J.-P. WANG AND L.-D. HUANG
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MEROMORPHIC FUNCTIONS SHARING THREE VALUES CM*

Jian-Ping Wang and Ling-Di Huang

Abstract

This paper studies the uniqueness of meromorphic functions that share three values CM and obtain some results that are improvements and generalizations of that of H. Ueda, G. Brosch, etc.

1. Introduction and results

In this paper, by meromorphic function we always mean a function which is meromorphic in the whole complex plane **C**. It is assumed that the reader is familiar with the usual notations and the fundamental results of R. Nevanlinna theory of meromorphic function as found in [6]. In particular, $S(r, f)$ will denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to +\infty$, possibly outside a set of r of finite linear measure.

Let f, g be nonconstant meromorphic functions. We say that a meromorphic function $a(z)(\not\equiv \infty)$ is a small function of f if $T(r, a) = S(r, f)$. If $N(r, 1/(f-a)) = S(r, f)$, then we say that a is an exceptional function of f. Moreover, we denote by $N(r, f = a = g)$ the counting function of those common zeros of $f - a$ and $g - a$, where z_0 is counted $\min\{p, q\}$ times if z_0 is a common zero of $f - a$ and $g - a$ with multiplicity p and q respectively; as usual, by $\overline{N}(r, f = a = g)$ the corresponding reduced counting function; and by $N_E(r, f = a = g)$ the counting function which "counts" only those common zeros of $f - a$ and $g - a$ with the same multiplicity in $N(r, f = a = g)$. These notations will be used throughout the paper.

Let f, g be two nonconstant meromorphic functions, and let a be a small function of f and g or a be a constant. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with the same multiplicity; if we ignore the multiplicity, then we say that f and g share a IM. In addition, we say that f and g share ∞ CM (resp. IM) if $1/f$ and $1/g$ share the value 0 CM (resp. IM).

For the statement of our results, we need a slight generalization of the definitions of CM and IM.

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Let f and g be two meromorphic functions, and let a be a small function of f and g or a be a constant.

Definition 1 (see [7, p. 799]). *We say that f and g share a "CM" if* $N\left(r,\frac{1}{f-a}\right) - N_E(f = a = g) = S(r,f)$ *and* $N\left(r,\frac{1}{g-a}\right) - N_E(r,f = a = g) = S(r,g)$.

Definition 2 (see [9, p. 317]). *We say that f and g share a "IM" if* $\overline{N}\left(r,\frac{1}{f-a}\right) - \overline{N}(r,f = a = g) = S(r,f)$ *and* $\overline{N}\left(r,\frac{1}{g-a}\right) - \overline{N}(r,f = a = g) = S(r,g)$.

Clearly, if a is shared "CM" by f and g, then a must be shared "IM" by f and g.

R. Nevanlinna proved the following famous four-point theorem.

Theorem A [12]. *If two nonconstant meromorphic functions f and g share four distinct values CM, then f is a Möbius transformation of g.*

Since then, many authors studied the uniqueness of meromorphic functions that share three or four values and obtained a series of results (see [2–5, 8, 10, 11, 13–16] etc). As we have known, in such problems of uniqueness of meromorphic function, some results can be usually generalized from sharing value to sharing small function. But, these generalizations are sometimes very difficult (see [7–9], [18] etc).

In 1997, Hua and Fang proved the following result.

Theorem B [7]. *Let f and g be two nonconstant meromorphic functions, and let $a_j(z)$ $(j = 1,\dots,4)$ be distinct small functions of f and g. If f and g share $a_j(z)$ $(j = 1,2,3)$ CM, and share $a_4(z)$ IM. Then f and g satisfy one of the following cases.*

(i) $f \equiv g$, (ii) $F \equiv -G$ *with* $a(z) \equiv -1$, (iii) $F + G \equiv 2$ *with* $a(z) \equiv 2$,

(iv) $(F - 1/2)(G - 1/2) \equiv 1/4$ *with* $a(z) \equiv 1/2$, (v) $F \cdot G \equiv 1$ *with* $a(z) \equiv -1$,

(vi) $(F - 1)(G - 1) \equiv 1$ *with* $a(z) \equiv 2$, (vii) $F + G \equiv 1$ *with* $a(z) \equiv 1/2$,

where $F \equiv \frac{f - a_1}{f - a_3}\frac{a_2 - a_3}{a_2 - a_1}$, $G \equiv \frac{g - a_1}{g - a_3}\frac{a_2 - a_3}{a_2 - a_1}$, *and* $a(z) \equiv \frac{a_4 - a_1}{a_4 - a_3}\frac{a_2 - a_3}{a_2 - a_1}$.

