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ON COMPLETE SPACELIKE SUBMANIFOLDS IN SEMI-RIEMANNIAN SPACE FORMS WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR

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Abstract

In this paper, by modifying Cheng-Yau's technique to complete spacelike submanifolds in $Q_p^{n+p}(c)$, we prove a rigidity theorem for complete spacelike submanifolds in the de Sitter space with parallel normalized mean curvature vector. As a corollary, we have the Corollary 1.1 of [7].

1. Introduction

Let $Q_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of index p and of constant curvature c, which is called an *indefinite space form of index p*. If $c > 0$, we call it the *De Sitter space of index p* and denote it by $S_p^{n+p}(c)$. If $c < 0$, we call it the *semi-Hyperbolic space of index p* and denote it by $H_p^{n+p}(c)$. A smooth immersion $\varphi : M^n \to Q_p^{n+p}(c)$ of an n dimensional connected manifold M^n is said to be a *spacelike* if the induced metric via φ is a Riemannian metric on M^n. As is usual, the spacelike submanifold is said to be complete if the Riemannian induced metric is a complete metric on M^n.

The interest in the study of spacelike hypersurfaces immersed in the de Sitter space is motivated by their nice Bernstein-type properties. It was proved by E. Calabi [5] (for $n \le 4$) and by S. Y. Cheng and S. T. Yau [15] (for all n) that a complete maximal spacelike hypersurface in L^{n+2} is totally geodesic. In [22], S. Nishikawa obtained similar results for others Lorentzian manifolds. In particular, he proved that a complete maximal spacelike hypersurface in $S_1^{n+1}(1)$ is totally geodesic.

Goddard [16] conjectured that a complete spacelike hypersurface with constant mean curvature in de Sitter S_1^{n+1} should be umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under

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appropriate additional hypotheses. For instance, in 1987 Akutagawa [2] proved the Goddard conjecture when $H^2 < 1$ if $n = 2$ and $H^2 < \frac{4(n-1)}{n^2}$ if $n > 2$. He also showed that when $n = 2$, for any constant $H^2 > c^2$ there exists a non-umbilical surface of mean curvature H in the de Sitter space $S_1^3(c)$ of constant curvature $c > 0$. One year later S. Montiel [20] solved Goddard's problem in the compact case in S_1^{n+1} without restriction over the range of H. He also gave examples of non-umbilical complete spacelike hypersurfaces in S_1^{n+1} with constant H satisying $H^2 \geq \frac{4(n-1)}{n^2}$ if $n > 2$, including the so-called hyperbolic cylinders. In [21], Montiel proved that the only complete spacelike hypersurface in S_1^{n+1} with constant $H = \frac{2\sqrt{n-1}}{n}$ with more than one topological end is a hyperbolic cylinder. At the same time, the complete hypersurfaces in the de Sitter space have been characterized by Cheng [9] under the hypothesis of the mean curvature and the scalar curvature being linearly related.

In order to study spacelike hypersurfaces with constant scalar curvature in de Sitter space, Y. Zheng [29] proved that a compact spacelike hypersurface in $S_1^{n+1}(1)$ with constant normalized scalar curvature r, $r < 1$ and non-negative sectional curvatures is totally umbilical. Later, Q. M. Cheng and S. Ishikawa [11] showed that Zhengs result in [29] is also true without additional assumptions on the sectional curvatures of the hypersurface. In [19], H. Li proposed the following problem: Let M^n be a complete spacelike hypersurface in $S_1^{n+1}(1)$, $n \geq 3$, with constant normalized scalar curvature r satisfying $\frac{n-2}{n} \leq r \leq 1$. Is M^n totally umbilical? A. Caminha [8] answered that question affirmatively under the additional condition that the supremum of H is attained on M^n. Recently, Camargo-Chaves-Sousa [6] showed that Li's question is also true if the mean curvature is bounded.

