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INVARIANTS OF AMPLE LINE BUNDLES ON PROJECTIVE VARIETIES AND THEIR APPLICATIONS, I*†‡

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Abstract

Let X be a projective variety of dimension n defined over the field of complex numbers and let $L_1, \ldots, L_{n-i}$ be ample line bundles on X, where i is an integer with $0 \leq i \leq n$. In this paper, first, we define some invariants called the ith sectional H-arithmetic genus, the ith sectional geometric genus and the ith sectional arithmetic genus of $(X, L_1, \ldots, L_{n-i})$. These are considered to be a generalization of invariants which have been defined in our previous papers. Moreover we investigate some basic properties of these, which are used in the second part and the third part of this work.

Introduction

Let X be a projective variety of dimension n which is defined over the field of complex numbers and let L be an ample line bundle on X. Then the pair (X, L) is called a *polarized variety*. Moreover if X is smooth, then (X, L) is called a polarized *manifold*.

At present, the following three invariants of (X, L) are well-known.

(1) The degree L^n.

(2) The sectional genus $g(L)$, which is defined by

$$g(L) := 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

(Here K_X denotes the canonical line bundle of X.)

(3) The Δ-genus $\Delta(L)$, which is defined by

$$\Delta(L) := n + L^n - h^0(L).$$

* *Key words and phrases.* Polarized varieties, ample line bundles, nef and big line bundles, sectional genus, ith sectional geometric genus, ith sectional H-arithmetic genus, ith sectional arithmetic genus, adjoint bundles.

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Concerning classifications of polarized manifolds by using these invariants, the following results are obtained.

(1) The case of the degree.
A classification of (X, L) such that L is very ample with $L^n \leq 11$ has been obtained by Weil [27] for the case where $L^n \leq 3$, by Swinnerton Dyer [25] for the case where $L^n = 4$, by Okonek [21], [22], [23], [24] and Ionescu [15], [16] for the case where $L^n \leq 8$, by Fania-Livorni [2], [3] for the case where $n \geq 3$ and $L^n = 9, 10$, and by Besana-Biancofiore [1] for the case where $n \geq 3$ and $L^n = 11$.

(2) The case of the sectional genus.
If L is ample in general, then it is proved that $g(L)$ is non-negative and a classification of (X, L) with $0 \leq g(L) \leq 2$ has been obtained by Fujita [4], [5], [6] for the case where $g(L) \leq 2$ and Ionescu [17] for the case where $g(L) \leq 1$.

(3) The case of the Δ-genus.
In this case, we can prove that $\Delta(L) \geq 0$ and a classification of (X, L) with $\Delta(L) \leq 2$ has been obtained (see [6] in detail).

By using the above results, various interesting results about polarized manifolds are obtained and applied to another researches. But there is a limit to study polarized manifolds by using only above three invariants.

So, in order to study polarized manifolds more deeply, for every integer i with $0 \leq i \leq n$, the author introduced new invariants of (X, L). Let i be an integer with $0 \leq i \leq n$. In [7], [8] and [10], we proposed invariants of (X, L), the *ith sectional geometric genus* $g_i(X, L)$, the *ith sectional arithmetic genus* $p_a^i(X, L)$ and the *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$. (See Definition 1.3.) Here we note that the ith sectional geometric genus is thought to be a generalization of the degree and the sectional genus of polarized manifolds (see Remark 1.3 (2) and (3)).

Here we recall the reason why $g_i(X, L)$ (resp. $p_a^i(X, L)$, $\chi_i^H(X, L)$) is called the ith sectional geometric genus (resp. the ith sectional arithmetic genus, the ith sectional H-arithmetic genus) of (X, L). Let (X, L) be a polarized manifold of dimension $n \geq 2$ with $\mathrm{Bs}|L| = \emptyset$, where $\mathrm{Bs}|L|$ is the base locus of the complete linear system $|L|$. Let i be an integer with $1 \leq i \leq n$. Let X_{n-i} be the transversal intersection of general $n - i$ members of $|L|$. In this case X_{n-i} is a smooth projective variety of dimension i. Then we can prove that $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$ (resp. $p_a^i(X, L) = p_a(X_{n-i})$, $\chi_i^H(X, L) = \chi(\mathcal{O}_{X_{n-i}})$), that is, $g_i(X, L)$ (resp. $p_a^i(X, L)$, $\chi_i^H(X, L)$) is the geometric genus (resp. the arithmetic genus, the H-arithmetic genus (see Remark 1.2)) of X_{n-i}.

Furthermore, by this consideration, the ith sectional geometric genus (resp. the ith sectional arithmetic genus, the ith sectional H-arithmetic genus) is expected to have properties similar to those of the geometric genus (resp. the arithmetic genus, the H-arithmetic genus) of i-dimensional projective manifolds (see [8, Section 3]).

On the other hand, let $L_1, \ldots, L_n$ be ample line bundles on X and let $\mathcal{F}$ be a coherent sheaf on X. In [19], Kleiman defined the intersection number $(L_1 \cdots L_n, \mathcal{F})$ by using the coefficient of polynomial $\chi(L_1^{\otimes t_1} \otimes \cdots \otimes L_n^{\otimes t_n} \otimes \mathcal{F})$.

If $L_1 = \cdots = L_n = L$ and $\mathscr{F} = \mathcal{O}_X$, then $(L \cdots L, \mathcal{O}_X)$ is the degree of (X, L). Namely $(L_1 \cdots L_n, \mathscr{F})$ is a generalization of the degree. So the author thought that we can generalize the ith sectional invariants of polarized varieties.

In this paper, we define new invariants of $(X, L_1, \ldots, L_{n-i}; \mathscr{F})$, where X is a projective variety, $L_1, \ldots, L_{n-i}$ are line bundles on X and $\mathscr{F}$ is a coherent sheaf on X. In particular, we treat the *ith sectional H-arithmetic genus*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}),$$

the *ith sectional geometric genus*

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$$

and the *ith sectional arithmetic genus*

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$$

of $(X, L_1, \ldots, L_{n-i}; \mathscr{F})$, and we will study some properties of these when $L_1, \ldots, L_{n-i}$ are ample.

Here we explain the meaning of invariants introduced in this paper. Let X be a smooth projective variety of dimension n and set $\mathscr{F} = \mathcal{O}_X$ and let $L_1, \ldots, L_{n-i}$ be ample line bundles on X, where i is an integer with $1 \leq i \leq n$. Assume that $\mathrm{Bs}|L_j|$ is empty for every j with $1 \leq j \leq n - i$. Then by the Bertini theorem, there exists a smooth member $X_j \in |L_j|$ for any j such that $Y := X_1 \cap \cdots \cap X_{n-i}$ is a smooth projective variety of dimension i. Then

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X) = \chi(\mathcal{O}_Y),$$
$$g_i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X) = h^i(\mathcal{O}_Y)$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X) = p_a(Y).$$

This is the reason why we call these invariants the ith sectional H-arithmetic genus, the ith sectional geometric genus and the ith sectional arithmetic genus of $(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X)$. In particular if $L = L_1 = \cdots = L_{n-i}$, then

$$\chi_i^H(X, L, \ldots, L; \mathcal{O}_X) = \chi_i^H(X, L),$$
$$g_i(X, L, \ldots, L; \mathcal{O}_X) = g_i(X, L)$$

and

$$p_a^i(X, L, \ldots, L; \mathcal{O}_X) = p_a^i(X, L).$$

So these invariants are generalization of the invariants introduced in [7] and [10].

The purpose of this work is to give fundamental results about these invariants which will become cornerstones for further investigation of polarized manifolds, and to give some applications by using these invariants. Results in this work will be used in order to investigate various problems about polarized manifolds.

This work consists of three parts. This paper is the first part. The contents of the first part are the following.

In section 1, we remind the definition of the ith sectional H-arithmetic genus, the ith sectional geometric genus and the ith sectional arithmetic genus of polarized varieties and we pick up some definitions and results which are used in this paper.

In section 2, first of all, for every integer i with $0 \leq i \leq n$ we will define the ith sectional H-arithmetic genus

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}),$$

the ith sectional geometric genus

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$$

and the ith sectional arithmetic genus

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$$

for any multi-prepolarized variety $(X, L_1, \ldots, L_{n-i})$ and any coherent sheaf $\mathscr{F}$ on X. Next we will study some fundamental and general properties of these. Furthermore we will calculate these invariants of some multi-polarized manifolds, which often appear in the adjunction theory, and $\mathscr{F} = \mathscr{O}_X$. In this section we also give an explicit formula of $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{E})$, $g_i(X, L_1, \ldots, L_{n-i}; \mathscr{E})$ and $p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{E})$ for multi-quasi-polarized *manifolds* $(X, L_1, \ldots, L_{n-i})$ and vector bundles $\mathscr{E}$ by using intersection numbers.

In the second part [11] and the third part [12], we will investigate some properties of these invariants more deeply for the case where $i = 1$ and 2, and give some applications of these invariants. See [11] and [12] for further details.

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Notation and Conventions

We say that X is a *variety* if X is an integral separated scheme of finite type. In particular X is irreducible and reduced if X is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly a smooth projective variety. The words "line bundles" and "Cartier divisors" are used interchangeably. Sometimes the tensor products of line bundles are denoted additively.

$\mathscr{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.
$\mathscr{O}_X$: the structure sheaf of X.
$\chi(\mathscr{F})$: the Euler-Poincaré characteristic of a coherent sheaf $\mathscr{F}$.
$h^i(\mathscr{F}) := \dim H^i(X, \mathscr{F})$ for a coherent sheaf $\mathscr{F}$ on X.
$h^i(D) := h^i(\mathscr{O}(D))$ for a divisor D.
$D|_C$: the restriction of D to C.
$|D|$: the complete linear system associated with a divisor D.
K_X: the canonical divisor of a smooth projective variety X.
$q(X)$ (or q): the irregularity $h^1(\mathscr{O}_X)$ of a smooth projective variety X.
$\kappa(D)$: the Iitaka dimension of a Cartier divisor D on X.
$\kappa(X)$: the Kodaira dimension of X.

$\mathbf{P}^n$: the projective space with $\dim X = n$.
$\mathbf{Q}^n$: a quadric hypersurface in $\mathbf{P}^{n+1}$.
$\mathbf{P}_Y(\mathscr{E})$: the $\mathbf{P}^{r-m+1}$-bundle associated with a locally free sheaf $\mathscr{E}$ of rank r over an m-dimensional variety Y.
$H(\mathscr{E})$: the tautological invertible sheaf of $\mathbf{P}_Y(\mathscr{E})$.
$\sim$ (or $=$): linear equivalence.
$\equiv$: numerical equivalence.
$\mathrm{Pic}(X)$: the Picard group of X.
For a real number m and a non-negative integer n, let

$$[m]^n := \begin{cases} m(m+1)\cdots(m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

$$[m]_n := \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[m]^n$ and $[m]_n$ are polynomials in m whose degree are n.

For any non-negative integer n,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers with $n \geq 0$. Then we put

$$\binom{m}{n} := \frac{[m]_n}{n!}$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$, and $\binom{m}{0} = 1$.

1. Preliminaries for the first part

Here we will give some definitions and results which are used in this paper.

DEFINITION 1.1. Let X be a complex projective variety of dimension n, and let $L_1, \ldots, L_k$ be line bundles (resp. ample line bundles, nef and big line bundles) on X. Then $(X, L_1, \ldots, L_k)$ is called a *multi-prepolarized* (resp. *multi-polarized*, *multi-quasi-polarized*) variety of type k. If $k = 1$, then it is called a *prepolarized* (resp. *polarized*, *quasi-polarized*) *variety*.

If X is smooth, then $(X, L_1, \ldots, L_k)$ is called a multi-prepolarized (resp. multi-polarized, multi-quasi-polarized) *manifold* of type k.

NOTATION 1.1. Let (X, L) be a prepolarized variety of dimension n. Then the Euler-Poincaré characteristic $\chi(tL)$ is a polynomial in t of at most total degree n. We set

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \frac{[t]^j}{j!}.$$

DEFINITION 1.2. Let (X, L) be a quasi-polarized variety of dimension n.
(1) The degree L^n of (X, L) is defined by $L^n := \chi_n(X, L)$.
(2) The sectional genus $g(L)$ of (X, L) is defined by $g(L) := 1 - \chi_{n-1}(X, L)$.

