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KODAI MATH. J.
31 (2008), 263–271

STABILITY AND INSTABILITY OF STANDING WAVES FOR 1-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH MULTIPLE-POWER NONLINEARITY

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1. Introduction and main results

In the present paper we consider the stability and instability of standing waves for the following nonlinear Schrödinger equation:

$$iu_t + u_{xx} + f(u) = 0, \quad t \geq 0,\ x \in \mathbf{R}, \tag{1}$$

where $f(u) = \sum_{j=1}^{m} a_j |u|^{p_j - 1} u$ with $a_j \in \mathbf{R}$ and $1 < p_1 < \cdots < p_m < \infty$. Equation (1) arises in various regions of mathematical physics.

The unique local existence of (1) is well known. That is for any $u_0 \in H^1(\mathbf{R})$, there exists a positive constant T and a unique local solution $u \in C([0,T); H^1(\mathbf{R})) \cap C^1([0,T); H^{-1}(\mathbf{R}))$ of (1) with $u(0) = u_0$. Furthermore, $u(t)$ satisfies the two conservation laws $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $E(u(t)) = E(u_0)$, where $E(v) := \frac{1}{2}\|v_x\|_{L^2} - \int_{\mathbf{R}} F(|v(x)|)\,dx$ and $F(s) = \int_0^s f(\sigma)\,d\sigma$. For details, see, e.g., [4], [11] and [15].

We say that the solution of (1) is a standing wave if it has a form $u(t,x) = e^{i\omega t}\varphi_\omega(x)$, where $\omega > 0$. Here φ_ω is a solution of the following equation:

$$\varphi_{xx} - \omega\varphi + f(\varphi) = 0, \quad x \in \mathbf{R},\ \varphi \in H^1(\mathbf{R})\backslash\{0\}. \tag{2}$$

The existence and uniqueness of the solution of (2) is well known: Set

$$\omega^* = \sup\left\{\omega > 0\text{: there exists } s > 0, \text{ s.t. } \frac{1}{2}\omega s^2 - F(s) < 0\right\},$$

then for any $\omega \in (0, \omega^*)$, there exists a solution φ_ω of (2). Further the solution is unique up to a translation and a phase change ([2]). In the present paper, we study how the stability of standing waves depends on frequency ω in the multiple-power nonlinearity case.

Stability and instability of standing waves is defined as follows.

Definition 1. *A standing wave $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is said to be stable if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property; for any $u_0 \in H^1(\mathbf{R})$ satisfying $\|u_0 - \varphi_\omega\|_{H^1(\mathbf{R})} < \delta$, the solution $u(t)$ of (1) with $u(0) = u_0$ can be continued to a solution in $0 \leq t < \infty$ and it satisfies the following condition*

Received September 4, 2007; revised December 21, 2007.

$$\sup_{0\le t<\infty}\inf_{\theta,y\in\mathbf{R}}\|u(t)-e^{i\theta}\varphi_\omega(\cdot-y)\|_{H^1(\mathbf{R})}<\varepsilon.$$

Otherwise, u_ω *is called unstable*.

Remark 1. We note that the conception of stronger stability which does not involve the translation

$$\sup_{0\le t<\infty}\inf_{\theta\in\mathbf{R}}\|u(t)-e^{i\theta}\varphi_\omega\|_{H^1(\mathbf{R})},$$

cannot be used in studying the stability of (1). It is because if $u(x,t)=e^{i\omega t}\varphi_\omega(x)$ is a solution of (1), then by a simple calculation, we observe that $u_c(x,t)=e^{i(\omega t-cx-c^2t)}\varphi_\omega(x+2ct)$ is also a solution. It is not hard to see that $u_c(x,0)$ can be taken arbitrary near $u(x,0)$ by taking c small, and if c is not zero, then $u_c(t)$ always goes away from $u(t)$ (see [14]).

Recently, many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [1, 2, 6, 7, 8, 9, 10, 12, 14, 17, 18]).