In higher codimension, the condition on the mean curvature is replaced by a condition on the mean curvature vector. Let $Q_p^{n+p}(c)$ be the complete connected semi-Riemannian manifolds of index p with constant curvature c and M^n be a spacelike submanifold of $Q_p^{n+p}(c)$ with parallel mean curvature vector h. When M^n is maximal, i.e., $h \equiv 0$, T. Ishihara [17] established a inequality for the squared norm $|B|^2$ of the second fundamental form B of M^n: $\frac{1}{2}\triangle|B|^2 \geq |B|^2(nc + |B|^2/2)$. As an important application, Ishihara proved that maximal complete spacelike submanifolds in $Q_p^{n+p}(c)$, $c \geq 0$, are totally umbilical and, if $c < 0$, then $0 \leq |B|^2 \leq -npc$. Moreover, he determined all the complete spacelike maximal submanifolds M^n of $Q_p^{n+p}(c)$, $c < 0$, satisfying $|B|^2 = -npc$. R. Aiyama [1] studied compact spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. She also proved that compact spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector and non-negative sectional curvatures are also totally umbilical. Q. M. Cheng [10] showed that Akutagawa's result [2] is valid for complete spacelike submanifolds in $S_p^{n+p}(c)$ with parallel mean curvature vector.

In [12] and [13], Chaves-Sousa obtained a Simon type formula for the squared norm of the traceless tensor $\phi = B - Hg$, where g stands for the induced metric on a spacelike submanifold in $Q_p^{n+p}(c)$ with parallel mean curvature vector. As an application of this formula, Brasil-Chaves-Mariano [3] obtained an other limitation for the supremum of the mean curvature $\sup H^2 < \dfrac{4(n-1)c}{(n-2)^2 p + 4(n-1)}$ as an extension of results of [2] and [10].

Recently, Camargo-Chaves-Sousa [7] considered complete spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector (which is much weaker than the condition to have parallel mean curvature vector) and obtained

THEOREM 1.1. *Let M^n be a complete spacelike submanifold in $Q_p^{n+p}(c)$, $n \geq 3$, with parallel normalized mean curvature vector and constant normalized scalar curvature r satisfying $r \leq c$. If the mean curvature H of M^n satisfies*

$$\sup H^2 < \frac{4(n-1)c}{(n-2)^2 p + 4(n-1)},$$

then M^n is totally umbilical.

In this paper, in order to improve Theorem 1.1, we modify Cheng-Yau's technique to complete spacelike submanifold in $Q_p^{n+p}(c)$ and prove a rigidity theorem under the hypothesis of the mean curvature and the normalized scalar curvature being linearly related. More precisely, we have

THEOREM 1.2. *Let M^n be a complete spacelike submanifold in $Q_p^{n+p}(c)$, $n \geq 3$ with parallel normalized mean curvature vector. If $r = aH + b$, $a, b \in \mathbf{R}$, $a \geq 0$, $(n-1)a^2 + 4n(c-b) \geq 0$ and the mean curvature H of M^n satisfies*

$$\sup H^2 < \frac{4(n-1)c}{(n-2)^2 p + 4(n-1)},$$

then M^n is totally umbilical.

Remark 1.3. *If we choose $a = 0$ and $b \leq c$ in Theorem* 1.2, *we obtain the Theorem* 1.1.

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2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold immersed in $Q_p^{n+p}(c)$. For any $p \in M$, we choose a local orthonormal frame $e_1, \ldots, e_{n+p}$ in $Q_p^{n+p}(c)$

around p such that $e_1, \ldots, e_n$ are tangent to M^n. Take the corresponding dual coframe $\omega_1, \ldots, \omega_{n+p}$. We use the following standard convention for indices:

$$1 \le A, B, C, \ldots \le n+p, \quad 1 \le i, j, k, \ldots \le n, \quad n+1 \le \alpha, \beta, \gamma, \ldots \le n+p.$$

Let $\varepsilon_i = 1$, $\varepsilon_\alpha = -1$, then the structure equations of $Q_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D R_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$R_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting those forms to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \le \alpha \le n+p. \tag{2.4}$$

So the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$, from Cartan lemma, we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.5}$$

Let $B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$ be the second fundamental form. We will denote by $h = \frac{1}{n} \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha$ and by $H = |h| = \frac{1}{n}\sqrt{\sum_\alpha (\sum_i h_{ii}^\alpha)^2}$ the mean curvature vector and the mean curvature of M^n, respectively.

The structure equations of M^n are

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_k \wedge \omega_l. \tag{2.7}$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.8}$$

$$n(n-1)r = n(n-1)c - n^2 H^2 + |B|^2, \tag{2.9}$$

where r is the normalized scalar curvature of M^n and $|B|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the norm square of the second fundamental form of M^n.