Remark 1.1. If X is smooth, then by the Riemann-Roch theorem we see that

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

Remark 1.2. Let X be a projective variety of dimension n. Then

$$\chi(\mathcal{O}_X) := \sum_{i=0}^{n} (-1)^i h^i(\mathcal{O}_X)$$

is called the *arithmetic genus of X in the sense of Hirzebruch*. In this paper, we also call this $\chi(\mathcal{O}_X)$ the *H-arithmetic genus of X*.

DEFINITION 1.3 ([7], [8], [10]). Let (X, L) be a polarized variety of dimension n and let i be an integer with $0 \leq i \leq n$.
(1) The *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$ is defined by the following:

$$\chi_i^H(X, L) = \chi_{n-i}(X, L).$$

(2) The *ith sectional geometric genus* $g_i(X, L)$ is defined by the following:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(3) The *ith sectional arithmetic genus* $p_a^i(X, L)$ is defined by the following:

$$p_a^i(X, L) = (-1)^i(\chi_{n-i}(X, L) - h^0(\mathcal{O}_X)).$$

Remark 1.3. (0) Since X is a complex projective variety, we have $h^0(\mathcal{O}_X) = 1$. Hence $p_a^i(X, L) = (-1)^i(\chi_{n-i}(X, L) - 1)$.
(1) Since $\chi_j(X, L) \in \mathbf{Z}$ for every integer j with $0 \leq j \leq n$, by definition we get $\chi_i^H(X, L) \in \mathbf{Z}$, $g_i(X, L) \in \mathbf{Z}$, and $p_a^i(X, L) \in \mathbf{Z}$ for every integer i with $0 \leq i \leq n$.
(2) If $i = 0$, then $g_0(X, L) = \chi_0^H(X, L) = L^n$ and $p_a^0(X, L) = L^n - 1$.
(3) If $i = 1$, then

$$\chi_1^H(X, L) = 1 - g(L),$$
$$g_1(X, L) = g(L),$$
$$p_a^1(X, L) = g(L).$$

(4) If $i = n$, then

$$\chi_n^H(X,L) = \chi(\mathcal{O}_X),$$
$$g_n(X,L) = h^n(\mathcal{O}_X),$$
$$p_a^n(X,L) = (-1)^n(\chi(\mathcal{O}_X) - 1).$$

DEFINITION 1.4. Let $(X, L_1, \ldots, L_k)$ be an n-dimensional multi-polarized manifold of type k, where k is a positive integer. Then $(X, L_1, \ldots, L_k)$ is called a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal variety* W if there exists a fiber space $f : X \to W$ such that $\dim W = n - k + 1$ (resp. $n-k$, $n-k-1$) and $K_X + L_1 + \cdots + L_k = f^*(A)$ for an ample line bundle A on W. We say that a polarized manifold (X, L) is a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal variety* Y *with* $\dim Y = m$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)L = f^*(A)$ (resp. $K_X + (n-m)L = f^*(A)$, $K_X + (n-m-1)L = f^*(A)$) for some ample line bundle A on Y.

DEFINITION 1.5. Let k be a positive integer.

(1) Let $(X, L_1, \ldots, L_k)$ and $(Y, A_1, \ldots, A_k)$ be n-dimensional multi-polarized manifolds of type k. Then $(X, L_1, \ldots, L_k)$ is called a *simple blowing up of a multi-polarized manifold* $(Y, A_1, \ldots, A_k)$ *of type* k if there exists a blowing up $\pi : X \to Y$ at a point $y \in Y$ such that $L_j = \pi^*(A_j) - E$ and $E|_E \cong \mathcal{O}_{\mathbf{P}^{n-1}}(-1)$ for every integer j with $1 \le j \le k$, where $E \cong \mathbf{P}^{n-1}$ is the exceptional effective divisor.

(2) A multi-polarized manifold $(\tilde{X}, \widetilde{L_1}, \ldots, \widetilde{L_k})$ of type k is called a *reduction of* $(X, L_1, \ldots, L_k)$ if there exists a birational morphism

$$\pi : (X, L_1, \ldots, L_k) \to (\tilde{X}, \widetilde{L_1}, \ldots, \widetilde{L_k})$$

such that π is a composite of simple blowing ups and $(\tilde{X}, \widetilde{L_1}, \ldots, \widetilde{L_k})$ is not simple blowing up of another multi-polarized manifold of type k. This π is called the *reduction map*.

Remark 1.4. Let $(X, L_1, \ldots, L_k)$ be a multi-polarized manifold of type k, where k is an integer with $1 \le k \le n-1$.

(1) If $(X, L_1, \ldots, L_k)$ is not a simple blowing up of another multi-polarized manifold of type k, then we regard $(X, L_1, \ldots, L_k)$ as a reduction of itself. Then there always exists a reduction of $(X, L_1, \ldots, L_k)$.

(2) Let $(M, A_1, \ldots, A_k)$ be a reduction of $(X, L_1, \ldots, L_k)$. Let ϕ be its reduction map. Assume that L_j is spanned for every integer j with $1 \le j \le k$. Then by Bertini's theorem there exist $Y_j \in |L_j|$ such that Y_j and X_j are smooth for every integer j with $1 \le j \le k$, where $X_j := Y_1 \cap \cdots \cap Y_j$. Let $N_j := \phi(Y_j)$ and $M_j := \phi(X_j)$. Then $N_j \in |A_j|$, and M_j and N_j are smooth. Furthermore $(M_j, A_{j+1}|_{M_j}, \ldots, A_k|_{M_j})$ is a reduction of $(X_j, L_{j+1}|_{X_j}, \ldots, L_k|_{X_j})$ for every integer j with $1 \le j \le k-1$.

DEFINITION 1.6 (See e.g. [13, Chapter 3, 3.2 Chern Classes]). Let X be a smooth projective variety of dimension n, and let $\mathscr{E}$ be a vector bundle of rank r on X.

(1) The *Chern polynomial* $c_t(\mathscr{E})$ is defined by the following.

$$c_t(\mathscr{E}) := \sum_{i=0}^{r} c_i(\mathscr{E})t^i.$$

(2) If the Chern polynomial of $\mathscr{E}$ is factored as follows

$$c_t(\mathscr{E}) = \Pi_{i=1}^{r}(1 + \alpha_i t),$$

then $\alpha_1, \ldots, \alpha_r$ are called *Chern roots of* $\mathscr{E}$.

(3) The *Chern character* $\mathrm{ch}(\mathscr{E})$ *of* $\mathscr{E}$ is defined by the formula:

$$\mathrm{ch}(\mathscr{E}) := \sum_{i=1}^{r} \exp(\alpha_i),$$

where $\exp(\alpha_i) := e^{\alpha_i}$. We denote $\mathrm{ch}(\mathscr{E})_l$ as the lth part of $\mathrm{ch}(\mathscr{E})$. For example $\mathrm{ch}(\mathscr{E})_0 = r$, $\mathrm{ch}(\mathscr{E})_1 = c_1(\mathscr{E})$, $\mathrm{ch}(\mathscr{E})_2 = (1/2)(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E}))$, and so on.

Remark 1.5. Let X be as in Definition 1.6. Let $\mathscr{E}$ and $\mathscr{E}'$ be vector bundles on X. Then

$$\mathrm{ch}(\mathscr{E} \otimes \mathscr{E}') = \mathrm{ch}(\mathscr{E})\,\mathrm{ch}(\mathscr{E}').$$

PROPOSITION 1.1. *Let* $x_0 = 1$ *and let* x_i *be an indeterminate of weight* i *for every integer* i *with* $i \geq 1$. *For any non-negative integer* k, *there exist unique polynomials of weight* k, $T_k \in \mathbf{Q}[x_1, \ldots, x_k]$, *such that the following properties hold*:

(1) $T_0 = 1$.

(2) *For any formal power series* $\sum_{i=0}^{\infty} x_i t^i$, *we set*

$$td_t\left(\sum_{i=0}^{\infty} x_i t^i\right) = \sum_{k=0}^{\infty} T_k(x_1, \ldots, x_k)t^k,$$

where t *is an indeterminate.*

If

$$\sum_{i=0}^{\infty} x_i t^i = \left(\sum_{i=0}^{\infty} y_i t^i\right)\left(\sum_{i=0}^{\infty} z_i t^i\right),$$

then

$$td_t\left(\sum_{i=0}^{\infty} x_i t^i\right) = \left(td_t\left(\sum_{i=0}^{\infty} y_i t^i\right)\right)\left(td_t\left(\sum_{i=0}^{\infty} z_i t^i\right)\right).$$

(3) *For the linear expression* $1 + xt$,

$$td_t(1 + xt) = \frac{xt}{1 - \exp(-xt)},$$

where $\exp(-xt) := e^{-xt}$.

Proof. See [14, Chapter I, §1]. □

DEFINITION 1.7. (1) Polynomials $T_k \in \mathbf{Q}[x_1, \ldots, x_k]$ in Proposition 1.1 is called the *Todd polynomial of weight k*.

(2) Let X be a smooth projective variety and let $\mathscr{F}$ be a vector bundle on X. Let $c_t(\mathscr{F}) = \sum_{i \geq 0} c_i(\mathscr{F})t^i$ be the Chern polynomial of $\mathscr{F}$. We set

$$\mathrm{td}_t(\mathscr{F}) = \mathrm{td}\left(\sum_{i \geq 0} c_i(\mathscr{F})t^i\right) = \sum_{k=0}^{\infty} T_k(c_1(\mathscr{F}), \ldots, c_k(\mathscr{F}))t^k,$$

where t is an indeterminate. Furthermore, we set

$$\mathrm{td}(\mathscr{F}) := \sum_{k=0}^{\infty} T_k(c_1(\mathscr{F}), \ldots, c_k(\mathscr{F})).$$

Then $\mathrm{td}(\mathscr{F})$ (resp. $T_k(c_1(\mathscr{F}), \ldots, c_k(\mathscr{F}))$) is called the *Todd class of* $\mathscr{F}$ (resp. *Todd polynomial of weight k of* $\mathscr{F}$).

(3) Let X be a smooth projective variety. Then we set

$$T_k(X) := T_k(c_1(\mathscr{T}_X), \ldots, c_k(\mathscr{T}_X)),$$

where $\mathscr{T}_X$ is the tangent bundle of X.

Remark 1.6. For a formal power series $\sum_{k=0}^{\infty} p_k t^k$, the Todd polynomial of weight 1 (resp. 2 and 3) $T_1(p_1)$ (resp. $T_2(p_1, p_2)$ and $T_3(p_1, p_2, p_3)$) is the following:

$$T_1(p_1) = \frac{1}{2}p_1.$$

$$T_2(p_1, p_2) = \frac{1}{12}(p_2 + p_1^2).$$

$$T_3(p_1, p_2, p_3) = \frac{1}{24}p_1p_2.$$

$$T_4(p_1, p_2, p_3, p_4) = \frac{1}{720}(-p_1^4 + 4p_1^2p_2 + 3p_2^2 + p_1p_3 - p_4).$$

(See [14, Chapter I].)

THEOREM 1.1 (Hirzebruch-Riemann-Roch). *Let X be a smooth complete variety and let $\mathscr{E}$ be a locally free sheaf on X. Then*

$$\chi(\mathscr{E}) = \int_X \mathrm{ch}(\mathscr{E})\ \mathrm{td}(\mathscr{T}_X),$$

where $\mathscr{T}_X$ *is the tangent bundle of* X, $\mathrm{ch}(\mathscr{E})$ (*resp.* $\mathrm{td}(\mathscr{T}_X)$) *is the Chern character of* $\mathscr{E}$ (*resp. the Todd class of* $\mathscr{T}_X$), *and* $\int_X$ *denotes the degree of the zero-dimensional component of* $(\mathrm{ch}(\mathscr{E})\ \mathrm{td}(\mathscr{T}_X)) \cap [X]$.

Proof. See [14, Chapter IV]. □

NOTATION 1.2. Let X be a smooth projective variety of dimension n and let i be an integer with $1 \leq i \leq n-1$. Let $L_1, \ldots, L_{n-i}$ be nef and big line bundles on X. Assume that $\mathrm{Bs}|L_j| = \emptyset$ for every integer j with $1 \leq j \leq n-i$. Then by Bertini's theorem, for every integer j with $1 \leq j \leq n-i$, there exists a general member $X_j \in |L_j|_{X_{j-1}}|$ such that X_j is a smooth projective variety of dimension $n-j$. (Here we set $X_0 := X$.)

