *The above inner product* $\langle f, g\rangle$ *is constant for any solutions* f *and* g *of the fundamental system of equations* (F) *of* (U, σ, η).

Proof. Let f and g be solutions of (F). Then the differential of $\langle f, g\rangle$ by Z_σ is equal to

$$Z_\sigma(\langle f, g\rangle) = -\sqrt{-1}(Z_\sigma(f)T(\bar{g}) + fZ_\sigma T(\bar{g}) - Z_\sigma T(f)\bar{g} - T(f)Z_\sigma(\bar{g})) - h^{-2}Z_\sigma(h)Z_\sigma(f)\bar{Z}_\sigma(\bar{g}) + h^{-1}sZ_\sigma(f)\bar{Z}_\sigma(\bar{g}) + h^{-1}Z_\sigma(f)Z_\sigma\bar{Z}_\sigma(\bar{g}).$$

By equations (19), (21) and (22), we have $\bar{Z}_\sigma Z_\sigma(f) = mZ_\sigma(f) + \sqrt{-1}hT(f)$, $Z_\sigma T(f) = 0$ and $Z_\sigma T(f) = 0$. Hence

$$Z_\sigma(\langle f, g\rangle) = -h^{-2}(Z_\sigma(h) - h(s + \bar{m}))Z_\sigma(f)\bar{Z}_\sigma(\bar{g}).$$

Since it identically holds that $s + \bar{m} = h^{-1}Z_\sigma(h)$, we get $Z_\sigma(\langle f, g\rangle) = 0$. The inner product satisfies $\overline{\langle f, g\rangle} = \langle g, f\rangle$. Thus, by taking the complex conjugate of $Z_\sigma(\langle f, g\rangle) = 0$, we also get $\bar{Z}_\sigma(\langle f, g\rangle) = 0$ for any solutions f and g of (F). Any function f that satisfies $Z_\sigma(f) = \bar{Z}_\sigma(f) = 0$ must be constant. Therefore $\langle f, g\rangle$ must be constant. □

For each function f, we denote by $\tilde{f}$ the column vector $(fZ_\sigma(f)T(f))^t$. By using a Hermitian matrix $\mathscr{H}$ defined by

$$\mathscr{H} := \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & h^{-1} & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix},$$

we may write

$$\langle f, g\rangle = \tilde{f}^t \mathscr{H} \bar{\tilde{g}}.$$

We will say a basis $\{f_1, f_2, f_3\}$ of the solution space of (F) is *normal* if the inner products satisfy

$$(\langle f_i, f_j\rangle)_{i,j} = \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}. \tag{32}$$

5.2. Construction. By using a normal basis, we construct, in the following theorem, a pseudo-Hermitian map. Let $\Re(f)$ denote the real part of f.

THEOREM 2. *Suppose that a pseudo-Hermitian structure* (U, σ, η) *on an open set* U *in the Heisenberg group* H *has* $\tau = 0$ *and* $\kappa = 0$. *Then there exists a normal basis* $\{f_1, f_2, f_3\}$ *of the solution space of* (F) *with* f_1 *a constant function equal to* $\frac{1}{2}$. *For such solutions* f_2 *and* f_3, *the map* $\Phi := (T(f_3)^{-1/2}f_2, T(f_3)^{-1}\Re(f_3)) : (U, \sigma, \eta) \to (H, -\sqrt{-1}, 0)$ *is a pseudo-Hermitian map.*

Proof. Solutions f of the system (F) are uniquely determined by the initial value $\tilde{f}(p) = (f(p) Z_\sigma(f)(p) T(f)(p))^t$ at an arbitrarily chosen point $p \in U$. Since (U, σ, η) has $\tau = 0$ and $\kappa = 0$, the solution space of (F) is of dimension 3 (see Theorem 1 and Corollary 3). Thus for any linear basis $\{f_1, f_2, f_3\}$ of solutions of (F), the initial values of these solutions at p form a regular three-by-three matrix $(\tilde{f}_1(p)\ \tilde{f}_2(p)\ \tilde{f}_3(p))$. Therefore we may choose the solutions f_1, f_2, and f_3 so that their initial values at p have the form

$$(\tilde{f}_1(p)\ \tilde{f}_2(p)\ \tilde{f}_3(p)) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & \sqrt{h(p)} & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{33}$$