The Codazzi equations are

$$h_{ijk}^\alpha = h_{ikj}^\alpha = h_{jik}^\alpha, \tag{2.10}$$

where the covariant derivative of h_{ij}^{α} is defined by

$$\sum_k h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} + \sum_k h_{kj}^{\alpha}\omega_{ki} + \sum_k h_{ik}^{\alpha}\omega_{kj} - \sum_\beta h_{ij}^{\beta}\omega_{\beta\alpha}. \tag{2.11}$$

Similarly, the components h_{ijkl}^{α} of the second derivative $\nabla^2 h$ are given by

$$\sum_l h_{ijkl}^{\alpha}\omega_l = dh_{ijk}^{\alpha} + \sum_l h_{ljk}^{\alpha}\omega_{li} + \sum_l h_{ilk}^{\alpha}\omega_{lj} + \sum_l h_{ijl}^{\alpha}\omega_{lk} - \sum_\beta h_{ijk}^{\beta}\omega_{\beta\alpha}. \tag{2.12}$$

By exterior differentiation of (2.11), we can get the following *Ricci formula*

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m h_{im}^{\alpha}R_{mjkl} + \sum_m h_{jm}^{\alpha}R_{mikl} + \sum_\beta h_{ij}^{\beta}R_{\alpha\beta kl}. \tag{2.13}$$

The Laplacian $\triangle h_{ij}^{\alpha}$ of h_{ij}^{α} is defined by $\triangle h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha}$, from the Codazzi equation and Ricci formula, we have

$$\triangle h_{ij}^{\alpha} = \sum_k h_{kkij}^{\alpha} + \sum_{m,k} h_{km}^{\alpha}R_{mijk} + \sum_{m,k} h_{im}^{\alpha}R_{mkjk} + \sum_{k,\beta} h_{ik}^{\beta}R_{\alpha\beta jk}. \tag{2.14}$$

If $H \neq 0$, we choose $e_{n+1} = \dfrac{h}{H}$, then it follows that

$$H^{n+1} := \frac{1}{n}tr(h^{n+1}) = H; \quad H^{\alpha} := \frac{1}{n}tr(h^{\alpha}) = -H\omega_{n+1\alpha}, \quad \forall\alpha \geq n+2, \tag{2.15}$$

where h^{α} denotes the matrix (h_{ij}^{α}). From (2.11) and (2.15), we can see that

$$\sum_k H_k^{n+1}\omega_k = dH; \quad \sum_k H_k^{\alpha}\omega_k = -H\omega_{n+1\alpha}, \quad \forall\alpha \geq n+2. \tag{2.16}$$

From (2.12), (2.15) and (2.16) we have

$$H_{kl}^{n+1} = H_{kl} - \frac{1}{H}\sum_{\beta>n+1} H_k^{\beta}H_l^{\beta}, \tag{2.17}$$

where

$$dH = \sum_i H_i\omega_i, \quad \nabla H_k = \sum_l H_{kl}\omega_l = dH_k + \sum_l H_l\omega_{lk}.$$

If M^n has parallel normalized mean curvature vector, we have

$$\omega_{n+1\alpha} = 0, \quad h^{n+1}h^{\alpha} = h^{\alpha}h^{n+1}, \quad \forall\alpha. \tag{2.18}$$

Then (2.16) and (2.17) yield

$$H_k^{\alpha} = 0, \quad \forall k,\ \alpha \geq n+2; \quad H_{kl}^{n+1} = H_{kl}. \tag{2.19}$$

From (2.12) and (2.19) we obtain

$$H^\alpha_{kl} = 0, \quad \alpha \geq n+2. \tag{2.20}$$

From (2.24) of [7] we have

$$\begin{aligned}\frac{1}{2}\triangle|B|^2 &= \frac{1}{2}\sum_{\alpha,i,j}\triangle(h^\alpha_{ij})^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + \sum_{\alpha,i,j} h^\alpha_{ij}\triangle h^\alpha_{ij} \\ &= \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 + n\sum_{\alpha,i,j} h^\alpha_{ij}H^\alpha_{ij} + nc(|B|^2 - nH^2) \\ &\quad - nH\sum_\alpha tr(h^{n+1}(h^\alpha)^2) + \sum_{\alpha,\beta}(tr(h^\alpha h^\beta))^2 + \sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha),\end{aligned} \tag{2.21}$$

where $N(A) = tr(AA^t)$, for all matrix $A = (a_{ij})$.