2. Definitions and fundamental properties

NOTATION 2.1. Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X. Let $\mathscr{F}$ be a coherent sheaf on X and let $s = \dim \mathrm{Supp}(\mathscr{F})$. Then $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F})$ is a polynomial in $t_1, \ldots, t_{n-i}$ of total degree at most s. We note that $s \leq n$. So we can write $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F})$ uniquely as follows.

$$\begin{aligned}
&\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F}) \\
&\quad = \sum_{p=0}^{n} \sum_{\substack{p_1 \geq 0, \ldots, p_{n-i} \geq 0 \\ p_1 + \cdots + p_{n-i} = p}} \chi_{p_1, \ldots, p_{n-i}}(L_1, \ldots, L_{n-i}; \mathscr{F}) \\
&\qquad \times \binom{t_1 + p_1 - 1}{p_1} \cdots \binom{t_{n-i} + p_{n-i} - 1}{p_{n-i}}.
\end{aligned}$$

Remark 2.1. Let X be a *complete algebraic scheme*, let $L_1, \ldots, L_u$ be invertible sheaves on X, and let $\mathscr{F}$ be a coherent sheaf on X. Let $s := \dim \mathrm{Supp}(\mathscr{F})$. Then $\chi(L_1^{t_1} \otimes \cdots \otimes L_u^{t_u} \otimes \mathscr{F})$ is a numerical polynomial in $t_1, \ldots, t_u$ of at most total degree s and can be written as follows.

$$\begin{aligned}
&\chi(L_1^{t_1} \otimes \cdots \otimes L_u^{t_u} \otimes \mathscr{F}) \\
&\quad = \sum_{p=0}^{n} \sum_{\substack{p_1 \geq 0, \ldots, p_u \geq 0 \\ p_1 + \cdots + p_u = p}} \chi_{p_1, \ldots, p_u}(L_1, \ldots, L_u; \mathscr{F}) \binom{t_1 + p_1 - 1}{p_1} \cdots \binom{t_u + p_u - 1}{p_u}.
\end{aligned}$$

DEFINITION 2.1. Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X. Let $\mathscr{F}$ be a coherent sheaf on X.

(1) The *ith sectional H-arithmetic genus* $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F})$ is defined by the following:

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = \begin{cases} \chi_{\underbrace{1, \ldots, 1}_{n-i}}(L_1, \ldots, L_{n-i}; \mathscr{F}) & \text{if } 0 \le i \le n-1, \\ \chi(\mathscr{F}) & \text{if } i = n. \end{cases}$$

(2) The *ith sectional geometric genus* $g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$ is defined by the following:

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}) - \chi(\mathscr{F})) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathscr{F}).$$

(3) The *ith sectional arithmetic genus* $p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$ is defined by the following:

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}) - h^0(\mathscr{F})).$$

Remark 2.2. (1) We can prove that $\chi_{p_1, \ldots, p_{n-i}}(L_1, \ldots, L_{n-i}; \mathscr{F})$ is an integer for every non-negative integers $p_1, \ldots, p_{n-i}$ with $0 \le p_1 + \cdots + p_{n-i} \le n$ (see Theorem 2.1 and Theorem 2.2). So in particular we see that

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}), \quad g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F}), \quad \text{and} \quad p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$$

are integers (see also Corollary 2.2).

(2) We set

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) := \chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{O}_X),$$
$$g_i(X, L_1, \ldots, L_{n-i}) := g_i(X, L_1, \ldots, L_{n-i}; \mathscr{O}_X),$$
$$p_a^i(X, L_1, \ldots, L_{n-i}) := p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{O}_X).$$

(3) If $i = 0$, then

$$\chi_0^H(X, L_1, \ldots, L_n) = g_0(X, L_1, \ldots, L_n) = p_a^0(X, L_1, \ldots, L_n) + 1 = L_1 \cdots L_n.$$

(4) If $i = 1$, then $g_1(X, L_1, \ldots, L_{n-1}) = p_a^1(X, L_1, \ldots, L_{n-1})$.

(5) If $i = n$, then for any coherent sheaf $\mathscr{F}$ on X we get

$$\chi_n^H(X; \mathscr{F}) = \chi(\mathscr{F}),$$
$$g_n(X; \mathscr{F}) = h^n(\mathscr{F}),$$
$$p_a^n(X; \mathscr{F}) = (-1)^n(\chi(\mathscr{F}) - h^0(\mathscr{F})).$$

Hence if $\mathscr{F} = \mathscr{O}_X$, then

$$\chi_n^H(X;\mathcal{O}_X) = \chi(\mathcal{O}_X),$$
$$g_n(X;\mathcal{O}_X) = h^n(\mathcal{O}_X),$$
$$p_a^n(X;\mathcal{O}_X) = (-1)^n(\chi(\mathcal{O}_X) - h^0(\mathcal{O}_X)).$$

(6) We note that we can define $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{F})$, $g_i(X, L_1, \ldots, L_{n-i}; \mathcal{F})$ and $p_a^i(X, L_1, \ldots, L_{n-i}; \mathcal{F})$ on any complete algebraic scheme (see Remark 2.1 and Definition 2.1).

LEMMA 2.1. *Let X be a projective variety of dimension n and let L be a line bundle on X. Let i be an integer with $0 \le i \le n-1$ and let p be an integer with $n - i \le p$. Then*

$$\chi_{p_1,\ldots,p_{n-i}}(L, \ldots, L; \mathcal{O}_X) = \chi_p(X, L)$$

for any non-negative integers $p_1, \ldots, p_{n-i}$ with $p_1 + \cdots + p_{n-i} = p$.

Proof. We set $t := t_1 + \cdots + t_{n-i}$. If we can prove that

$$\frac{1}{p!}t(t+1)\cdots(t+p-1) = \sum_{\substack{p_1 \ge 0,\ldots,p_{n-i}\ge 0 \\ p_1+\cdots+p_{n-i}=p}} \binom{t_1+p_1-1}{p_1} \cdots \binom{t_{n-i}+p_{n-i}-1}{p_{n-i}},$$

then

$$\begin{aligned}
&\chi(L^{t_1} \otimes \cdots \otimes L^{t_{n-i}}) \\
&\quad = \chi(L^t) \\
&\quad = \sum_{p=0}^{n} \frac{\chi_p(L)}{p!} t(t+1)\cdots(t+p-1) \\
&\quad = \sum_{p=0}^{n} \sum_{\substack{p_1 \ge 0,\ldots,p_{n-i}\ge 0 \\ p_1+\cdots+p_{n-i}=p}} \chi_p(L) \binom{t_1+p_1-1}{p_1} \cdots \binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.
\end{aligned}$$

Hence by the definition of $\chi_{p_1,\ldots,p_{n-i}}(L, \ldots, L; \mathcal{O}_X)$ we get the assertion. So we will prove the above equality.

First we prove the following.

CLAIM 2.1. $[t_1 + t_2]^p = \sum_{k=0}^{p} \binom{p}{k} [t_1]^k [t_2]^{p-k}$.

Proof. (See [20, Chapter 1, Excercise 1.42 (i)].)

$$\begin{aligned}
\sum_{k=0}^{p} \binom{p}{k} [t_1]^k [t_2]^{p-k} &= \sum_{k=0}^{p} p! \binom{t_1+k-1}{k} \binom{t_2+p-k-1}{p-k} \\
&= p! \sum_{k=0}^{p} \binom{t_1-1+k}{k} \binom{t_2+p-1-k}{p-k}
\end{aligned}$$

$$= p!\binom{t_1 + t_2 + p - 1}{p}$$
$$= [t_1 + t_2]^p. \qquad \square$$

By using this formula, we have

$$\begin{aligned}
&t(t+1)\cdots(t+p-1)\\
&\quad = [t]^p\\
&\quad = [t_1 + \cdots + t_{n-i}]^p\\
&\quad = \sum_{p_1=0}^{p}\binom{p}{p_1}[t_1]^{p_1}[t_2 + \cdots + t_{n-i}]^{p-p_1}\\
&\quad = \sum_{p_1=0}^{p}\binom{p}{p_1}[t_1]^{p_1}\left(\sum_{p_2=0}^{p-p_1}\binom{p-p_1}{p_2}[t_2]^{p_2}[t_3 + \cdots + t_{n-i}]^{p-p_1-p_2}\right)\\
&\quad = \cdots\\
&\quad = \sum_{p_1=0}^{p}\binom{p}{p_1}[t_1]^{p_1}\left(\sum_{p_2=0}^{p-p_1}\binom{p-p_1}{p_2}[t_2]^{p_2}\left(\cdots\right.\right.\\
&\qquad \left.\left.\times\left(\sum_{p_{n-i-1}=0}^{p-p_1-\cdots-p_{n-i-2}}\binom{p-p_1-\cdots-p_{n-i-2}}{p_{n-i-1}}[t_{n-i-1}]^{p_{n-i-1}}[t_{n-i}]^{p_{n-i}}\right)\right)\right)\\
&\quad = \sum_{\substack{p_1\geq 0,\ldots,p_{n-i}\geq 0\\ p_1+\cdots+p_{n-i}=p}} p!\,\frac{[t_1]^{p_1}}{p_1!}\cdots\frac{[t_{n-i}]^{p_{n-i}}}{p_{n-i}!}\\
&\quad = p!\sum_{\substack{p_1\geq 0,\ldots,p_{n-i}\geq 0\\ p_1+\cdots+p_{n-i}=p}}\binom{t_1+p_1-1}{p_1}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.
\end{aligned}$$

Therefore we get the assertion. $\square$

By Corollary 2.1 below we see that the above notion is a generalization of the sectional H-arithmetic (resp. geometric, arithmetic) genus of polarized varieties.

COROLLARY 2.1. *Let X be a projective variety of dimension n and let L be a line bundle on X. Let i be an integer with $0 \leq i \leq n$. Then*

$$\chi_i^H(X, L, \ldots, L) = \chi_i^H(X, L),$$
$$g_i(X, L, \ldots, L) = g_i(X, L),$$
$$p_a^i(X, L, \ldots, L) = p_a^i(X, L).$$

In general we get the following theorem.

THEOREM 2.1. *Let X be a projective variety of dimension n. Let $L_1, \dots, L_q$ be line bundles on X, where q is a positive integer, and let $\mathscr{F}$ be a coherent sheaf on X. Then we write $\chi(L_1^{t_1} \otimes \cdots \otimes L_q^{t_q} \otimes \mathscr{F})$ as follows*:

$$\chi(L_1^{t_1} \otimes \cdots \otimes L_q^{t_q} \otimes \mathscr{F}) = \sum_{u=0}^{n} \sum_{\substack{w_1 \geq 0, \dots, w_q \geq 0 \\ w_1 + \cdots + w_q = u}} \chi_{w_1, \dots, w_q}(L_1, \dots, L_q; \mathscr{F}) \binom{t_1 + w_1 - 1}{w_1} \cdots \binom{t_q + w_q - 1}{w_q}.$$

Then

$$\chi_{w_1, \dots, w_q}(L_1, \dots, L_q; \mathscr{F}) = \chi^H_{n - w(q)}(\underbrace{L_1, \dots, L_1}_{w_1}, \dots, \underbrace{L_q, \dots, L_q}_{w_q}; \mathscr{F}),$$

where $w(q) := \sum_{j=1}^{q} w_j$.