At first, we will introduce the results in the single power case $f(u)=a|u|^{p-1}u$ with $a>0$ and $p>1$. For this case, if $1<p<5$, then u_ω is stable for every $\omega\in(0,\infty)$, and if $5\le p$, then u_ω is unstable for every $\omega\in(0,\infty)$ (see [1], [3] and [18]). For the single power case, (1) has scaling invariance, and using it, one can verify the stability. Note that the stability of standing waves is independent of the frequency ω in the single power case. Although it is not the case with the double power nonlinearity. In this case, there is no scaling invariance in (1), so the problem to investigate the stability of standing waves becomes more complicated.

Although, when $f(u)=a_1|u|^{p_1-1}u+a_2|u|^{p_2-1}u$, Ohta [16] proved the following theorem.

Theorem A (Ohta [16]). *Let* $1<p_1<p_2$.

(I) *Let* $a_1,a_2>0$.
- (I.1) *If* $p_2\le 5$, *then* u_ω *is stable for any* $\omega\in(0,\infty)$.
- (I.2) *If* $p_1\ge 5$, *then* u_ω *is unstable for any* $\omega\in(0,\infty)$.
- (I.3) *If* $p_1<5<p_2$, *then there exist positive constants* ω_1 *and* ω_2 *such that* u_ω *is stable for any* $\omega\in(0,\omega_1)$, *and unstable for any* $\omega\in(\omega_2,\infty)$.

(II) *Let* $a_1>0$, $a_2<0$.
- (II.1) *If* $p_1\le 5$, *then* u_ω *is stable for any* $\omega\in(0,\omega^*)$.
- (II.2) *If* $p_1>5$, *then there exist positive constants* ω_3 *and* ω_4 *such that* u_ω *is unstable for any* $\omega\in(0,\omega_3)$, *and stable for any* $\omega\in(\omega_4,\omega^*)$.

(III) *Let* $a_1<0$, $a_2>0$.
- (III.1) *If* $p_2\ge 5$, *then* u_ω *is unstable for any* $\omega\in(0,\infty)$.
- (III.2) *If* $p_2<5$, *then there exists a positive constant* ω_5 *such that* u_ω *is stable for any* $\omega\in(\omega_5,\infty)$. *Furthermore if* $p_1+p_2>6$, *then*

there exists a positive constant ω_6 such that u_ω is unstable for any $\omega \in (0, \omega_6)$.

Theorem A shows that, in the double power case, the stability of standing waves can change when the frequency ω varies. In Theorem A, there are gaps in (I.3) (for $\omega \in [\omega_1, \omega_2]$), (II.2) (for $\omega \in [\omega_3, \omega_4]$) and (III.2) (for $\omega \in [0, \omega_5]$ in the case of $p_1 + p_2 < 6$ and for $\omega \in [\omega_6, \omega_5]$ in the case of $p_1 + p_2 > 6$). It seems difficult to verify whether the standing wave u_ω is stable or not if the equation does not have scaling invariance. Our first target is to fill these gaps.

Our main results are the following.

THEOREM 1. *Let $f(u) = a_1|u|^{p_1-1}u + a_2|u|^{p_2-1}u$.*

(1) *Suppose $a_1, a_2 > 0$ and $1 < p_1 < 5 < p_2$. Then there exists $\omega_1 > 0$ such that for $\omega \in (0, \omega_1)$, u_ω is stable, and for $\omega \in [\omega_1, \infty)$, u_ω is unstable.*
(2) *Suppose $a_1 > 0$, $a_2 < 0$ and $5 < p_1 < p_2$. Then there exists $\omega_2 > 0$ such that for $\omega \in (0, \omega_2]$, u_ω is unstable, and for $\omega \in (\omega_2, \omega^*)$, u_ω is stable.*
(3) *Suppose $a_1 < 0$, $a_2 > 0$, $7/3 < p_1 < p_2 < 5$ and $p_1 + p_2 > 6$. Then there exists $\omega_3 > 0$ such that for $\omega \in (0, \omega_3]$, u_ω is unstable, and for $\omega \in (\omega_3, \infty)$ then u_ω is stable.*

Remark 2. Since $a_2 > 0$ for the cases (1) and (3), we observe that $\omega^* = \infty$. On the other hand, since $a_2 < 0$ for the case (2), we have $\omega^* < \infty$.