Set $\phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha\delta_{ij}$, it is easy to check that ϕ^α is traceless and

$$\begin{aligned}|\phi|^2 &= \sum_{\alpha,i,j}(\phi^\alpha_{ij})^2 = |B|^2 - nH^2 \\ N(\phi^\alpha) &= N(h^\alpha) - n(H^\alpha)^2, \quad n+1 \leq \alpha \leq n+p,\end{aligned} \tag{2.22}$$

where ϕ^α denotes the matrix (ϕ^α_{ij}). Following Cheng-Yau [15], we introduce a modified operator $\square$ acting on any C^2-function f by

$$\square(f) = \sum_{i,j}\left(\left(nH + \frac{n-1}{2}a\right)\delta_{ij} - h^{n+1}_{ij}\right)f_{ij}, \tag{2.23}$$

where f_{ij} is given by the following

$$\sum_j f_{ij}\omega_j = df_i + f_j\omega_{ij}.$$

LEMMA 2.1. *Let M^n be a complete spacelike submanifold of $Q^{n+p}_p(c)$ with $r = aH + b$, $a, b \in \mathbf{R}$ and $(n-1)a^2 + 4nc - 4nb \geq 0$. Then we have*

$$|\nabla B|^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 \geq n^2|\nabla H|^2. \tag{2.24}$$

Proof. From Gauss equation, we have

$$|B|^2 = n^2H^2 + n(n-1)(r-c) = n^2H^2 + n(n-1)(aH + b - c).$$

Taking the covariant derivative of the above equation, we have

$$2\sum_{\alpha,i,j} h^\alpha_{ij}h^\alpha_{ijk} = 2n^2HH_k + n(n-1)aH_k.$$

Therefore,

$$4|B|^2|\nabla B|^2 \geq 4\sum_k \left(\sum_{\alpha,i,j} h_{ij}^{\alpha}h_{ijk}^{\alpha}\right)^2 = [2n^2H + n(n-1)a]^2|\nabla H|^2.$$

Since we know

$$\begin{aligned}[2n^2H + n(n-1)a]^2 - 4n^2|B|^2 &= 4n^4H^2 + n^2(n-1)^2a^2 + 4n^3(n-1)aH \\ &\quad - 4n^2(n^2H^2 + n(n-1)(aH + b - c)) \\ &= n^2(n-1)^2a^2 - 4n^3(n-1)(b-c) \\ &= n^2(n-1)[(n-1)a^2 + 4nc - 4nb] \geq 0,\end{aligned}$$

it follows that

$$|\nabla B|^2 \geq n^2|\nabla H|^2. \qquad \square$$

We will need the following algebraic lemma, whose proof can be found in [27].

Lemma 2.2. *Let $A, B : \mathbf{R}^n \to \mathbf{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $tr(A) = tr(B) = 0$. Then*

$$|tr\, A^2B| \leq \frac{n-2}{\sqrt{n(n-1)}}N(A)\sqrt{N(B)}.$$

We also will need the well known generalized Maximum Principle due to H. Omori [25].

Lemma 2.3. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to \mathbf{R}$ be a smooth function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}$ in M^n such that*

$$\lim_{k\to\infty} f(p_k) = \sup f; \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0; \quad \limsup_{k\to\infty}(\triangle f(p_k)) \leq 0.$$

Proposition 2.4. *Let M^n be a complete spacelike submanifold in $Q_p^{n+p}(c)$ with parallel normalized mean curvature vector. If $r = aH + b$, $a, b \in \mathbf{R}$ and $(n-1)a^2 + 4nc - 4nb \geq 0$, then the following inequality holds*

$$\square(nH) \geq |\phi|^2\left(\frac{|\phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c - H^2)\right). \tag{2.25}$$

Proof. From (2.23) we have

$$\Box(nH) = \sum_{i,j}\left(\left(nH + \frac{1}{2}(n-1)a\right)\delta_{ij} - h_{ij}^{n+1}\right)(nH)_{ij} \tag{2.26}$$

$$= \left(nH + \frac{1}{2}(n-1)a\right)\triangle(nH) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}$$

$$= \left(nH + \frac{1}{2}(n-1)a\right)\triangle\left(nH + \frac{1}{2}(n-1)a\right) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}$$

$$= \frac{1}{2}\triangle\left(nH + \frac{1}{2}(n-1)a\right)^2$$

$$- \left|\nabla\left(nH + \frac{1}{2}(n-1)a\right)\right|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}$$

$$= \frac{1}{2}\triangle\left(nH + \frac{1}{2}(n-1)a\right)^2 - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}.$$