Proof. We consider

$$\chi(\underbrace{L_1^{s_1} \otimes \cdots \otimes L_1^{s_{w_1}}}_{w_1} \otimes \underbrace{L_2^{s_{w_1+1}} \otimes \cdots \otimes L_2^{s_{w_1+w_2}}}_{w_2} \otimes \cdots \otimes \underbrace{L_q^{s_{w(q-1)+1}} \otimes \cdots \otimes L_q^{s_{w(q)}}}_{w_q} \otimes \mathscr{F}).$$

Then

$$\chi(L_1^{s_1} \otimes \cdots \otimes L_1^{s_{w_1}} \otimes L_2^{s_{w_1+1}} \otimes \cdots \otimes L_2^{s_{w_1+w_2}} \otimes \cdots \otimes L_q^{s_{w(q-1)+1}} \otimes \cdots \otimes L_q^{s_{w(q)}} \otimes \mathscr{F})$$
$$= \sum_{u=0}^{n} \sum_{\substack{r_1 \geq 0, \dots, r_{w(q)} \geq 0 \\ r_1 + \cdots + r_{w(q)} = u}} \chi_{r_1, \dots, r_{w(q)}}(\underbrace{L_1, \dots, L_1}_{w_1}, \dots, \underbrace{L_q, \dots, L_q}_{w_q}; \mathscr{F})$$
$$\times \binom{s_1 + r_1 - 1}{r_1} \cdots \binom{s_{w(q)} + r_{w(q)} - 1}{r_{w(q)}}.$$

Here we set

$$t_j := s_{w(j-1)+1} + s_{w(j-1)+2} + \cdots + s_{w(j)},$$

where $w(0) := 0$. Then by Claim 2.1 for every j with $1 \leq j \leq q$

$$[t_j]^{w_j} = \sum_{r_{w(j-1)+1}=0}^{w_j} \binom{w_j}{r_{w(j-1)+1}} [s_{w(j-1)+1}]^{r_{w(j-1)+1}} [s_{w(j-1)+2} + \cdots + s_{w(j)}]^{w_j - r_{w(j-1)+1}}$$
$$= \sum_{r_{w(j-1)+1}=0}^{w_j} \binom{w_j}{r_{w(j-1)+1}} [s_{w(j-1)+1}]^{r_{w(j-1)+1}}$$
$$\times \left(\sum_{r_{w(j-1)+2}=0}^{w_j - r_{w(j-1)+1}} \binom{w_j - r_{w(j-1)+1}}{r_{w(j-1)+2}} [s_{w(j-1)+2}]^{r_{w(j-1)+2}} \right.$$
$$\left. \times [s_{w(j-1)+3} + \cdots + s_{w(j)}]^{w_j - r_{w(j-1)+1} - r_{w(j-1)+2}} \right)$$

$$= \cdots$$

$$= (w_j)! \sum_{\substack{r_{w(j-1)+1} \geq 0, \ldots, r_{w(j)} \geq 0 \\ r_{w(j-1)+1} + \cdots + r_{w(j)} = w_j}} \frac{[s_{w(j-1)+1}]^{r_{w(j-1)+1}}}{(r_{w(j-1)+1})!} \cdots \frac{[s_{w(j)}]^{r_{w(j)}}}{(r_{w(j)})!}$$

$$= (w_j)! \sum_{\substack{r_{w(j-1)+1} \geq 0, \ldots, r_{w(j)} \geq 0 \\ r_{w(j-1)+1} + \cdots + r_{w(j)} = w_j}} \binom{s_{w(j-1)+1} + r_{w(j-1)+1} - 1}{r_{w(j-1)+1}} \cdots \binom{s_{w(j)} + r_{w(j)} - 1}{r_{w(j)}}.$$

Hence if $r_{w(j-1)+1} + \cdots + r_{w(j)} = w_j$ for every j with $1 \leq j \leq q$, then

$$\chi_{w_1, \ldots, w_q}(L_1, \ldots, L_q; \mathscr{F}) = \chi_{r_1, \ldots, r_{w(q)}}(\underbrace{L_1, \ldots, L_1}_{w_1}, \ldots, \underbrace{L_q, \ldots, L_q}_{w_q}; \mathscr{F}).$$

In particular,

$$\chi_{w_1, \ldots, w_q}(L_1, \ldots, L_q; \mathscr{F}) = \chi_{\underbrace{1, \ldots, 1}_{w(q)}}(\underbrace{L_1, \ldots, L_1}_{w_1}, \ldots, \underbrace{L_q, \ldots, L_q}_{w_q}; \mathscr{F}).$$

Therefore we get the assertion. □

For a given projective variety X and line bundles $L_1, \ldots, L_{n-i}$, we can calculate the ith sectional H-arithmetic genus, the ith sectional geometric genus and the ith sectional arithmetic genus by the following theorem and corollaries. Here we give the following notation which is used below.

NOTATION 2.2. We set

$$S(j)_u = \{(p_1, \ldots, p_j) \mid p_m \in \mathbf{Z}, 0 \leq p_m \leq 1, \#\{m \mid p_m = 1\} = u\}.$$

THEOREM 2.2. *Let X be a projective variety of dimension n, and let i be an integer with $0 \leq i \leq n-1$. Let $L_1, \ldots, L_{n-i}$ be line bundles on X, and let $\mathscr{F}$ be a coherent sheaf on X. Then for any integer j with $0 \leq j \leq n-i$*

$$(-1)^j \sum_{(q_1, \ldots, q_{n-i}) \in S(n-i)_j} \chi_{q_1, \ldots, q_{n-i}}(L_1, \ldots, L_{n-i}; \mathscr{F})$$

$$= \sum_{u=0}^{j} \left\{ (-1)^{j-u} \binom{n-i-u}{j-u} \sum_{(p_1, \ldots, p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\}.$$

Proof. By using the equation in Notation 2.1 and taking sums of $\chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F})$ for any $(p_1, \ldots, p_{n-i}) \in S(n-i)_j$ we get the following.

$$(1)_j \quad \sum_{r=0}^{j} \left\{ (-1)^{j-r} \binom{n-i-j+r}{r} \sum_{(q_1, \ldots, q_{n-i}) \in S(n-i)_{j-r}} \chi_{q_1, \ldots, q_{n-i}}(L_1, \ldots, L_{n-i}; \mathscr{F}) \right\}$$

$$= \sum_{(p_1, \ldots, p_{n-i}) \in S(n-i)_j} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}).$$

We prove this theorem by induction on j. If $j = 0$, then this is true. Assume that the equation is true for $j \leq k-1$ with an integer k with $1 \leq k \leq n-i$. We consider the case where $j = k$. Then by $(1)_j$ we get

$$(1)_k \qquad \sum_{r=0}^{k} \left\{ (-1)^{k-r} \binom{n-i-k+r}{r} \sum_{(q_1,\ldots,q_{n-i}) \in S(n-i)_{k-r}} \chi_{q_1,\ldots,q_{n-i}}(L_1,\ldots,L_{n-i};\mathscr{F}) \right\}$$
$$= \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_k} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}).$$

Let p be an integer with $1 \leq p \leq k$. By induction hypothesis we get

$$(2) \qquad (-1)^{k-p} \binom{n-i-k+p}{p} \sum_{(q_1,\ldots,q_{n-i}) \in S(n-i)_{k-p}} \chi_{q_1,\ldots,q_{n-i}}(L_1,\ldots,L_{n-i};\mathscr{F})$$
$$= \sum_{u=0}^{k-p} \left\{ (-1)^{k-p-u} \binom{n-i-u}{k-p-u} \right.$$
$$\left. \times \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\}$$
$$\times \binom{n-i-k+p}{p}.$$

Here we note that

$$(-1)^0 \binom{n-i-k+0}{0} = 1.$$

By $(1)_k$ and (2) we see that

$$(-1)^k \sum_{(q_1,\ldots,q_{n-i}) \in S(n-i)_k} \chi_{q_1,\ldots,q_{n-i}}(L_1,\ldots,L_{n-i};\mathscr{F})$$
$$= \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_k} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F})$$
$$- \sum_{p=1}^{k} \left\{ (-1)^{k-p} \binom{n-i-k+p}{p} \right.$$
$$\left. \times \sum_{(q_1,\ldots,q_{n-i}) \in S(n-i)_{k-p}} \chi_{q_1,\ldots,q_{n-i}}(L_1,\ldots,L_{n-i};\mathscr{F}) \right\}$$

$$
\begin{aligned}
&= \sum_{(p_1,\dots,p_{n-i})\in S(n-i)_k} \chi(L_1^{-p_1}\otimes\cdots\otimes L_{n-i}^{-p_{n-i}}\otimes\mathscr{F}) \\
&\quad - \sum_{p=1}^{k}\left[\sum_{u=0}^{k-p}\left\{(-1)^{k-p-u}\binom{n-i-u}{k-p-u}\right.\right. \\
&\qquad \left.\left.\times \sum_{(p_1,\dots,p_{n-i})\in S(n-i)_u} \chi(L_1^{-p_1}\otimes\cdots\otimes L_{n-i}^{-p_{n-i}}\otimes\mathscr{F})\right\}\right. \\
&\qquad \left.\times\binom{n-i-k+p}{p}\right] \\
&= \sum_{(p_1,\dots,p_{n-i})\in S(n-i)_k} \chi(L_1^{-p_1}\otimes\cdots\otimes L_{n-i}^{-p_{n-i}}\otimes\mathscr{F}) \\
&\quad - \sum_{q=1}^{k}\left\{\sum_{p=1}^{q}(-1)^{q-p}\binom{n-i-k+p}{p}\binom{n-i-k+q}{q-p}\right\} \\
&\quad \times \sum_{(p_1,\dots,p_{n-i})\in S(n-i)_{k-q}} \chi(L_1^{-p_1}\otimes\cdots\otimes L_{n-i}^{-p_{n-i}}\otimes\mathscr{F}).
\end{aligned}
$$

CLAIM 2.2.

$$
\sum_{p=1}^{q}(-1)^{q-p}\binom{n-i-k+p}{p}\binom{n-i-k+q}{q-p} = (-1)^{q-1}\binom{n-i-k+q}{q}.
$$

Proof of this claim. First we get

$$
\begin{aligned}
\binom{n-i-k+p}{p}\binom{n-i-k+q}{q-p} &= \frac{(n-i-k+p)!}{(n-i-k)!p!}\times\frac{(n-i-k+q)!}{(n-i-k+p)!(q-p)!} \\
&= \frac{q!}{(q-p)!p!}\times\frac{(n-i-k+q)!}{(n-i-k)!q!} \\
&= \binom{q}{q-p}\binom{n-i-k+q}{q}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{p=1}^{q}(-1)^{q-p}\binom{n-i-k+p}{p}\binom{n-i-k+q}{q-p} \\
&\quad = \sum_{p=1}^{q}(-1)^{q-p}\binom{q}{q-p}\binom{n-i-k+q}{q}
\end{aligned}
$$

$$= \binom{n-i-k+q}{q} \sum_{p=1}^{q} (-1)^{q-p} \binom{q}{q-p}$$

$$= (-1)^{q-1} \binom{n-i-k+q}{q}.$$

This completes the proof of Claim 2.2. □

Hence

$$\begin{aligned}
&(-1)^k \sum_{(q_1,\dots,q_{n-i}) \in S(n-i)_k} \chi_{q_1,\dots,q_{n-i}}(L_1,\dots,L_{n-i};\mathscr{F}) \\
&\quad = \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_k} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \\
&\qquad + \sum_{q=1}^{k} \left\{ (-1)^q \binom{n-i-k+q}{q} \right. \\
&\qquad\qquad \left. \times \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_{k-q}} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\} \\
&\quad = \sum_{q=0}^{k} \left\{ (-1)^q \binom{n-i-k+q}{q} \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_{k-q}} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\} \\
&\quad = \sum_{u=0}^{k} \left\{ (-1)^{k-u} \binom{n-i-u}{k-u} \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\}.
\end{aligned}$$

Hence we get the assertion. □

COROLLARY 2.2. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1,\dots,L_{n-i}$ be line bundles on X. Then*

$$\begin{aligned}
&\chi_i^H(X,L_1,\dots,L_{n-i};\mathscr{F}) \\
&\qquad = \sum_{u=0}^{n-i} \left\{ (-1)^u \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\},
\end{aligned}$$

$$\begin{aligned}
&g_i(X,L_1,\dots,L_{n-i};\mathscr{F}) \\
&\quad = (-1)^i \left(\sum_{u=0}^{n-i} \left\{ (-1)^u \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\} - \chi(\mathscr{F}) \right) \\
&\qquad + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathscr{F}),
\end{aligned}$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = (-1)^i \left(\sum_{u=0}^{n-i} \left\{ (-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\} - h^0(\mathscr{F}) \right).$$

Proof. By Theorem 2.2 for $j = n - i$, we get

$$(-1)^{n-i} \chi_{1,\ldots,1}(L_1, \ldots, L_{n-i}; \mathscr{F}) = \sum_{u=0}^{n-i} \left\{ (-1)^{n-i-u} \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\}.$$

Hence

$$\chi_{1,\ldots,1}(L_1, \ldots, L_{n-i}; \mathscr{F}) = \sum_{u=0}^{n-i} \left\{ (-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{F}) \right\}.$$

So we get the assertion by definition. □

Remark 2.3. By Corollary 2.2, we see that

$$\chi_i^H(X, L_1^{s_1}, \ldots, L_{n-i}^{s_{n-i}}; \mathscr{F})$$

(resp. $g_i(X, L_1^{s_1}, \ldots, L_{n-i}^{s_{n-i}}; \mathscr{F})$ and $p_a^i(X, L_1^{s_1}, \ldots, L_{n-i}^{s_{n-i}}; \mathscr{F})$)

is a polynomial of $s_1, \ldots, s_{n-i}$.