Remark 3. There are still gaps in the cases of Theorem 1 (3). However Ohta [16] showed, when $a_1 < 0$, $a_2 > 0$ and $p_1 = 2$, $p_2 = 3$, u_ω is stable for any $\omega \in (0, \infty)$. So, in Theorem 2 (3), the condition $p_1 + p_2 > 6$ is needed, although it may be not optimal.

In the single power case, the stability of standing waves does not change by ω, and in the double power case, stability of standing waves change at most once. So, the natural question arises: if the equation has more powers, then could we get standing waves that change its stability more than once? The next theorem gives examples of standing waves that change its stability, by ω, two and three times.

THEOREM 2. (1) *Let $f(u) = a_1|u|^2u + |u|^6u - |u|^8u$, let $a_1 > 0$ be sufficiently small. Then there exist five real numbers $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5$ such that for $\omega \in (0, \omega_1) \cup (\omega_4, \omega_5)$, u_ω is stable, and for $\omega \in (\omega_2, \omega_3)$, u_ω is unstable.*
(2) *Let $f(u) = a_1|u|^2u + |u|^6u - a_3|u|^8u + |u|^{10}u$, let $a_1 > 0$ be sufficiently small and $a_3 > 0$ sufficiently large. Then there exist six real numbers $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$ such that for $\omega \in (0, \omega_1) \cup (\omega_4, \omega_5)$, u_ω is stable, and for $\omega \in (\omega_2, \omega_3) \cup (\omega_6, \infty)$, u_ω is unstable.*

Remark 4. The conditions in Theorem 2 are for only technical reasons. Our motivation was to show there are equations whose standing waves change its stability several times when the frequency ω varies.

2. Proofs of Theorems 1 and 2

We first summarize three lemmas needed for the proof of Theorems 1 and 2.

Lemma 1 (Grillakis, Shatah and Strauss [12]). *Set*

$$I(\omega) = \|\varphi_\omega\|_2^2 = \int_{-\infty}^{\infty} |\varphi_\omega(x)|^2 \, dx.$$

If $I'(\omega) > 0$, then $u_\omega(t) = e^{i\omega t}\varphi_\omega$ is stable, and if $I'(\omega) < 0$, then u_ω is unstable.

For the case $I'(0) = 0$, Comech and Pelinovski proved the following theorem.

Theorem B (Comech and Pelinovski [5]). *Let $e^{i\omega t}\varphi_\omega$ be the standing wave solution of* (1). *Assume that $I'(\omega_*) = 0$ and $I''(\omega_*) \neq 0$ for some $\omega_* \in (0, \omega^*)$. Then there is a positive number ε such that for any $\delta > 0$, there exists $t_1 = t_1(\delta, \varepsilon) < \infty$ and a pair of functions $(\omega, \rho) \in C^1([0, t_1]; (0, \omega^*)) \times C^1([0, t_1]; H^1(\mathbf{R}))$, such that $u(t) = e^{i\int_0^t \omega(t')\,dt'}(\varphi_{\omega(t)} + \rho(t))$ is a solution to* (1) *and such that $|\omega(0) - \omega_*| < \delta$, $\|\rho(t)\|_{H^1(\mathbf{R})} \le 0$ and $|\omega(t_1) - \omega_*| > \varepsilon$.*

The following lemma is a direct consequence of Theorem B.