Remark 1. From the proof of Lemma 6 and Lemma 7 in [7], it is easy to see that the conclusion is still true if we replace IM with "IM" in Theorem B.

For the meromorphic functions that share three values, H. Ueda, G. Brosch proved the following results respectively.

Theorem C (see [2] or [11, p. 36]). *Let two meromorphic functions f and g share* 0, 1, ∞ *CM. If there exists a finite value $a(\neq 0,1)$ such that $g(z)=a$ whenever $f(z)=a$, then f is a Möbius transformation of g.*

In order to state Ueda's result, we need the following notations and definitions.

Let f be a meromorphic function, let a be a small function of f or be a constant, and let p, k be positive integers. We denote by $f(z_0)\overset{(p)}{=}a$ that z_0 is a zero of $f-a$ with multiplicity p, and by $E(a,k,f)=\{z\in\mathbf{C}: f(z)\overset{(p)}{=}a, p\le k\}$ the set of all zeros (counting multiplicity) of $f-a$ with multiplicity less than or equal to k.

Theorem D [13]. *Let two meromorphic functions f and g share* 0, 1, ∞ *CM. If there exists a finite value a $(\neq 0,1)$ and an integer $k(\ge 2)$ such that $E(a,k,f)=E(a,k,g)$, then f is a Möbius transformation of g.*

The main purpose of this paper is further to study the uniqueness of meromorphic functions that share three values CM, and to prove the following results.

Theorem 1. *Let two nonconstant meromorphic functions f and g share* 0, 1, ∞ *CM. If there exists a small entire function $a(z)(\not\equiv 0,1)$ of f and g such that $N(r,f=a=g)\neq S(r,f)$, then f and g satisfy one of the following five cases.*

(i) $f\equiv g$, (ii) $f\cdot g\equiv 1$ *with* $a(z)\equiv -1$, (iii) $f+g\equiv 1$ *with* $a(z)\equiv \frac{1}{2}$,

(iv) $(f-1)(g-1)\equiv 1$ *with* $a(z)\equiv 2$,

(v) $$f(z)=\frac{e^{\int a(z)\gamma'(z)\,dz}-1}{e^{\gamma(z)}-1},\quad g(z)=\frac{e^{-\int a(z)\gamma'(z)\,dz}-1}{e^{-\gamma(z)}-1},$$

where $\gamma(z)$ is a nonconstant entire function, and $a(z)\not\equiv -1,\frac{1}{2},2$.

By Theorem 1, we can prove the following results which generalizes Theorem C and improves Theorem D.

Theorem 2. *Let two nonconstant meromorphic functions f and g share* 0, 1, ∞ *CM. If there exists a small entire function $a(z)(\not\equiv 0,1)$ of f and g such that $g(z)-a(z)=0$ whenever $f(z)\overset{(p)}{=}a(z)$ for $p=1,2$. Then f and g must satisfy one of the following ten cases.*

(i) $f\equiv g$, (ii) $f\equiv ag$, *where* $a(z)(\not\equiv -1)$, 1 *are exceptional functions of* f,

(iii) $f-1\equiv(1-a)(g-1)$, *where* $a(z)(\not\equiv 2)$, 0 *are exceptional functions of* f,

(iv) $(f-a)(g-1+a)\equiv a(1-a)$, *where* $a(z)\left(\not\equiv \frac{1}{2}\right)$, ∞ *are exceptional functions of* f, (v) $f\equiv -g$ *with* $a(z)\equiv -1$, (vi) $f+g\equiv 2$ *with* $a(z)\equiv 2$,

(vii) $\left(f-\frac{1}{2}\right)\left(g-\frac{1}{2}\right)\equiv\frac{1}{4}$ *with* $a(z)\equiv\frac{1}{2}$, (viii) $f\cdot g\equiv 1$ *with* $a(z)\equiv -1$,

(ix) $(f-1)(g-1)\equiv 1$ *with* $a(z)\equiv 2$, (x) $f+g\equiv 1$ *with* $a(z)\equiv\frac{1}{2}$.