COROLLARY 2.3. *Let X be a smooth projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \ldots, L_{n-i}$ be nef and big line bundles on X. Then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \chi(\mathscr{O}_X) + \sum_{u=1}^{n-i} \left\{ (-1)^{n+u} \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) \right\},$$

$$g_i(X, L_1, \ldots, L_{n-i}) = \sum_{u=1}^{n-i} \left\{ (-1)^{n-i-u} \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) \right\} + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathscr{O}_X),$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i})$$
$$= (-1)^i \left(\sum_{u=1}^{n-i} \left\{ (-1)^{n+u} \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) \right\} + \chi(\mathcal{O}_X) - h^0(\mathcal{O}_X) \right).$$

Proof. By the Kawamata-Viehweg vanishing theorem (see [18] and [26]), we get

$$\chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}}) = (-1)^n h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i})$$

for any $(p_1, \ldots, p_{n-i}) \in S(n-i)_u$ with $1 \leq u$.

If $u = 0$, then $S(n-i)_0 = \{(0, \ldots, 0)\}$. Hence $(-1)^0 \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_0} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}}) = \chi(\mathcal{O}_X)$. Hence

$$\sum_{u=0}^{n-i} \left\{ (-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}}) \right\}$$
$$= \chi(\mathcal{O}_X) + \sum_{u=1}^{n-i} (-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}}).$$

So we get the assertion by Corollary 2.2. □

Remark 2.4. This corollary is a generalization of [7, Theorem 2.3].

Here we calculate ith sectional invariants of some multi-polarized manifolds which will appear in the second part [11] and the third part [12].

Example 2.1. Let X be a smooth projective variety of dimension n, and let i be an integer with $1 \leq i \leq n-1$. Let $L_1, \ldots, L_{n-i}$ be ample line bundles on X.

(A) Suppose that $(X, L_1, \ldots, L_{n-i}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$. Then

$$\sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) = 0$$

for any integer u with $1 \leq u \leq n-i$. Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence for every integer i with $i \geq 1$

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = 1, \\ g_i(X, L_1, \ldots, L_{n-i}) = 0, \\ p_a^i(X, L_1, \ldots, L_{n-i}) = 0. \end{cases}$$

(B) Suppose that $(X, L_1, L_2, \ldots, L_{n-i}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$. Then since $i \geq 1$ we get

$$\sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) = 0$$

for any integer u with $1 \leq u \leq n - i$. Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence for every integer i with $i \geq 1$

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = 1, \\ g_i(X, L_1, \ldots, L_{n-i}) = 0, \\ p_a^i(X, L_1, \ldots, L_{n-i}) = 0. \end{cases}$$

(C) Suppose that $(X, L_1, L_2, \ldots, L_{n-i}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(3), \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$. Then

$$\sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2 \text{ and } 1 \leq u \leq n - i, \\ 0 & \text{if } i = 1 \text{ and } 1 \leq u \leq n - 2, \\ 1 & \text{if } i = 1 \text{ and } u = n - 1. \end{cases}$$

Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence for every integer i with $i \geq 1$

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = \begin{cases} 1 & \text{if } i \geq 2, \\ 0 & \text{if } i = 1, \end{cases} \\ g_i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1, \end{cases} \\ p_a^i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1. \end{cases} \end{cases}$$

(D) Suppose that $(X, L_1, L_2, L_3, \ldots, L_{n-i}) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(2), \mathcal{O}_{\mathbf{P}^n}(1), \ldots, \mathcal{O}_{\mathbf{P}^n}(1))$. Then

$$\sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i})$$

$$= \begin{cases} 0 & \text{if } (i \geq 2 \text{ and } 1 \leq u \leq n - i) \\ & \text{or } (i = 1 \text{ and } 1 \leq u \leq n - 2), \\ 1 & \text{if } i = 1 \text{ and } u = n - 1. \end{cases}$$

Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = \begin{cases} 1 & \text{if } i \geq 2, \\ 0 & \text{if } i = 1, \end{cases} \\ g_i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1, \end{cases} \\ p_a^i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1. \end{cases} \end{cases}$$

(E) Suppose that $(X, L_1, \ldots, L_{n-i}) = (\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1), \ldots, \mathcal{O}_{\mathbf{Q}^n}(1))$. Since $i \geq 1$, we get

$$\sum_{(p_1, \ldots, p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i}) = 0$$

for any integer u with $1 \leq u \leq n - i$. Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence for $1 \leq i$

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = 1, \\ g_i(X, L_1, \ldots, L_{n-i}) = 0, \\ p_a^i(X, L_1, \ldots, L_{n-i}) = 0. \end{cases}$$

(F) Suppose that $(X, L_1, L_2, \ldots, L_{n-i}) = (\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(2), \mathcal{O}_{\mathbf{Q}^n}(1), \ldots, \mathcal{O}_{\mathbf{Q}^n}(1))$. Then

$$\sum_{(p_1, \ldots, p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_1 + \cdots + p_{n-i} L_{n-i})$$

$$= \begin{cases} 0 & \text{if } (i \geq 2 \text{ and } 1 \leq u \leq n - i) \\ & \text{or } (i = 1 \text{ and } 1 \leq u \leq n - 2), \\ 1 & \text{if } i = 1 \text{ and } u = n - 1. \end{cases}$$

Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

Hence

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = \begin{cases} 1 & \text{if } i \geq 2, \\ 0 & \text{if } i = 1, \end{cases} \\ g_i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1, \end{cases} \\ p_a^i(X, L_1, \ldots, L_{n-i}) = \begin{cases} 0 & \text{if } i \geq 2, \\ 1 & \text{if } i = 1. \end{cases} \end{cases}$$

(G) Suppose that $(X, L_1, \ldots, L_{n-i})$ satisfies $K_X + L_1 + \cdots + L_{n-i} = \mathcal{O}_X$. Then

$$\sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1L_1 + \cdots + p_{n-i}L_{n-i}) = \begin{cases} 0 & \text{if } 1 \le u \le n-i-1, \\ 1 & \text{if } u = n-i. \end{cases}$$

Moreover

$$h^j(\mathcal{O}_X) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \ge 1. \end{cases} \tag{G.1}$$

Hence for $1 \le i$

$$\begin{cases} \chi_i^H(X, L_1, \ldots, L_{n-i}) = 1 + (-1)^i, \\ g_i(X, L_1, \ldots, L_{n-i}) = 1, \\ p_a^i(X, L_1, \ldots, L_{n-i}) = 1. \end{cases}$$

Let j be an integer with $2 \le i+1 \le j \le n-1$ and let $\{b_1, \ldots, b_{n-j}\} \subset \{1, \ldots, n-i\}$. Here we assume that $b_k \ne b_l$ for $k \ne l$. Then

$$\sum_{(p_1,\ldots,p_{n-j}) \in S(n-j)_u} h^0(K_X + p_1L_{b_1} + \cdots + p_{n-j}L_{b_{n-j}}) = 0$$

for any integer u with $1 \le u \le n-j$. By (G.1), we obtain

$$\begin{cases} \chi_j^H(X, L_{b_1}, \ldots, L_{b_{n-j}}) = 1, \\ g_j(X, L_{b_1}, \ldots, L_{b_{n-j}}) = 0, \\ p_a^j(X, L_{b_1}, \ldots, L_{b_{n-j}}) = 0 \end{cases}$$

for $2 \le i+1 \le j \le n-1$ and any $\{b_1, \ldots, b_{n-j}\} \subset \{1, \ldots, n-i\}$.

(H) Let n and m be integers with $n \ge 3$ and $n > m \ge 1$. Let $(X, L_1, \ldots, L_{n-m+1})$ be an n-dimensional scroll over a normal variety Y of dimension m. Namely $K_X + L_1 + \cdots + L_{n-m+1} = \Phi^*(H)$, where H is an ample line bundle on Y and Φ is its fibration. Let i be an integer with $i \ge m-1$. Then for $1 \le b_1 < \cdots < b_{n-i} \le n-m+1$

$$\sum_{u=1}^{n-i} (-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1L_{b_1} + \cdots + p_{n-i}L_{b_{n-i}})$$

$$= \begin{cases} 0 & \text{if } m \le i \le n-1, \\ (-1)^{n-m+1} h^0(H) & \text{if } i = m-1. \end{cases}$$

Since $h^j(\mathcal{O}_X) = 0$ if $j \ge m+1$, we have

$$\sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X) = \begin{cases} 0 & \text{if } m+1 \le i \le n-1, \\ h^m(\mathcal{O}_X) & \text{if } i = m, \\ h^{m-1}(\mathcal{O}_X) - h^m(\mathcal{O}_X) & \text{if } i = m-1. \end{cases}$$

Therefore by Corollary 2.3

$$\chi_i^H(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} \chi(\mathcal{O}_X) & \text{if } m \le i \le n-1, \\ (-1)^{m-1}h^0(H) + \chi(\mathcal{O}_X) & \text{if } i = m-1, \end{cases}$$

$$g_i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} 0 & \text{if } m+1 \le i \le n-1, \\ h^m(\mathcal{O}_X) & \text{if } i = m, \\ h^0(H) + h^{m-1}(\mathcal{O}_X) - h^m(\mathcal{O}_X) & \text{if } i = m-1, \end{cases}$$

$$p_a^i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} (-1)^i(\chi(\mathcal{O}_X) - 1) & \text{if } m \le i, \\ h^0(H) + (-1)^{m-1}(\chi(\mathcal{O}_X) - 1) & \text{if } i = m-1. \end{cases}$$

(I) Let n and m be integers with $n \ge 3$ and $m \ge 1$. Let $(X, L_1, \ldots, L_{n-m})$ be an n-dimensional quadric fibration over a normal variety Y of dimension m. Namely $K_X + L_1 + \cdots + L_{n-m} = \Phi^*(H)$, where H is an ample line bundle on Y and Φ is its fibration. Let i be an integer with $i \ge m$. Then for $1 \le b_1 < \cdots < b_{n-i} \le n-m$

$$\sum_{u=1}^{n-i}(-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_{b_1} + \cdots + p_{n-i}L_{b_{n-i}})$$
$$= \begin{cases} 0 & \text{if } m+1 \le i \le n-1, \\ (-1)^{n-m}h^0(H) & \text{if } i = m. \end{cases}$$

Since $h^j(\mathcal{O}_X) = 0$ if $j \ge m+1$, we have

$$\sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X) = \begin{cases} 0 & \text{if } m+1 \le i \le n-1, \\ h^m(\mathcal{O}_X) & \text{if } i = m. \end{cases}$$

Therefore by Corollary 2.3

$$\chi_i^H(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} \chi(\mathcal{O}_X) & \text{if } m+1 \le i \le n-1, \\ (-1)^m h^0(H) + \chi(\mathcal{O}_X) & \text{if } i = m, \end{cases}$$

$$g_i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} 0 & \text{if } m+1 \le i \le n-1, \\ h^0(H) + h^m(\mathcal{O}_X) & \text{if } i = m, \end{cases}$$

$$p_a^i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} (-1)^i(\chi(\mathcal{O}_X) - 1) & \text{if } m+1 \le i \le n-1, \\ h^0(H) + (-1)^m(\chi(\mathcal{O}_X) - 1) & \text{if } i = m. \end{cases}$$

(J) Let n and m be integers with $n \ge 3$ and $m \ge 1$. Let $(X, L_1, \ldots, L_{n-m-1})$ be an n-dimensional Del Pezzo fibration over a normal variety Y of dimension m. Namely $K_X + L_1 + \cdots + L_{n-m-1} = \Phi^*(H)$, where H is an ample line bundle on Y and Φ is its fibration. Let i be an integer with $i \ge m+1$. Then for $1 \le b_1 < \cdots < b_{n-i} \le n-m-1$

$$\sum_{u=1}^{n-i}(-1)^u \sum_{(p_1,\ldots,p_{n-i}) \in S(n-i)_u} h^0(K_X + p_1 L_{b_1} + \cdots + p_{n-i}L_{b_{n-i}})$$
$$= \begin{cases} 0 & \text{if } m+2 \le i \le n-1, \\ (-1)^{n-m-1}h^0(H) & \text{if } i = m+1. \end{cases}$$

Since $h^j(\mathcal{O}_X) = 0$ if $j \geq m+1$, we have

$$\sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X) = 0$$

if $i \geq m+1$. Therefore by Corollary 2.3

$$\chi_i^H(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} \chi(\mathcal{O}_X) & \text{if } m+2 \leq i \leq n-1, \\ (-1)^{m+1} h^0(H) + \chi(\mathcal{O}_X) & \text{if } i = m+1, \end{cases}$$

$$g_i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} 0 & \text{if } m+2 \leq i \leq n-1, \\ h^0(H) & \text{if } i = m+1, \end{cases}$$

$$p_a^i(X, L_{b_1}, \ldots, L_{b_{n-i}}) = \begin{cases} (-1)^i(\chi(\mathcal{O}_X) - 1) & \text{if } m+2 \leq i \leq n-1, \\ h^0(H) + (-1)^{m+1}(\chi(\mathcal{O}_X) - 1) & \text{if } i = m+1. \end{cases}$$

PROPOSITION 2.1. *Let i and n be integers with $n \geq 2$ and $0 \leq i \leq n$, and let $(X, L_1, \ldots, L_{n-i})$ be an n-dimensional multi-prepolarized variety of type $n-i$.*

(1) *If* $\dim \operatorname{Supp}(\mathscr{F}) < n-i$, *then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = 0,$$

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = \begin{cases} (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^j h^j(\mathscr{F}) & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i = 0, \end{cases}$$

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = (-1)^{i+1} h^0(\mathscr{F}).$$

(2) $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F})$, $g_i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$ *and* $p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F})$ *are symmetric with respect to* $L_1, \ldots, L_{n-i}$.