Lemma 2. *If $I'(\omega_*) = 0$ and $I''(\omega_*) \neq 0$, then u_{ω_*} is unstable.*

Proof. Because φ_ω is an even real valued function, $\partial_\omega \varphi_\omega$ is an even real valued function and $\partial_x \varphi_\omega$ is an odd real valued function. It follows that $\partial_\omega \varphi_\omega \perp \partial_x \varphi_\omega$ and $\partial_\omega \varphi_\omega \perp i\varphi_\omega$ in $H^1(\mathbf{R})$. Now, note that the tangent space of the orbit $\{e^{is}\varphi_{\omega_*}(\cdot + y) \mid s, y \in \mathbf{R}\}$ is spanned by $\partial_x \varphi_{\omega_*}$ and $i\varphi_{\omega_*}$. So by Theorem B,

$$u(t) = \exp\left(i \int_0^t \omega(\tau)\, d\tau\right)(\varphi_{\omega(t)} + \rho(t)) \sim \varphi_{\omega_*} + t\partial\varphi_{\omega_*}.$$

Therefore $u(t)$, which was initially close to the orbit, leaves the ε-tubular neighborhood of the orbit in finite time. ■

Lemma 3 (Iliev and Kirchev [14]). *Suppose $f(u) = \sum_{j=1}^m a_j |u|^{p_j - 1} u$, then we have*

$$I'(\omega) = -\frac{1}{W'(h)} \int_0^h \frac{\sum_{j=1}^m c_j (h^{q_j} - s^{q_j})}{(\sum_{j=1}^m d_j (h^{q_j} - s^{q_j}))^{3/2}} \, ds,$$

where $q_j = \dfrac{p_j - 1}{2}$, $c_j = \dfrac{a_j(5 - p_j)}{p_j + 1}$, $d_j = \dfrac{2a_j}{p_j + 1}$, $W(s) = \omega s - \sum_{j=1}^m \dfrac{2a_j}{p_j + 1} s^{(p_j + 1)/2}$ and $h = h(\omega)$ is a positive number satisfying $W(h) = 0$, $W'(h) < 0$ and $W(s) > 0$ for all $s \in (0, h)$.

Remark 5. Function $h(\omega)$ can be defined as

$$h(\omega) := \sup\{h > 0 \mid W(s) > 0 \text{ for all } s \in (0, h)\}.$$

Recall the definition of ω^*. Since $\omega \in (0, \omega^*)$, we have $W(h(\omega)) = 0$. Further, by

$$W(s) > 0 \Leftrightarrow \omega > V(s) := \sum_{j=1}^{m} \frac{2a_j}{p_j + 1} s^{(p_j - 1)/2},$$

we see that

$$h(\omega) = \sup\{h > 0 \mid \omega > V(s) \text{ for all } s \in (0, h)\}. \tag{3}$$

So we have that $h(\omega)$ is a monotone increasing function. Furthermore by (3), for $a_1 > 0$, $h(0) = 0$ and for $a_1 < 0$, $h(0) > 0$. Also for $a_m > 0$, $\lim_{\omega \to \infty} h(\omega) = \infty$ and for $a_m < 0$, $\lim_{\omega \to \omega^*} h(\omega) < \infty$.

We now describe the proof of Theorem 1

(*Proof of Theorem* 1). By Lemmas 1 and 2, we have only to check the sign of $I'(\omega)$ given by lemma 3.

In the case $m = 2$, $I'(\omega)$ can be written as

$$I'(\omega) = -\frac{h^{(5-p_1)/4}}{2W'(h)} \int_0^1 \frac{H(h, s)}{(d_1(1 - s^{q_1}) + d_2(1 - s^{q_2})h^{q_2 - q_1})^{3/2}}\, ds,$$

where $H(h, s) := c_1(1 - s^{q_1}) + c_2(1 - s^{q_2})h^{q_2 - q_1}$. Because $-h^{(5-p_1)/4}/2W'(h)$ is always positive and we only care about the sign of I', it suffices to consider

$$F(h) = \int_0^1 \frac{H(h, s)}{(d_1(1 - s^{q_1}) + d_2(1 - s^{q_2})h^{q_2 - q_1})^{3/2}}\, ds.$$

By a simple calculation, we have

$$F'(h) = \frac{a_2(p_2 - p_1)}{2(p_2 + 1)} h^{q_2 - q_1 - 1} \times \int_0^1 \frac{(1 - s^{q_2})\tilde{H}(h, s)}{(d_1(1 - s^{q_1}) + d_2(1 - s^{q_2})h^{q_2 - q_1})^{5/2}}\, ds,$$

where $\tilde{H}(h, s) := -r(1 - s^{q_1}) - c_2(1 - s^{q_2})h^{q_2 - q_1}$ and $r = c_1 + 2d_1(q_2 - q_1)$.