On the other side, from Gauss equation and $r = aH + b$, we have

$$\triangle|B|^2 = \triangle(n^2H^2 + n(n-1)(r-c)) \tag{2.27}$$

$$= \triangle(n^2H^2 + n(n-1)(aH + b - c))$$

$$= \triangle(n^2H^2 + (n-1)anH)$$

$$= \triangle\left(nH + \frac{1}{2}(n-1)a\right)^2.$$

From (2.21), (2.26) and (2.27) we get

$$\Box(nH) = \frac{1}{2}\triangle|B|^2 - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \tag{2.28}$$

$$= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + n\sum_{\alpha,i,j} h_{ij}^{\alpha}H_{ij}^{\alpha} - n\sum_{i,j} h_{ij}^{n+1}H_{ij}$$

$$+ nc(|B|^2 - nH^2) - nH\sum_{\alpha} tr(h^{n+1}(h^{\alpha})^2)$$

$$+ \sum_{\alpha,\beta}(tr(h^{\alpha}h^{\beta}))^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}).$$

Since M^n has parallel normalized mean curvature vector, (2.19), (2.20) and (2.28) yield

$$\square(nH) = \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 - n^2|\nabla H|^2 + nc(|B|^2 - nH^2) \tag{2.29}$$
$$- nH \sum_{\alpha} tr(h^{n+1}(h^{\alpha})^2) + \sum_{\alpha,\beta} (tr(h^{\alpha}h^{\beta}))^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}).$$

From (2.15) and (2.22), we have

$$\begin{aligned} &\phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \\ &N(\phi^{n+1}) = tr(\phi^{n+1})^2 = tr(h^{n+1})^2 - nH^2 = N(h^{n+1}) - nH^2, \\ &tr(h^{n+1})^3 = tr(\phi^{n+1})^3 + 3HN(\phi^{n+1}) + nH^3, \\ &\phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \quad N(\phi^{\alpha}) = N(h^{\alpha}), \quad \alpha \geq n+2. \end{aligned} \tag{2.30}$$

By (2.29), (2.30) and Lemma 2.1, we see that

$$\square(nH) \geq n|\phi|^2(c - H^2) - nH \sum_{\alpha} tr(\phi^{n+1}(\phi^{\alpha})^2) \tag{2.31}$$
$$+ \sum_{\alpha,\beta} (tr(h^{\alpha}h^{\beta}))^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}).$$

By (2.18) we know that the traceless matrix ϕ^{n+1} commutes with the traceless matrices ϕ^{α}, for all α. Hence we can apply Lemma 2.2 in order to obtain

$$\sum_{\alpha} tr(\phi^{n+1}(\phi^{\alpha})^2) \leq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{N(\phi^{n+1})}|\phi|^2 \leq \frac{n-2}{\sqrt{n(n-1)}} |\phi|^3. \tag{2.32}$$

Moreover, Cauchy-Schwarz inequality implies that

$$|\phi|^4 \leq p \sum_{\alpha} (N(\phi^{\alpha}))^2 \leq p \sum_{\alpha,\beta} (tr(h^{\alpha}h^{\beta}))^2. \tag{2.33}$$

Inserting (2.32) and (2.33) into (2.31), we arrive to (2.25). □

PROPOSITION 2.5. *Let M^n be a complete spacelike submanifold in $Q^{n+p}_p(c)$ with bounded mean curvature. If $r = aH + b$, $a, b \in \mathbf{R}$, $a \geq 0$ and $(n-1)a^2 + 4nc - 4nb \geq 0$, then there is sequence of points $\{p_k\} \in M^n$ such that*

$$\lim_{k\to\infty} nH(p_k) = n \sup H; \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0; \quad \limsup_{k\to\infty} (\square(nH)(p_k)) \leq 0.$$

Proof. Choose a local orthonormal frame field $e_1, \dots, e_n$ at $p \in M^n$ such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$. Thus

$$\square(nH) = \sum_i \left[\left(nH + \frac{1}{2}(n-1)a\right) - \lambda_i^{n+1}\right](nH)_{ii}.$$