Notice that the function f_1 is a constant function equal to $1/2$. From Lemma 6, it follows that the inner products of those functions satisfy equation (32), that is, $\{f_1, f_2, f_3\}$ is a normal basis. We denote by I the matrix on the right-hand side of (32); $(\langle f_i, \overline{f_j} \rangle)_{i,j} = I$. Now consider the matrix $\mathscr{F} := (\tilde{f}_1\ \tilde{f}_2\ \tilde{f}_3)$ for these solutions. Then we have $\mathscr{F}^t \mathscr{H} \overline{\mathscr{F}} = I$, and thus

$$\mathscr{F} \bar{I} \overline{\mathscr{F}}^t = \mathscr{F} \bar{I} (\mathscr{H}^{-1} (\mathscr{F}^t)^{-1} I)^t = \mathscr{F} I^2 \mathscr{F}^{-1} (\mathscr{H}^{-1})^t = (\mathscr{H}^{-1})^t. \tag{34}$$

Among the componentwise equations, we use the following

$$\begin{aligned}
&(1,1)\text{-component} : -\Im(f_3) + |f_2|^2 = 0 \\
&(1,2)\text{-component} : -\frac{\sqrt{-1}}{2} \bar{Z}_\sigma(\bar{f}_3) + f_2 \bar{Z}_\sigma(\bar{f}_2) = 0 \\
&(1,3)\text{-component} : -\frac{\sqrt{-1}}{2} T(\bar{f}_3) + f_2 T(\bar{f}_2) = -\sqrt{-1} \\
&(3,3)\text{-component} : |T(f_2)|^2 = 0.
\end{aligned}$$

By using those solutions f_2 and f_3, we define a map $\Phi : U \to H$ by

$$\Phi(z,t) := (f_2(z,t), \Re(f_3(z,t))). \tag{35}$$

and will show that Φ is a pseudo-Hermitian map from (U, σ, η) to the standard model $(H, -\sqrt{-1}, 0)$. We will prove this in two steps.

Step 1. We show that *the map* Φ *in* (35) *satisfies* $\Phi^*(\alpha_3) = e^\eta \alpha_3$.

Let $\{\theta^1, \theta^{\bar{1}}, \theta\}$ be the dual coframe to the frame $\{Z_\sigma, \bar{Z}_\sigma, T\}$ of the complexified tangent space $\mathbf{C} \otimes TH$ of the Heisenberg group H. Then the exterior derivative of any function f is expressed as $df = Z_\sigma(f)\theta^1 + \bar{Z}_\sigma(f)\theta^{\bar{1}} + T(f)\theta$. Putting $w = f_2(z,t)$ and $u = \Re(f_3(z,t)) = \frac{1}{2}(f_3(z,t) + \overline{f_3(z,t)})$, we may calculate the pullback $\Phi^*(\alpha_3) = du + \sqrt{-1}(w\, d\bar{w} - \bar{w}\, dw)$ of the contact form $\alpha_3 =$

$dt + \sqrt{-1}(z\,d\bar{z} - \bar{z}\,dz)$ and find that the θ, θ^1, and $\theta^{\bar{1}}$-components of $du + \sqrt{-1}(w\,d\overline{w} - \overline{w}\,dw)$ are respectively as follows:

$$\theta\text{-component}: \frac{1}{2}T(f_3 + \bar{f}_3) + \sqrt{-1}(f_2 T(\bar{f}_2) - \bar{f}_2 T(f_2))$$

$$\theta^1\text{-component}: \frac{1}{2}Z_\sigma(f_3 + \bar{f}_3) + \sqrt{-1}(f_2 Z_\sigma(\bar{f}_2) - \bar{f}_2 Z_\sigma(f_2))$$

$$\theta^{\bar{1}}\text{-component}: \frac{1}{2}\bar{Z}_\sigma(f_3 + \bar{f}_3) + \sqrt{-1}(f_2 \bar{Z}_\sigma(\bar{f}_2) - \bar{f}_2 \bar{Z}_\sigma(f_2)).$$