Proof. This is easy and we omit the proof. (See also [19, Proposition 1 and Proposition 2 in Chapter I, §2].) □

LEMMA 2.2. *Let X be a projective variety of dimension n and $\mathscr{F}$ a coherent sheaf on X. Let i be an integer with $1 \leq i \leq n-1$. Let A_1, B_1, $L_2, \ldots, L_{n-i}$ be line bundles on X. Then*

$$\begin{aligned} \chi_{\underbrace{1, \ldots, 1}_{n-i}}(A_1 + B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) &= \chi_{\underbrace{1, \ldots, 1}_{n-i}}(A_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\ &\quad + \chi_{\underbrace{1, \ldots, 1}_{n-i}}(B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\ &\quad - \chi_{\underbrace{1, \ldots, 1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}). \end{aligned}$$

Proof. We set

$$\chi(A_1^{s_1}\otimes B_1^{s_2}\otimes L_2^{t_2}\otimes\cdots\otimes L_{n-i}^{t_{n-i}}\otimes\mathscr{F}) \tag{3}$$
$$=\sum_{p=0}^{n}\sum_{\substack{q_1\geq 0,q_2\geq 0,p_2\geq 0,\ldots,p_{n-i}\geq 0\\ q_1+q_2+p_2+\cdots+p_{n-i}=p}}\chi_{q_1,q_2,p_2,\ldots,p_{n-i}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$\times\binom{s_1+q_1-1}{q_1}\binom{s_2+q_2-1}{q_2}\binom{t_2+p_2-1}{p_2}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}$$

and

$$\chi((A_1\otimes B_1)^{s}\otimes L_2^{t_2}\otimes\cdots\otimes L_{n-i}^{t_{n-i}}\otimes\mathscr{F}) \tag{4}$$
$$=\sum_{p=0}^{n}\sum_{\substack{r_1\geq 0,p_2\geq 0,\ldots,p_{n-i}\geq 0\\ r_1+p_2+\cdots+p_{n-i}=p}}\chi_{r_1,p_2,\ldots,p_{n-i}}(A_1+B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$\times\binom{s+r_1-1}{r_1}\binom{t_2+p_2-1}{p_2}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.$$

Then by setting $(s_1,s_2,s)=(-1,-1,-1)$, and comparing the above two equalities, we obtain

$$\chi_{\underbrace{1,1,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})+\chi_{\underbrace{0,0,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$-\chi_{\underbrace{1,0,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$-\chi_{\underbrace{0,1,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$=-\chi_{\underbrace{1,1,1,\ldots,1}_{n-i}}(A_1+B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$+\chi_{\underbrace{0,1,1,\ldots,1}_{n-i}}(A_1+B_1,L_2,\ldots,L_{n-i};\mathscr{F}).$$

Hence

$$\chi_{\underbrace{1,1,1,\ldots,1}_{n-i}}(A_1+B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$=\chi_{\underbrace{1,0,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$
$$+\chi_{\underbrace{0,1,1,\ldots,1}_{n-i+1}}(A_1,B_1,L_2,\ldots,L_{n-i};\mathscr{F})$$

$$
\begin{aligned}
&- \chi_{\underbrace{1,1,1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\
&+ \chi_{\underbrace{0,1,1,\ldots,1}_{n-i}}(A_1 + B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\
&- \chi_{\underbrace{0,0,1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}).
\end{aligned}
$$

On the other hand, by using (3) and (4) above we see that

$$
\chi_{\underbrace{1,0,1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) = \chi_{\underbrace{1,1,\ldots,1}_{n-i}}(A_1, L_2, \ldots, L_{n-i}; \mathscr{F}),
$$

$$
\chi_{\underbrace{0,1,1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) = \chi_{\underbrace{1,1,\ldots,1}_{n-i}}(B_1, L_2, \ldots, L_{n-i}; \mathscr{F})
$$

and

$$
\chi_{\underbrace{0,1,1,\ldots,1}_{n-i}}(A_1 + B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) = \chi_{\underbrace{0,0,1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}).
$$

Therefore we get

$$
\begin{aligned}
&\chi_{\underbrace{1,\ldots,1}_{n-i}}(A_1 + B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\
&\quad = \chi_{\underbrace{1,\ldots,1}_{n-i}}(A_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\
&\qquad + \chi_{\underbrace{1,\ldots,1}_{n-i}}(B_1, L_2, \ldots, L_{n-i}; \mathscr{F}) \\
&\qquad - \chi_{\underbrace{1,\ldots,1}_{n-i+1}}(A_1, B_1, L_2, \ldots, L_{n-i}; \mathscr{F}). \qquad \square
\end{aligned}
$$

Remark 2.5. If $i = 0$ then

$$
\begin{aligned}
&\chi_{\underbrace{1,\ldots,1}_{n}}(A_1 + B_1, L_2, \ldots, L_n; \mathscr{F}) \\
&\quad = \chi_{\underbrace{1,\ldots,1}_{n}}(A_1, L_2, \ldots, L_n; \mathscr{F}) + \chi_{\underbrace{1,\ldots,1}_{n}}(B_1, L_2, \ldots, L_n; \mathscr{F}).
\end{aligned}
$$

See [19, Proposition 2 in Chapter I, §2].

COROLLARY 2.4. *Let X be a projective variety of dimension n and let i be an integer with $1 \leq i \leq n-1$. Let A, B, $L_1, \ldots, L_{n-i-1}$ be line bundles on X. Then*

$$\chi_i^H(X, A+B, L_1, \ldots, L_{n-i-1}) \tag{5}$$
$$= \chi_i^H(X, A, L_1, \ldots, L_{n-i-1}) + \chi_i^H(X, B, L_1, \ldots, L_{n-i-1})$$
$$- \chi_{i-1}^H(X, A, B, L_1, \ldots, L_{n-i-1}),$$

$$g_i(X, A+B, L_1, \ldots, L_{n-i-1}) \tag{6}$$
$$= g_i(X, A, L_1, \ldots, L_{n-i-1}) + g_i(X, B, L_1, \ldots, L_{n-i-1})$$
$$+ g_{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}) - h^{i-1}(\mathcal{O}_X),$$

$$p_a^i(X, A+B, L_1, \ldots, L_{n-i-1}) \tag{7}$$
$$= p_a^i(X, A, L_1, \ldots, L_{n-i-1}) + p_a^i(X, B, L_1, \ldots, L_{n-i-1})$$
$$+ p_a^{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}).$$

Proof. By Lemma 2.2 we get

$$\chi_{1,\ldots,1}(A+B, L_1, \ldots, L_{n-i-1})$$
$$= \chi_{1,\ldots,1}(A, L_1, \ldots, L_{n-i-1}) + \chi_{1,\ldots,1}(B, L_1, \ldots, L_{n-i-1})$$
$$- \chi_{1,\ldots,1}(A, B, L_1, \ldots, L_{n-i-1}).$$

Hence by the definition of the sectional H-arithmetic genus and the sectional arithmetic genus, we get the assertion (5) and (7).

For the sectional geometric genus, we have

$$\begin{aligned}
&g_i(X, A+B, L_1, \ldots, L_{n-i-1}) - g_i(X, A, L_1, \ldots, L_{n-i-1}) \\
&\quad - g_i(X, B, L_1, \ldots, L_{n-i-1}) - g_{i-1}(X, A, B, L_1, \ldots, L_{n-i-1}) \\
&= (-1)^i(\chi_{1,\ldots,1}(A+B, L_1, \ldots, L_{n-i-1}) - \chi_{1,\ldots,1}(A, L_1, \ldots, L_{n-i-1}) \\
&\quad - \chi_{1,\ldots,1}(B, L_1, \ldots, L_{n-i-1}) + \chi_{1,\ldots,1}(A, B, L_1, \ldots, L_{n-i-1})) \\
&\quad + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X) - 2\sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X) \\
&\quad - \sum_{j=0}^{n-i+1}(-1)^{n-i+1-j}h^{n-j}(\mathcal{O}_X) \\
&= -h^{i-1}(\mathcal{O}_X).
\end{aligned}$$

Therefore we get the assertion (6). □

Remark 2.6. If $i = 0$, then

$$g_0(X, A+B, L_1, \ldots, L_{n-1})$$
$$= g_0(X, A, L_1, \ldots, L_{n-1}) + g_0(X, B, L_1, \ldots, L_{n-1}).$$

LEMMA 2.3. *Let X be a projective variety of dimension n and let $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}''$ be coherent sheaves on X such that the following is exact*:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Let i be an integer with $0 \leq i \leq n-1$. Let $L_1, L_2, \ldots, L_{n-i}$ be line bundles on X. Then

$$\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F}) = \chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F}') + \chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathscr{F}'')$$

and

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

Proof. This is easy and we omit the proof. □

COROLLARY 2.5. *Let X be a projective variety of dimension n, and let $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}''$ be coherent sheaves on X such that the following is exact*:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Let i be an integer with $0 \leq i \leq n-1$. Let $L_1, L_2, \ldots, L_{n-i}$ be line bundles on X. Then

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = \chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}') + \chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{F}'')$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}) = p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}') + p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{F}'') + (-1)^i(h^0(\mathscr{F}') + h^0(\mathscr{F}'') - h^0(\mathscr{F})).$$

LEMMA 2.4. *Let X be a projective variety of dimension n, i an integer with $0 \leq i \leq n-1$ and $L_1, L_2, \ldots, L_{n-i}$ line bundles on X. Assume that there exists an effective divisor $\Delta \in |L_1|$. Then*

$$\chi_{\underbrace{1, \ldots, 1}_{n-i}}(L_1, L_2, \ldots, L_{n-i}; \mathscr{O}_X) = \chi_{\underbrace{1, \ldots, 1}_{n-i-1}}(L_2|_\Delta, \ldots, L_{n-i}|_\Delta; \mathscr{O}_\Delta).$$

Proof. We consider the following exact sequence.

$$0 \to L_1^{t_1-1} \otimes L_2^{t_2} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \to L_1^{t_1} \otimes L_2^{t_2} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \to L_1^{t_1} \otimes L_2^{t_2} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes j_*(\mathscr{O}_\Delta) \to 0,$$

where $j : \Delta \hookrightarrow X$ is a closed embedding. On the other hand,

$$
\begin{aligned}
(8)\qquad & \chi(L_1^{t_1}\otimes L_2^{t_2}\otimes\cdots\otimes L_{n-i}^{t_{n-i}})-\chi(L_1^{t_1-1}\otimes L_2^{t_2}\otimes\cdots\otimes L_{n-i}^{t_{n-i}})\\
&=\sum_{p=0}^{n}\sum_{\substack{p_1\geq 0,\ldots,p_{n-i}\geq 0\\ p_1+\cdots+p_{n-i}=p}}\chi_{p_1,\ldots,p_{n-i}}(L_1,\ldots,L_{n-i};\mathcal{O}_X)\\
&\quad\times\binom{t_1+p_1-1}{p_1}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}\\
&\quad-\sum_{p=0}^{n}\sum_{\substack{p_1\geq 0,\ldots,p_{n-i}\geq 0\\ p_1+\cdots+p_{n-i}=p}}\chi_{p_1,\ldots,p_{n-i}}(L_1,\ldots,L_{n-i};\mathcal{O}_X)\\
&\quad\times\binom{t_1+p_1-2}{p_1}\binom{t_2+p_2-1}{p_2}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}\\
&=\sum_{p=0}^{n}\sum_{\substack{p_1\geq 0,p_2\geq 0\ldots,p_{n-i}\geq 0\\ p_1+p_2+\cdots+p_{n-i}=p}}\chi_{p_1,p_2,\ldots,p_{n-i}}(L_1,L_2,\ldots,L_{n-i};\mathcal{O}_X)\\
&\quad\times\binom{t_1+p_1-2}{p_1-1}\binom{t_2+p_2-1}{p_2}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
&\chi(L_1^{t_1}\otimes L_2^{t_2}\otimes\cdots\otimes L_{n-i}^{t_{n-i}}\otimes j_*(\mathcal{O}_\Delta))\\
&\quad=\chi(j_*((j^*L_1)^{t_1}\otimes(j^*L_2)^{t_2}\otimes\cdots\otimes(j^*L_{n-i})^{t_{n-i}}))\\
&\quad=\chi((L_1|_\Delta)^{t_1}\otimes(L_2|_\Delta)^{t_2}\otimes\cdots\otimes(L_{n-i}|_\Delta)^{t_{n-i}}).
\end{aligned}
$$