Remark 2. In Theorem 2, if f and g share $a(z)$ "IM", then the cases (ii)–(iv) of the conclusion of Theorem 2 cannot occur. Since under these three cases, we always have $\overline{N}\left(r, \frac{1}{f-a}\right) = S(r,f)$. Suppose that f and g satisfy the case (ii), that is to say, $f \equiv ag$ with $a(z) \not\equiv -1$. This and Lemma 2 which will be stated in the next section imply that

$$N\left(r, \frac{1}{f-a^2}\right) \le 2\overline{N}\left(r, \frac{1}{f-a^2}\right) + S(r,f) = 2\overline{N}\left(r, \frac{1}{ag-a^2}\right) + S(r,f)$$
$$\le 2\overline{N}\left(r, \frac{1}{g-a}\right) + S(r,f) = 2\overline{N}\left(r, \frac{1}{f-a}\right) + S(r,f) = S(r,f),$$

which means that a^2 is also an exceptional function of f. But in the case (ii), f has already two exceptional functions a and 1, thus we must have $a^2 \equiv a$ or $a^2 \equiv 1$. Since $a \not\equiv 0, 1$, so we obtain $a \equiv -1$, a contradiction. Similarly, if f and g satisfy the case (iii) or (iv), then we can deduce that $a(2-a)$ or $a^2/(2a-1)$ is an exceptional function of f, and then get a contradiction.

2. Lemmas

Let k be a positive integer, and let $a \in \mathbf{C} \cup \{\infty\}$. We denote by $N_{(k+1}(r,a,f)$ the counting function of those a-points of f with multiplicity at least $k+1$, and write $N_{k)}(r,a,f) = N(r,a,f) - N_{(k+1}(r,a,f)$.

Lemma 1 (see [7, Lemma 2]). *Let f and g be two nonconstant meromorphic functions that share* 0, 1, ∞ *CM. If there exists a small function $a(z)(\not\equiv 0,1,\infty)$ of f and g such that $T(r,f) \neq N\left(r, \frac{1}{f-a}\right) + S(r,f)$, then one of the following cases holds.*

(i) $f \equiv g$, (ii) $f \equiv ag$, *where* $a(z)$, 1 *are exceptional functions of* f,

(iii) $f - 1 \equiv (1-a)(g-1)$, *where* $a(z)$, 0 *are exceptional functions of* f,

(iv) $(f-a)(g-1+a) \equiv a(1-a)$, *where* $a(z)$, ∞ *are exceptional functions of* f.

Lemma 2 (see [7, Lemma 5]). *Let f and g be two nonconstant meromorphic functions that share* 0, 1, ∞ *CM. If $f \not\equiv g$, then for any small function $a(z)(\not\equiv 0,1,\infty)$ of f and g, we have*

$$N_{(3}(r,a,f) + N_{(3}(r,a,g) = S(r,f).$$

Lemma 3 (see [1] or [17, p. 77, Theorem 1.52]). *Let $f_j(z)$, $g_j(z)$ $(j = 1,\dots,n)$ be two groups of entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$, *and*

(ii) $\rho(f_j) < \rho(e^{g_h-g_k})$ *for* $1 \le j \le n$ *and* $1 \le h < k \le n$, *where* $\rho(f_j)$ *and* $\rho(e^{g_h-g_k})$ *denote the orders of growth of $f_j(z)$ and $e^{g_h(z)-g_k(z)}$ respectively. Then $f_j(z) \equiv 0$ for $j = 1,\dots,n$.*