Now define

$$l(s) := \frac{1 - s^{q_1}}{1 - s^{q_2}}.$$

Then, $l(s)$ is a monotone decreasing function in $(0, 1)$ and $l(s)$ satisfies

$$\frac{q_1}{q_2} < l(s) < 1, \quad \forall s \in (0, 1). \tag{4}$$

Part (1). In this case, we have $c_1 > 0$, $c_2 < 0$ and $r > 0$. Put

$$A_{p_1,p_2} := \left(-\frac{c_1q_1}{c_2q_2}\right)^{1/(q_2-q_1)}, \quad B_{p_1,p_2} := \left(-\frac{rq_1}{c_2q_2}\right)^{1/(q_2-q_1)}.$$

Taking $h = A_{p_1,p_2}\alpha^{1/(q_2-q_1)}$ for $\alpha > 0$, $H(h,s)$ can be rewritten as $H(h,s) = c_1(1-s^{q_2})\{l(s) - \alpha q_1/q_2\}$. From (4), if $\alpha < 1$, i.e. $h < A_{p_1,p_2}$, then $F(h) > 0$, and if $\alpha > q_2/q_1$, i.e. $h > A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}$, then $F(h) < 0$. In the same way, we see that if $h < B_{p_1,p_2}$, we have $F'(h) < 0$.

Now, $A_{p_1,p_2} < A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}$ always holds since $q_1 < q_2$. Also if $7/3 \le p_1$, then by a simple calculation we have $A_{p_1,p_2} < A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} \le B_{p_1,p_2}$. Since $F(A_{p_1,p_2}) > 0$, $F(A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}) < 0$ and F is a monotone decreasing function at $(A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)})$, there exists an $\omega_1 > 0$ such that if $\omega \in (0,\omega_1)$, then $\partial_\omega \|\varphi_\omega\|^2 < 0$. So, by lemmas 1 and 2, we have the conclusion.

For the case $1 < p_1 < 7/3$, it suffices to prove that, if $B_{p_1,p_2} \le h$, then $F(h) < 0$. If $h \in (A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)})$, there exists a solution of $l(s^*) - \alpha q_1/q_2 = 0$, since $\alpha \in (1, q_2/q_1)$ and $l(s)$ decreases from 1 to q_2/q_1. Furthermore if $s \in (0,s^*)$, $H(h,s)$ is positive and if $s \in (s^*,1)$, it is negative. Also because the denominator of the integrand of F is monotonically decreasing function, we see that $\int_0^1 H(h,s)\,ds < 0$ implies $F(h) < 0$. Now we note that

$$\int_0^1 H(h,s)\,ds = \frac{c_1q_1}{q_1+1} + \frac{c_2q_2}{q_2+1}h^{q_2-q_1},$$

and

$$\frac{c_1q_1}{q_1+1} + \frac{c_2q_2}{q_2+1}h^{q_2-q_1} < 0 \Leftrightarrow \left(-\frac{c_1q_1(q_2+1)}{c_2q_2(q_1+1)}\right)^{1/(q_2-q_1)} < h.$$

Therefore, since

$$\left(-\frac{c_1q_1(q_2+1)}{c_2q_2(q_1+1)}\right)^{1/(q_2-q_1)} < B_{p_1,p_2} \Leftrightarrow q_1 < q_2,$$

we see that $F(h) < 0$ for $B_{p_1,p_2} \le h$.

Part (2). In this case we have $c_1 < 0$, $c_2 > 0$ and $h(0) = 0$, $h(\omega^*) < \infty$. Since the signs of c_1, c_2 are opposite from part (1), we see that if $A_{p_1,p_2} > h$, then $F(h) < 0$, and if $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h$, then $F(h) > 0$. Also by a simple calculation we see that $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h(\omega^*)$.