By Remark 2.1 we can write $\chi((L_1|_\Delta)^{t_1}\otimes(L_2|_\Delta)^{t_2}\otimes\cdots\otimes(L_{n-i}|_\Delta)^{t_{n-i}})$ as

$$
\begin{aligned}
(9)\qquad & \chi((L_1|_\Delta)^{t_1}\otimes(L_2|_\Delta)^{t_2}\otimes\cdots\otimes(L_{n-i}|_\Delta)^{t_{n-i}})\\
&=\sum_{p=0}^{n-1}\sum_{\substack{q\geq 0,p_2\geq 0\ldots,p_{n-i}\geq 0\\ q+p_2+\cdots+p_{n-i}=p}}\chi_{q,p_2,\ldots,p_{n-i}}(L_1|_\Delta,L_2|_\Delta,\ldots,L_{n-i}|_\Delta;\mathcal{O}_\Delta)\\
&\quad\times\binom{t_1+q-1}{q}\binom{t_2+p_2-1}{p_2}\cdots\binom{t_{n-i}+p_{n-i}-1}{p_{n-i}}.
\end{aligned}
$$

By (8) and (9), we get

$$
\chi_{1,1,\ldots,1}(L_1,L_2,\ldots,L_{n-i};\mathcal{O}_X)=\chi_{0,1,\ldots,1}(L_1|_\Delta,L_2|_\Delta,\ldots,L_{n-i}|_\Delta;\mathcal{O}_\Delta).
$$

On the other hand,

$$\chi((L_2|_\Delta)^{t_2} \otimes \cdots \otimes (L_{n-i}|_\Delta)^{t_{n-i}}) \tag{10}$$
$$= \sum_{p=0}^{n-1} \sum_{\substack{p_2 \geq 0, \ldots, p_{n-i} \geq 0 \\ p_2 + \cdots + p_{n-i} = p}} \chi_{p_2, \ldots, p_{n-i}}(L_2|_\Delta, \ldots, L_{n-i}|_\Delta; \mathcal{O}_\Delta)$$
$$\times \binom{t_2 + p_2 - 1}{p_2} \cdots \binom{t_{n-i} + p_{n-i} - 1}{p_{n-i}}.$$

By putting $t_1 = 0$ in (9), we obtain

$$\chi_{0,1,\ldots,1}(L_1|_\Delta, L_2|_\Delta, \ldots, L_{n-i}|_\Delta; \mathcal{O}_\Delta) = \chi_{1,\ldots,1}(L_2|_\Delta, \ldots, L_{n-i}|_\Delta; \mathcal{O}_\Delta).$$

Therefore we get the assertion. □

THEOREM 2.3. *Let n and i be integers such that $n \geq 2$ and $1 \leq i \leq n-1$. Let $(X, L_1, \ldots, L_{n-i})$ be an n-dimensional multi-polarized manifold of type $(n-i)$.*

(1) *Assume that* $\mathrm{Bs}|L_1| = \emptyset$. *Then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \chi_i^H(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}),$$
$$g_i(X, L_1, \ldots, L_{n-i}) = g_i(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}),$$
$$p_a^i(X, L_1, \ldots, L_{n-i}) = p_a^i(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}).$$

(*Here $X_1 \in |L_1|$ is a smooth member.*)

(2) *Assume that* $\mathrm{Bs}|L_j| = \emptyset$ *for every integer j with $1 \leq j \leq n-i$. We use Notation* 1.2. *Then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \chi_i(\mathcal{O}_{X_{n-i}}),$$
$$g_i(X, L_1, \ldots, L_{n-i}) = h^i(\mathcal{O}_{X_{n-i}}),$$
$$p_a^i(X, L_1, \ldots, L_{n-i}) = p_a(X_{n-i}).$$

Proof. (1) By Lemma 2.4 we get

$$\chi_{1,\ldots,1}(L_1, \ldots, L_{n-i}; \mathcal{O}_X) = \chi_{1,\ldots,1}(L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}; \mathcal{O}_{X_1}).$$

Hence

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \chi_i^H(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1})$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}) = (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}) - 1)$$
$$= (-1)^i(\chi_i^H(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}) - 1)$$
$$= p_a^i(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}).$$

By the Lefschetz theorem $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{X_1})$ for every integer j with $0 \leq j \leq i-1$. Therefore

$$(-1)^i\chi(\mathcal{O}_X) - \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X)$$

$$= (-1)^i\chi(\mathcal{O}_{X_1}) - \sum_{j=0}^{n-1-i}(-1)^{n-1-i-j}h^{n-1-j}(\mathcal{O}_{X_1}).$$

Hence

$$g_i(X, L_1, \ldots, L_{n-i}) = g_i(X_1, L_2|_{X_1}, \ldots, L_{n-i}|_{X_1}).$$

So we get the assertion (1).

(2) By assumption and (1) above, we obtain

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \chi_i^H(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}).$$

On the other hand, by [10, Remark 2.1 (4)] we see that $\chi_i^H(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}) = \chi(\mathcal{O}_{X_{n-i}})$. Hence we get the first equality in (2).

By the Lefschetz theorem, we get $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{X_{n-i}})$ for every j with $0 \le j \le i-1$. Therefore

$$(-1)^i\chi(\mathcal{O}_X) - \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X)$$

$$= (-1)^i\chi(\mathcal{O}_{X_{n-i-1}}) - \sum_{j=0}^{1}(-1)^{1-j}h^{i+1-j}(\mathcal{O}_{X_{n-i-1}}).$$

Hence

$$g_i(X, L_1, \ldots, L_{n-i}) = g_i(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}).$$

On the other hand, by [7, Remark 2.5 (i)] we see that $g_i(X_{n-i-1}, L_{n-i}|_{X_{n-i-1}}) = h^i(\mathcal{O}_{X_{n-i}})$. So we get the second equality in (2).

Finally

$$\begin{aligned} p_a^i(X, L_1, \ldots, L_{n-i}) &= (-1)^i(\chi_i^H(X, L_1, \ldots, L_{n-i}) - 1) \\ &= (-1)^i(\chi(\mathcal{O}_{X_{n-i}}) - 1) \\ &= p_a(X_{n-i}). \end{aligned}$$

So we get the third equality in (2). □

PROPOSITION 2.2. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \ldots, L_{n-i}$, A and B be line bundles on X.*

(1) *If L_1 is numerically equivalent to zero, then $\chi_{1,\ldots,1}(L_1, \ldots, L_{n-i}; \mathcal{O}_X) = 0$.*

(2) *If A is numerically equivalent to B, then*

$$\chi_{1,\ldots,1}(A, L_2, \ldots, L_{n-i}; \mathcal{O}_X) = \chi_{1,\ldots,1}(B, L_2, \ldots, L_{n-i}; \mathcal{O}_X).$$

Proof. (1) Since

$$\chi(L_1^{\otimes t_1} \otimes L_2^{\otimes t_2} \otimes \cdots \otimes L_{n-i}^{\otimes t_{n-i}}) = \chi(L_2^{\otimes t_2} \otimes \cdots \otimes L_{n-i}^{\otimes t_{n-i}}),$$

we get the assertion.

(2) By assumption there exists a Cartier divisor D on X such that $A = B + D$ and D is numerically equivalent to zero. By Lemma 2.2 and (1) above, we obtain

$$\begin{aligned}
&\chi_{1,\dots,1}(A, L_2, \dots, L_{n-i}; \mathcal{O}_X) \\
&\quad = \chi_{1,\dots,1}(B + D, L_2, \dots, L_{n-i}; \mathcal{O}_X) \\
&\quad = \chi_{1,\dots,1}(B, L_2, \dots, L_{n-i}; \mathcal{O}_X) + \chi_{1,\dots,1}(D, L_2, \dots, L_{n-i}; \mathcal{O}_X) \\
&\qquad - \chi_{1,\dots,1}(B, D, L_2, \dots, L_{n-i}; \mathcal{O}_X) \\
&\quad = \chi_{1,\dots,1}(B, L_2, \dots, L_{n-i}; \mathcal{O}_X).
\end{aligned}$$

We get the assertion. □

By Proposition 2.2, we get the following corollary.

COROLLARY 2.6. *Let X be a projective variety of dimension n, and let i be an integer with $0 \le i \le n-1$. Let $L_1, \dots, L_{n-i}$, A and B be line bundles on X.*

(1) *If L_1 is numerically equivalent to zero, then*

$$\chi_i^H(X, L_1, \dots, L_{n-i}) = 0,$$

$$g_i(X, L_1, \dots, L_{n-i}) = \begin{cases} 0 & \text{if } i = 0, \\ \sum_{j=0}^{i-1} (-1)^{i-j-1} h^j(\mathcal{O}_X) & \text{if } i \neq 0, \end{cases}$$

$$p_a^i(X, L_1, \dots, L_{n-i}) = (-1)^{i+1}.$$

(2) *If A is numerically equivalent to B, then*

$$\chi_i^H(X, A, L_2, \dots, L_{n-i}) = \chi_i^H(X, B, L_2, \dots, L_{n-i}),$$
$$g_i(X, A, L_2, \dots, L_{n-i}) = g_i(X, B, L_2, \dots, L_{n-i}),$$
$$p_a^i(X, A, L_2, \dots, L_{n-i}) = p_a^i(X, B, L_2, \dots, L_{n-i}).$$

PROPOSITION 2.3. *Let n and i be integers with $n \ge 2$ and $0 \le i \le n-1$. Let $(X, L_1, \dots, L_{n-i})$ be an n-dimensional multi-polarized manifold of type $(n-i)$. Let $(M, A_1, \dots, A_{n-i})$ be a reduction of $(X, L_1, \dots, L_{n-i})$ (see Definition 1.5). Then for every integer i with $1 \le i \le n-1$*

$$\chi_i^H(X, L_1, \dots, L_{n-i}) = \chi_i^H(M, A_1, \dots, A_{n-i}),$$
$$g_i(X, L_1, \dots, L_{n-i}) = g_i(M, A_1, \dots, A_{n-i}),$$
$$p_a^i(X, L_1, \dots, L_{n-i}) = p_a^i(M, A_1, \dots, A_{n-i}).$$

Proof. By definition of a reduction, we see that

$$h^0(K_X + L_1^{p_1} + \cdots + L_{n-i}^{p_{n-i}}) = h^0(K_{M_1} + A_1^{p_1} + \cdots + A_{n-i}^{p_{n-i}})$$

for every $(p_1, \ldots, p_{n-i}) \in S(n-i)_u$, where i and u are integers with $1 \le i \le n-1$ and $1 \le u \le n-i$. Hence by Corollary 2.3, we obtain the assertion. □

The following result is a generalization of [9, Theorem 2.1] for the ith sectional geometric genus of quasi-polarized manifolds.