LEMMA 4 (see [19, Lemma 6]). *Let* f_1, f_2 *be nonconstant meromorphic functions satisfying* $\overline{N}(r, f_j) + \overline{N}\left(r, \dfrac{1}{f_j}\right) = S(r)$ *for* $j = 1, 2$. *If* $\overline{N}(r, f_1 = 1 = f_2) \neq S(r)$, *then there exist two integers* s, t *satisfying* $|s| + |t| > 0$ *such that* $f_1^s \cdot f_2^t \equiv 1$, *where* $S(r) = o(T(r))$ $(r \to +\infty, r \notin E)$, $T(r) = T(r, f_1) + T(r, f_2)$, *and* E *denotes a set of* r *of finite linear measure.*

3. The Proof of Theorem 1

We suppose first that $f \not\equiv g$. Since f and g share 0, 1, ∞ CM, by the second fundamental theorem due to R. Nevanlinna, we have

$$(3.1)\qquad (1 + o(1))T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right)$$
$$\leq N(r, g) + N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) \leq (3 + o(1))T(r, g).$$

Similarly, we obtain

$$(3.2)\qquad (1 + o(1))T(r, g) \leq (3 + o(1))T(r, f).$$

From (3.1) and (3.2), it follows that

$$(3.3)\qquad S(r,f) = S(r, g).$$

Set

$$(3.4)\qquad \varphi := \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}.$$

If $\varphi \not\equiv 0$, then from (3.3), (3.4), the fundamental estimate of the logarithmic derivative, and the hypothesis that f and g share 0, 1, ∞ CM, we have

$$(3.5)\qquad T(r, \varphi) = S(r, f) + S(r, g) = S(r, f).$$

Since f and g share 0, 1, ∞ CM, thus by (3.4) and (3.5) we deduce that

$$N(r, f = a = g) \leq N(r, 1/\varphi) + S(r, f) \leq T(r, \varphi) + S(r, f) = S(r, f),$$

which contradicts the hypothesis of Theorem 1. Hence, we must have $\varphi \equiv 0$, namely

$$(3.6)\qquad \frac{f'(f-a)}{f(f-1)} \equiv \frac{g'(g-a)}{g(g-1)}.$$

Noting that f and g share 0, 1, ∞ CM, thus there exist two entire functions α and β such that

$$(3.7)\qquad \frac{f}{g} = e^{\alpha}, \quad \frac{f-1}{g-1} = e^{\beta}.$$

Since $f \not\equiv g$, by (3.7) we can deduce that $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ and $e^{\beta-\alpha} \not\equiv 1$. Set $\gamma := \beta - \alpha$, then from (3.7) we have

$$f = \frac{e^{\beta} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-\beta} - 1}{e^{-\gamma} - 1}. \tag{3.8}$$

Rewriting (3.6) as

$$(1 - a)\left(\frac{f'}{f - 1} - \frac{g'}{g - 1}\right) \equiv a\left(\frac{g'}{g} - \frac{f'}{f}\right). \tag{3.9}$$

By (3.7) and the fact that $\alpha = \beta - \gamma$, we obtain

$$\frac{f}{g} = e^{\beta-\gamma}, \quad \frac{f - 1}{g - 1} = e^{\beta}, \tag{3.10}$$

from (3.10), it follows that

$$\frac{f'}{f} - \frac{g'}{g} = \beta' - \gamma', \quad \frac{f'}{f - 1} - \frac{g'}{g - 1} = \beta'. \tag{3.11}$$

Substitution (3.11) into (3.9) gives

$$\beta' \equiv a\gamma'. \tag{3.12}$$

From (3.8) and (3.12), we have

$$f = \frac{e^{\int a\gamma'} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-\int a\gamma'} - 1}{e^{-\gamma} - 1}. \tag{3.13}$$

We now claim that $[a(z) + 1]\left[a(z) - \frac{1}{2}\right][a(z) - 2] \equiv 0$ if and only if f and g satisfy one of the cases (ii)–(iv) of the conclusion of Theorem 1, and thus f is a Möbius transformation of g, where f and g are defined by (3.13).