First if $0 \le r := c_1 + 2d_1(q_2 - q_1)$, then $\tilde{H}(h,s)$ will be always negative. Consequently, since $a_2 < 0$, $F' > 0$ will always hold. So we have the conclusion in this case.

Next if $r < 0$, then we see that

$$\tilde{H}(h,s) < 0 \Leftrightarrow -rl(s) < c_2h^{q_2-q_1}.$$

Since $l(s) < 1$, if $h > (-r/c_2)^{1/(q_2-q_1)}$, then $\tilde{H}(h,s) < 0$ and $F'(h) > 0$ follows. By a simple calculation, we see that $(-r/c_2)^{1/(q_2-q_1)} < A_{p_1,p_2}$. So, for $h \in (0, A_{p_1,p_2}]$, F is negative, and for $h \in [A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}]$, F' is positive, and for $h \in [A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}), h(\omega^*))$, F is positive. From this, we have the conclusion.

Part (3). In this case we have $c_1 < 0$, $c_2 > 0$, $r < 0$, and $h(0) > 0$. Since the signs of c_1, c_2 are the same as in part (2), we see that if $A_{p_1,p_2} > h$, then $F(h) < 0$, and if $A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < h$, then $F(h) > 0$. Now, since $h(0) > 0$, we wish to make A_{p_1,p_2} larger than $h(0)$. By a simple calculation we see that if $q_1 + q_2 > 2$, i.e. $p_1 + p_2 > 6$, then $h(0) < A_{p_1,p_2}$ holds.

Next, observing $\tilde{H}(h,s)$, we see that if $h < (-rq_1/c_2q_2)^{1/(q_2-q_1)}$, then $F'(h) > 0$. Now, if $p_1 > 7/3$, by a simple calculation we see that

$$A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)} < (-rq_1/c_2q_2)^{1/(q_2-q_1)}.$$

So, for $h \in (0, A_{p_1,p_2}]$, F is negative, and for $h \in [A_{p_1,p_2}, A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}]$, F' is positive, and for $h \in [A_{p_1,p_2}(q_2/q_1)^{1/(q_2-q_1)}), h(\omega^*))$, F is positive. This gives us the conclusion. ∎

(*Proof of Theorem* 2). We will not consider the point where $I'(\omega) = 0$, so we will only use Lemmas 1 and 3, and will not use Lemma 2.

Part (1). Since $a_1 > 0$ and $a_3 := -1 < 0$, we have $h(0) = 0$ and $h(\omega^*) < 0$. Furthermore, calculating $h(\omega^*)$ from the definition, we see that $h(\omega^*) > 1$.

By Lemma 3,

$$I'(\omega) = \int_0^h \frac{\frac{1}{2}a_1(h-s) - \frac{1}{4}(h^3-s^3) + \frac{2}{5}(h^4-s^4)}{\left(\frac{1}{2}a_1(h-s) + \frac{1}{4}(h^3-s^3) - \frac{1}{5}(h^4-s^4)\right)^{3/2}}\, ds.$$

Set

$$H(h,s) := \frac{1}{2}a_1(h-s) - \frac{1}{4}(h^3-s^3) + \frac{2}{5}(h^4-s^4).$$

Then we have

$$H(h,s) > 0 \Leftrightarrow \frac{a_1}{2} - \frac{1}{4}(h^2+hs+s^2) + \frac{2}{5}(h^3+h^2s+hs^2+s^3) > 0.$$

So, by setting

$$G(h) := \frac{a_1}{2} - \frac{3}{4}h^2 + \frac{2}{5}h^3,$$

$$\tilde{G}(h) := \frac{a_1}{2} - \frac{1}{4}h^2 + \frac{8}{5}h^3,$$

we see that $G \le H \le \tilde{G}$. Therefore if $G(h) > 0$, then $H(h,s) > 0$ for $\forall s \in (0,h)$, and if $\tilde{G}(h) < 0$, then $H(h,s) < 0$ for $\forall s \in (0,h)$.