THEOREM 2.4. *Let n and i be integers with $n \ge 2$ and $0 \le i \le n-1$. Let $(X, L_1, \ldots, L_{n-i})$ be an n-dimensional multi-quasi-polarized manifold. Let $\mathscr{E}$ be a vector bundle of rank r on X. Then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{E}) = \sum_{k=0}^{i} \left(\sum_{(t_1,\ldots,t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}),$$

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{E}) = \begin{cases} rL_1 \cdots L_n & \text{if } i = 0, \\ \sum_{k=0}^{i} \left(\sum_{(t_1,\ldots,t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^k}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}) & \\ \quad + \sum_{j=0}^{i-1} (-1)^{i-j-1} h^j(\mathscr{E}) & \text{if } i \ne 0, \end{cases}$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{E}) = (-1)^i \left(\sum_{k=0}^{i} \left(\sum_{(t_1,\ldots,t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}) - h^0(\mathscr{E}) \right).$$

Here

$$S(u)_r^+ := \left\{ (t_1, \ldots, t_u) \in \mathbf{Z}^{\oplus u} \,\middle|\, \sum_{j=1}^{u} t_j = r \text{ and } t_k \ge 1 \text{ for any } k \right\}$$

and

$$R_k(X, \mathscr{E}) := \sum_{l=0}^{k} \mathrm{ch}(\mathscr{E})_l T_{k-l}(X).$$

Proof. First we are going to calculate $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{E})$. By Corollary 2.2, we get

$$\chi_i^H(X, L_1, \dots, L_{n-i}; \mathscr{E})$$
$$= \chi(\mathscr{E}) + \sum_{u=1}^{n-i} \left\{ (-1)^u \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} \chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{E}) \right\}.$$

By Theorem 1.1 and Remark 1.5,

$$\chi(L_1^{-p_1} \otimes \cdots \otimes L_{n-i}^{-p_{n-i}} \otimes \mathscr{E}) = \int_X \operatorname{ch}(-L_1^{p_1} - \cdots - L_{n-i}^{p_{n-i}}) \operatorname{ch}(\mathscr{E}) \operatorname{td}(\mathscr{T}_X)$$
$$= \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} (L_1^{p_1} + \cdots + L_{n-i}^{p_{n-i}})^{n-k} R_k(X, \mathscr{E}).$$

Hence

$$\chi_i^H(X, L_1, \dots, L_{n-i}; \mathscr{E})$$
$$= \sum_{u=1}^{n-i} (-1)^{n+u} \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} (L_1^{p_1} + \cdots + L_{n-i}^{p_{n-i}})^{n-k} R_k(X, \mathscr{E}) \right\}$$
$$+ \chi(\mathscr{E})$$
$$= \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \left[\sum_{u=1}^{n-i} (-1)^{n+u} \left\{ \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} (L_1^{p_1} + \cdots + L_{n-i}^{p_{n-i}})^{n-k} \right\} \right] R_k(X, \mathscr{E})$$
$$+ \chi(\mathscr{E}).$$

Here we note that if $k = n$, then

$$\frac{(-1)^k}{(n-k)!} \sum_{u=1}^{n-i} (-1)^{n+u} \left\{ \sum_{(p_1,\dots,p_{n-i}) \in S(n-i)_u} (L_1^{p_1} + \cdots + L_{n-i}^{p_{n-i}})^{n-k} \right\} R_k(X, \mathscr{E})$$
$$= (-1)^n \left(\sum_{u=1}^{n-i} (-1)^{n+u} \binom{n-i}{u} \right) R_n(X, \mathscr{E})$$
$$= \sum_{u=1}^{n-i} (-1)^u \binom{n-i}{u} R_n(X, \mathscr{E})$$
$$= -R_n(X, \mathscr{E}) = -\chi(\mathscr{E}).$$

Hence we can write $\chi_i^H(X, L_1, \dots, L_{n-i}; \mathscr{E})$ as follows.

$$\chi_i^H(X, L_1, \dots, L_{n-i}; \mathscr{E})$$
$$= \sum_{k=0}^{n-1} \left(\sum_{(t_1,\dots,t_{n-i}) \in S(n-i)_{n-k}} \alpha(t_1, \dots, t_{n-i}) L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}),$$

where $\alpha(t_1, \dots, t_{n-i}) \in \mathbf{Q}$. Next we calculate $\alpha(t_1, \dots, t_{n-i})$.

CLAIM 2.3. *Assume that* $0 \le k \le n-1$ *and* $(t_1, \dots, t_{n-i}) \in S(n-i)_{n-k}$. *Then*

$$\alpha(t_1, \dots, t_{n-i}) = \begin{cases} 0 & \text{if } t_j = 0 \text{ for some } j, \\ \dfrac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} & \text{if } t_j \ge 1 \text{ for every } j. \end{cases}$$

Proof. Let $(t_1, \dots, t_{n-i}) \in S(n-i)_{n-k}$, $A := \{j \mid t_j \ge 1\}$, and $s := \# A$.

(1) Assume that $s \le n-i-1$. Then $L_1^{t_1} \cdots L_{n-i}^{t_{n-i}}$ appears only in $(L_{j_1} + \cdots + L_{j_u})^{n-k}$, where $\{j_1, \dots, j_u\}$ satisfies $A \subset \{j_1, \dots, j_u\}$.

The coefficient of $L_1^{t_1} \cdots L_{n-i}^{t_{n-i}}$ in $(L_{j_1} + \cdots + L_{j_u})^{n-k}$ is $(n-k)!/(t_1! \cdots t_{n-i}!)$. Furthermore there are $\binom{n-i-s}{u-s}$ choices of $\{j_1, \dots, j_u\}$. Therefore

$$\begin{aligned} \alpha(t_1, \dots, t_{n-i}) &= \frac{(-1)^k}{(n-k)!} \sum_{u=s}^{n-i} (-1)^{n+u} \frac{(n-k)!}{t_1! \cdots t_{n-i}!} \binom{n-i-s}{u-s} \\ &= \frac{(-1)^k}{t_1! \cdots t_{n-i}!} \sum_{u=s}^{n-i} (-1)^{n+u} \binom{n-i-s}{u-s} \\ &= 0. \end{aligned}$$

(2) Assume that $s = n-i$. Then we note that $0 \le k \le i$ in this case. Then $L_1^{t_1} \cdots L_{n-i}^{t_{n-i}}$ appears only in $(L_1 + \cdots + L_{n-i})^{n-k}$ and the coefficient of $L_1^{t_1} \cdots L_{n-i}^{t_{n-i}}$ in $(L_1 + \cdots + L_{n-i})^{n-k}$ is $(n-k)!/(t_1! \cdots t_{n-i}!)$. Hence

$$\begin{aligned} \alpha(t_1, \dots, t_{n-i}) &= \frac{(-1)^k}{(n-k)!} (-1)^i \frac{(n-k)!}{t_1! \cdots t_{n-i}!} \\ &= \frac{(-1)^{k+i}}{t_1! \cdots t_{n-i}!}. \end{aligned}$$

So we get the assertion of this claim. □

Therefore by Claim 2.3

$$\begin{aligned} &\chi_i^H(X, L_1, \dots, L_{n-i}; \mathscr{E}) \\ &\quad = \sum_{k=0}^{n-1} \left(\sum_{(t_1, \dots, t_{n-i}) \in S(n-i)_{n-k}} \alpha(t_1, \dots, t_{n-i}) L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}) \\ &\quad = \sum_{k=0}^{i} \left(\sum_{(t_1, \dots, t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) R_k(X, \mathscr{E}). \end{aligned}$$

So we get the first assertion.

Since

$$g_i(X, L_1, \ldots, L_{n-i}; \mathscr{E})$$
$$:= \begin{cases} (-1)^i \chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{E}) + \sum_{j=0}^{i-1} (-1)^{i-j-1} h^j(\mathscr{E}) & \text{if } i \neq 0, \\ \chi_0^H(X, L_1, \ldots, L_n; \mathscr{E}) & \text{if } i = 0, \end{cases}$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}; \mathscr{E}) = (-1)^i \big(\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathscr{E}) - h^0(\mathscr{E})\big),$$

we get the second and third assertion. □

COROLLARY 2.7. *Let n and i be integers with $n \geq 2$ and $0 \leq i \leq n-1$. Let $(X, L_1, \ldots, L_{n-i})$ be an n-dimensional multi-quasi-polarized manifold. Then*

$$\chi_i^H(X, L_1, \ldots, L_{n-i}) = \sum_{k=0}^{i} \left(\sum_{(t_1, \ldots, t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) T_k(X),$$

$$g_i(X, L_1, \ldots, L_{n-i}) = \begin{cases} L_1 \cdots L_n & \text{if } i = 0, \\ \sum_{k=0}^{i} \left(\sum_{(t_1, \ldots, t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^k}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) T_k(X) & \\ \quad + \sum_{j=0}^{i-1} (-1)^{i-j-1} h^j(\mathscr{O}_X) & \text{if } i \neq 0, \end{cases}$$

and

$$p_a^i(X, L_1, \ldots, L_{n-i}) = (-1)^i \left(\sum_{k=0}^{i} \left(\sum_{(t_1, \ldots, t_{n-i}) \in S(n-i)_{n-k}^+} \frac{(-1)^{k+i}}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) T_k(X) - 1 \right).$$

Here $S(u)_r^+ := \{(t_1, \ldots, t_u) \in \mathbf{Z}^{\oplus u} \mid \sum_{j=1}^{u} t_j = r \text{ and } t_k \geq 1 \text{ for any } k\}$.

Proof. By setting $\mathscr{E} = \mathscr{O}_X$ in Theorem 2.4, we get the assertion since $R_k(X, \mathscr{O}_X) = T_k(X)$. □

REFERENCES

[1] G. BESANA AND A. BIANCOFIORE, Degree eleven projective manifolds of dimension greater than or equal to three, Forum Math. **17** (2005), 711–733.

[2] M. L. FANIA AND E. L. LIVORNI, Degree nine manifolds of dimension greater than or equal to 3, Math. Nachr. **169** (1994), 117–134.

[3] M. L. FANIA AND E. L. LIVORNI, Degree ten manifolds of dimension n greater than or equal to 3, Math. Nachr. **188** (1997), 79–108.

[4] T. FUJITA, On polarized manifolds whose adjoint bundles are not semipositive, Advanced Studies in Pure Math. **10** (1985), 167–178.

[5] T. FUJITA, Classification of polarized manifolds of sectional genus two, the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata, 1987, 73–98.

[6] T. Fujita, Classification theories of polarized varieties, London Math. Soc. lecture note series **155**, 1990.
[7] Y. Fukuma, On the sectional geometric genus of quasi-polarized varieties, I, Comm. Alg. **32** (2004), 1069–1100.
[8] Y. Fukuma, Problems on the second sectional invariants of polarized manifolds, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **25** (2004), 55–64.
[9] Y. Fukuma, A formula for the sectional geometric genus of quasi-polarized manifolds by using intersection numbers, J. Pure Appl. Algebra **194** (2004), 113–126.
[10] Y. Fukuma, On the second sectional H-arithmetic genus of polarized manifolds, Math. Z. **250** (2005), 573–597.
[11] Y. Fukuma, Invariants of ample line bundles on projective varieties and their applications, II, in preparation.
[12] Y. Fukuma, Invariants of ample line bundles on projective varieties and their applications, III, in preparation.
[13] W. Fulton, Intersection theory, Ergebnisse der Mathematik und ihrer Grenzgebiete **2**, Springer-Verlag, 1984.
[14] F. Hirzebruch, Topological methods in algebraic geometry, Grundlehren Math. Wiss. **131**, Springer-Verlag, New York, 1966.
[15] P. Ionescu, Embedded projective varieties of small invariants, Proceedings of the Week of Algebraic Geometry, Bucharest, 1982, Lecture notes in math. **1056**, 1984, 142–186.
[16] P. Ionescu, Embedded projective varieties of small invariants, III, Algebraic geometry, Proceedings of Conference on Hyperplane Sections, L'Aquila, Italy, 1988, Lecture notes in math. **1417**, 1990, 138–154.
[17] P. Ionescu, Generalized adjunction and applications, Math. Proc. Camb. Phil. Soc. **99** (1986), 457–472.
[18] Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann. **261** (1982), 43–46.
[19] S. L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. **84** (1966), 293–344.
[20] L. Lovász, Combinatorial problems and excercises, North-Holland.
[21] C. Okonek, 3-Mannigfaltigkeiten im $\mathbf{P}^5$ und ihre zugehörigen stabilen Garben, Manuscripta Math. **38** (1982), 175–199.
[22] C. Okonek, Moduli reflexiver Garben und Flachem von kleinem Grad im $\mathbf{P}^4$, Math. Z. **184** (1983), 549–572.
[23] C. Okonek, Über 2-codimensionale Untermannigfaltigkeiten vom Grad 7 in $\mathbf{P}^4$ und $\mathbf{P}^5$, Math. Z. **187** (1984), 209–219.
[24] C. Okonek, Flachem von Grad 8 im $\mathbf{P}^4$, Math. Z. **191** (1986), 207–223.
[25] H. P. F. Swinnerton Dyer, An enumeration of all varieties of degree 4, Amer. J. Math. **95** (1973), 403–418.
[26] E. Viehweg, Vanishing theorems, J. Reine Angew. Math. **335** (1982), 1–8.
[27] A. Weil, Sur les critères d'équivalence en géométrie algébrique, Math. Ann. **128** (1954), 95–127.

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