Since θ equals $e^\eta \alpha_3$, it suffices, for the proof of Step 1, to show that the θ^1-component and the $\theta^{\bar{1}}$-component vanish, and that the θ-component is equal to 1. Since the functions f_2 and f_3 are solutions of the system (F), they satisfy $\bar{Z}_\sigma(f_i) = 0 = Z_\sigma(\bar{f}_i)$. From the $(1,2)$-component of (34), we deduce that the θ^1- and the $\theta^{\bar{1}}$-components are equal to 0, that is, Φ is a contact map. From the $(1,3)$ and the $(3,3)$-component of (34), it follows that $T(f_2) = 0$ and $T(\bar{f}_3) = 2$. Thus we find that the θ-component is equal to 1. Therefore Φ satisfies $\Phi^*(\alpha_3) = e^\eta \alpha_3$.

Step 2. We show that *the map* $\Phi : (U, \sigma) \to (H, \sigma_0)$ *is a CR map.*

Since the functions f_2 and f_3 satisfy the equation $\bar{Z}_\sigma(f_i) = 0$, the map $\varphi = (f_2, f_3) : U \to \mathbf{C}^2$ is J_σ-holomorphic, namely φ satisfies $\varphi_*(J_\sigma(v)) = \sqrt{-1}\varphi_*(v)$ for all contact element v of U. By the $(1,1)$-component of (34), the image of φ is contained in the real hypersurface $M = \{(z_1, z_2) \in \mathbf{C}^2; \Im(z_2) = |z_1|^2\}$. On the other hand, for the standard CR structure σ_0, the map $\varphi_0 : H \to \mathbf{C}^2$ defined by $\varphi_0(z, t) = (z, t + \sqrt{-1}|z|^2)$ is J_{σ_0}-holomorphic, and is a bijection between H and the hypersurface M. Since Φ is equal to the composition $(\varphi_0)^{-1} \circ \varphi$, it commutes with J_σ and J_{σ_0}; $J_{\sigma_0}(\Phi_*(v)) = \Phi_*(J_\sigma(v))$ for all contact element v. This proves the claim of Step 2, and completes the proof. □

From Theorem 2, we directly deduce the following result (see, for example, Blair et al. [1] and Cho et al. [6]).

Corollary 1. *For a pseudo-Hermitian structure* (U, σ, η), *there exists a pseudo-Hermitian map from* (U, σ, η) *to the standard model* $(H, -\sqrt{-1}, 0)$ *if and only if* (U, σ, η) *has* $\tau = 0$ *and* $\kappa = 0$.

Corollary 2. *Any two* 3*-dimensional flat Sasakian structures are locally equivalent by a diffeomorphism which preserves the underlining contact structures.*

6. Appendix

In the proof of the integrability of (F) in §4, we used a basic fact on existence and uniqueness of solutions of a system of linear partial differential equations. Although it follows familiar lines, we give a proof here.

Let M be a connected and simply connected manifold of dimension n whose tangent bundle is trivial. Let $\{v_1, \dots, v_n\}$ be a frame field of the complexified tangent space $\mathbf{C} \otimes TM$, and γ_{ij}^k the functions defined by

$$[v_i, v_j] = \sum_{k=1}^{n} \gamma_{ij}^k v_k.$$

Consider the following system of linear partial differential equations:

$$v_i(f) = S_i f \quad \text{for } i = 1, \dots, n, \tag{36}$$

where the unknown function is a vector-valued function $f : M \to \mathbf{C}^m$, and S_i are square matrices of order m whose entries are smooth functions on U. Denote by $[S_i, S_j]_a^b$ the (a, b)-component of the commutator matrix $[S_i, S_j] = S_i S_j - S_j S_i$ for $a, b = 1, \dots, m$.