If $H \equiv 0$ the proposition is obvious. Let us suppose that H is not identically zero. By changing the orientation of M^n if necessary, we may assume $\sup H > 0$. From

$$\begin{aligned}(\lambda_i^{n+1})^2 \leq |B|^2 &= n^2H^2 + n(n-1)(aH + b - c)\\ &= (nH)^2 + (n-1)a(nH) + n(n-1)(b-c)\\ &= \left(nH + \frac{1}{2}(n-1)a\right)^2 - \frac{1}{4}(n-1)((n-1)a^2 + 4nc - 4nb)\\ &\leq \left(nH + \frac{1}{2}(n-1)a\right)^2,\end{aligned}$$

we have

$$|\lambda_i^{n+1}| \leq \left|nH + \frac{1}{2}(n-1)a\right|. \tag{2.34}$$

Then

$$R_{ijij} = c - \sum_\alpha (h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2) \geq c - p\left(nH + \frac{1}{2}(n-1)a\right)^2. \tag{2.35}$$

Because H is bounded, it follows from (2.35) that the sectional curvatures are bounded from below. Therefore we may apply Lemma 2.3 to nH, obtaining a sequence of points $\{p_k\} \in M^n$ such that

$$\lim_{k\to\infty} nH(p_k) = n\sup H; \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0; \quad \limsup_{k\to\infty}((nH)_{ii}(p_k)) \leq 0. \tag{2.36}$$

Since H is bounded, taking subsequences if necessary, we can arrive to a sequence $\{p_k\} \in M^n$ which satisfies (2.36) and such that $H(p_k) \geq 0$. Thus from (2.34) we get

$$\begin{aligned}0 \leq nH(p_k) + \frac{1}{2}(n-1)a - |\lambda_i^{n+1}(p_k)| &\leq nH(p_k) + \frac{1}{2}(n-1)a - \lambda_i^{n+1}(p_k)\\ &\leq nH(p_k) + \frac{1}{2}(n-1)a + |\lambda_i^{n+1}(p_k)|\\ &\leq 2nH(p_k) + (n-1)a.\end{aligned} \tag{2.37}$$

Using once more the fact that H is bounded, from (2.37) we infer that $nH(p_k) + \frac{1}{2}(n-1)a - \lambda_i^{n+1}(p_k)$ is non-negative and bounded. By applying $\square(nH)$ at p_k, taking the limit and using (2.36) and (2.37) we have

$$\begin{aligned}\limsup_{k\to\infty}(\square(nH)(p_k)) &\le \sum_i \limsup_{k\to\infty}\left[\left(nH+\frac{1}{2}(n-1)a\right)-\lambda_i^{n+1}\right](p_k)(nH)_{ii}(p_k)\\ &\le 0.\end{aligned}$$

□

3. Proof of the main result

Proof of theorem 1.2. If M^n is maximal, i.e., if $H\equiv 0$, due to Ishihara's result [17] we know that M^n is totally geodesic. Let us suppose that H is not identically zero. In this case, by Proposition 2.5 it is possible to obtain a sequence of points $\{p_k\}\in M^n$ such that

$$\limsup_{k\to\infty}(\square(nH)(p_k))\le 0,\quad \lim_{k\to\infty} H(p_k)=\sup H>0. \tag{3.1}$$

Moreover, using the Gauss equation, we have that

$$|\phi|^2=|B|^2-nH^2=n(n-1)(H^2+aH+b-c). \tag{3.2}$$

In view of $\lim_{k\to\infty} H(p_k)=\sup H$ and $a\ge 0$, (3.2) implies that $\lim_{k\to\infty}|\phi|^2(p_k)=\sup|\phi|^2$. Now we consider the following polynomial given by

$$P_{\sup H}(x)=\frac{x^2}{p}-\frac{n(n-2)}{\sqrt{n(n-1)}}\sup Hx+n(c-\sup H^2). \tag{3.3}$$

If $\sup H^2<\dfrac{4(n-1)c}{(n-2)^2p+4(n-1)}$, then the discriminant of $P_{\sup H}(x)$ is negative. Hence, $P_{\sup H}(\sup|\phi|)>0$.

Using Lemma 2.1 and evaluating (2.25) at the points p_k of the sequence, taking the limit and using (3.1), we obtain that

$$0\ge\limsup_{k\to\infty}(\square(nH)(p_k))\ge\sup|\phi|^2P_{\sup H}(\sup|\phi|)\ge 0,$$

and so $\sup|\phi|^2P_{\sup H}(\sup|\phi|)=0$. Therefore, since $P_{\sup H}(\sup|\phi|)>0$, we conclude that $\sup|\phi|^2=0$ which shows that M^n is totally umbilical. □

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