Now, $G(h)$ is positive near $h = 0$ and $\tilde{G}(h)$ takes negative values for some $h \in (0,1)$ when a_1 is small. So, we see that there exist $h_1 < h_2 < h_3$ (< 1) such that for $h \in (0,h_1)$, $I' > 0$ and for $h \in (h_2,h_3)$, $I' < 0$.

Next, we will show that for $h = 1$, $I' > 0$.

$$\begin{aligned} H(1,s) &= \frac{a_1}{2}(1-s) - \frac{1}{4}(1-s^3) + \frac{2}{5}(1-s^4) \\ &= \frac{a_1}{2}(1-s) + \frac{3}{20} + s^3\left(\frac{1}{4} - \frac{2}{5}s\right) \\ &> 0. \end{aligned}$$

So, there exist two numbers h_4 and h_5 such that $h_3 < h_4 < 1 < h_5 < h(\omega^*)$ and $I'(h) > 0$ for $h \in (h_4, h_5)$.

Part (2). In this case, we only have to calculate G and $\tilde{G}$ as in Part (1). ∎

Remark 6. We can make an example of standing waves that change its stability exactly $2m-1$ times when the frequency ω varies, by considering $2m$-power nonlinearity. In fact, by taking

$$f(u) = a_1|u|^2u + \sum_{j=2}^{2m}(-1)^j N^{-((2m-j)(2m+1-j)/2)}|u|^{2j+2}u,$$

and if N is sufficiently large and a_1 is sufficiently small, we see that $F(h)$ changes its sign as frequency ω varies in the same way as Theorem 1 and Theorem 2. This is possible because we can set $a_1 > 0$ and $a_{2m} > 0$, so that $h(0) = 0$ and $h(\infty) = \infty$. This makes computation simpler because we do not have to consider the situations like in the proof of part (3) of Theorem (1) (for example, the situation $h(0) < A_{p_1,p_2}$).

Acknowledgments. The author would like to express his deep gratitude to Professor Yoshio Tsutsumi for his helpful discussions and encouragement. He also wishes to express his sincere appreciation to Professor Masahito Ohta for his helpful advice.

References

[1] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, C. R. Acad. Sci. Paris **293** (1981), 489–492.

[2] H. Berestycki and P. L. Lions, Nonlinear scalar field equations, I, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[3] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. **85** (1982), 549–561.

[4] T. Cazenave and F. B. Weissler, The Cauchy problem for the nonlinear Schrödinger equation in H^1, Manuscripta Math. **61** (1988), 477–494.

[5] A. Comech and D. Pelinovski, Purely nonlinear instability of standing waves with minimal energy, Comm. Pure Appl. Math. **56** (2003), 1565–1607.

[6] R. Fukuizumi, Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential, Discrete Contin. Dynam. Systems. **7** (2001), 525–544.

[7] R. Fukuizumi, Stability of standing waves for nonlinear Schrödinger equations with critical power nonlinearity and potentials, Advances in Differential Equations **10** (2005), 259–276.

[8] R. Fukuizumi and M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with potentials, Differential and Integral Eqs. **16** (2003), 691–706.

[9] R. Fukuizumi and M. Ohta, Stability of standing waves for nonlinear Schrödinger equations with potentials, Differential and Integral Eqs. **16** (2003), 111–128.

[10] R. Fukuizumi and M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, J. Math. Kyoto Univ. **45** (2005), 145–158.

[11] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations, I, J. Funct. Anal. **32** (1979), 1–32.

[12] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry, I, J. Funct. Anal. **74** (1987), 160–197.

[13] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry, II, J. Funct. Anal. **94** (1990), 308–348.

[14] I. Iliev and K. Kirchev, Stability and instability of solitary waves for one-dimensional singular Schrödinger equations, Differential Integral Equations **6** (1993), 685–703.

[15] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), 113–129.

[16] M. Ohta, Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity, Kodai Math. J. **18** (1995), 68–74.

[17] J. Shatah and W. A. Strauss, Instability of nonlinear bound states, Comm. Math. Phys. **100** (1985), 173–190.

[18] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. **87** (1983), 567–576.

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