Proposition 2. *Suppose the matrices S_i satisfy*

$$v_i(S_j) - v_j(S_i) + [S_j, S_i] = \sum_{k=1}^{n} \gamma_{ij}^k S_k \quad \textit{for all } i, j = 1, \dots, n. \tag{37}$$

Then, for each $(p, y) \in M \times \mathbf{C}^m$, the initial value problem $f(p) = y$ of the system (36) *has a unique solution f on entire M. Therefore there exists a one-to-one correspondence between the solution space of the system* (36) *and the set of initial values $\{f(p) \in \mathbf{C}^m\}$ through the initial value problem.*

Proof. Let $(y_1, \dots, y_m)$ be the canonical coordinate system of $\mathbf{C}^m$, and s_{ia}^b the (a, b)-component of the matrix S_i; smooth functions on M. Define the vector fields $\tilde{v}_i$ on $M \times \mathbf{C}^m$ by

$$\tilde{v}_i = \left(v_i, \sum_{a,b=1}^{m} s_{ia}^b y^a \frac{\partial}{\partial y^b} \right) \in TM \oplus T\mathbf{C}^m = T(M \times \mathbf{C}^m) \quad \text{for } i = 1, \dots, n.$$

A function f satisfies equation $v_i(f) = S_i f$, if and only if the graph of f is tangent to the vector field $\tilde{v}_i$. Therefore, if the vector fields $\tilde{v}_i$ satisfy the Frobenius condition

$$[\tilde{v}_i, \tilde{v}_j] \equiv 0 \mod \tilde{v}_1, \dots, \tilde{v}_n \quad \text{for all } i, j = 1, \dots, n, \tag{38}$$

then the integral manifolds of the plane field spanned by the vector fields $\tilde{v}_i$ are graphs of solutions of the system (36). Now we calculate the Lie bracket $[\tilde{v}_i, \tilde{v}_j]$;

$$[\tilde{v}_i, \tilde{v}_j] = [v_i, v_j] + \sum_{a,b=1}^{m} (v_i(s_{ja}^b) - v_j(s_{ia}^b)) y^a \frac{\partial}{\partial y^b} + \sum_{a,b,c=1}^{m} (s_{jb}^c s_{ia}^b - s_{ib}^c s_{ja}^b) y^a \frac{\partial}{\partial y^c}$$

$$= \sum_{k=1}^{n} \gamma_{ij}^k v_k + \sum_{a,b=1}^{m} (v_i(s_{ja}^b) - v_j(s_{ia}^b) + [S_j, S_i]_a^b) y^a \frac{\partial}{\partial y^b}.$$

This shows that, if the matrices S_i satisfy the condition (37), we have

$$[\tilde{v}_i, \tilde{v}_j] = \sum_{k=1}^{m} \gamma_{ij}^k \tilde{v}_k$$

and thus find that the vector fields $\tilde{v}_i$ satisfy the Frobenius condition (38). The n-dimensional plane field spanned by $\tilde{v}_i$ is integrable if and only if they satisfy the Frobenius condition. On the other hand, the vector fields $\tilde{v}_i$ are projected down to the vector fields v_i by the projection $\pi : M \times \mathbf{C}^m \to M$ to the first factor. Therefore each integral manifold of the plane field spanned by the vector fields $\tilde{v}_i$ is projected down to M by the projection π, and is the graph of a solution of the system (36). For each point $(p, y) \in M \times \mathbf{C}^m$, there exists a unique integral manifold through (p, y), which means the existence and uniqueness of the initial value problem of the system (36). □

The above proof also shows the following:

Corollary 3. *The maximal dimension of the solution space of* (36) *is equal to* m.

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