In fact, suppose that there exist four finite complex numbers c_j $(j = 1, 2, 3, 4)$ such that $f = \dfrac{c_1 g + c_2}{c_3 g + c_4}$, where $c_1 c_4 \neq c_2 c_3$. By this and (3.13) we obtain

$$\begin{aligned} 2c_3 + c_4 - 2c_2 - c_1 = {} & c_1 e^{\gamma - \int a\gamma'} + (c_3 - c_1)e^{-\int a\gamma'} + (c_3 + c_4)e^{\int a\gamma'} \\ & - c_4 e^{-\gamma + \int a\gamma'} - (c_1 + c_2)e^{\gamma} + (c_4 - c_2)e^{-\gamma}. \end{aligned} \tag{3.14}$$

We note first that γ is not a constant. Otherwise, from (3.12) we know that β is also a constant, and thus by (3.8) we can deduce that f is a constant, a contradiction. So from this and the fact that $a(z) \not\equiv 0, 1$, we can derive that both $\gamma - \int a\gamma'$ and $\int a\gamma'$ are not constants. Thus, by applying Lemma 3 to (3.14), and noting the fact that $c_1 c_4 \neq c_2 c_3$ and that $a(z) \not\equiv 0, 1$, we find that one of the following cases holds.

(i) $\gamma - \int a\gamma' - \int a\gamma' \equiv$ constant, that is $a(z) \equiv \frac{1}{2}$,
(ii) $\gamma - \int a\gamma' + \gamma \equiv$ constant, that is $a(z) \equiv 2$, and
(iii) $-\int a\gamma' - \gamma \equiv$ constant, that is $a(z) \equiv -1$.

Otherwise, i.e., the above three cases are all not true, then by (3.14) and Lemma 3 we can deduce that $c_j = 0$ for $j = 1, 2, 3, 4$, which is impossible.

On the other hand, if $a(z) \equiv \frac{1}{2}$, then from (3.12) we have $\gamma \equiv 2\beta + c$, where c is a constant. Thus, by (3.7) and the fact that $\alpha = \beta - \gamma$, it follows that

$$\frac{g}{f} \equiv e^{\gamma-\beta} \equiv e^{\beta+c} \equiv e^c \frac{f-1}{g-1}. \tag{3.15}$$

From the hypothesis of Theorem 1 that $N(r, f = a = g) \neq S(r)$, we can deduce that there exists a point z_0 such that $f(z_0) = g(z_0) = a(z_0)$ $(\neq 0, 1)$, which and (3.15) implies that $e^c = 1$, and thus we obtain from (3.15) that $(g - f)(g + f - 1) \equiv 0$, that is $f + g \equiv 1$. Similarly, if $a(z) \equiv -1$ or $a(z) \equiv 2$, then from (3.7), (3.12), the fact $\alpha = \beta - \gamma$, and the hypothesis of Theorem 1, we can also deduce that $f \cdot g \equiv 1$ or $(f - 1)(g - 1) \equiv 1$, respectively. This proves Theorem 1.

4. The Proof of Theorem 2

We suppose first that $f \not\equiv g$. Otherwise, we have done in this case. Since $a(z)$ is a small function of f and g, from Lemma 2 and the hypothesis of Theorem 2, we have

$$N\left(r, \frac{1}{f-a}\right) = N_{2)}\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{4.1}$$

Form (4.1) and the fact that $g(z) = a(z)$ whenever $f(z) \overset{(p)}{=} a(z)$ for $p = 1, 2$, it follows that

$$\overline{N}(r, f = a = g) = \overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{4.2}$$

We shall divide our argument into two cases.

Case 1. $\overline{N}\left(r, \frac{1}{f-a}\right) = S(r, f)$.

From (4.1) we have

$$N\left(r, \frac{1}{f-a}\right) \le 2\overline{N}\left(r, \frac{1}{f-a}\right) + S(r, f) = S(r, f). \tag{4.3}$$

By (4.3) and Lemma 1, we know that f and g must assume one of the forms (ii)–(vii) of the conclusion of Theorem 2.

Case 2. $\overline{N}\left(r, \frac{1}{f-a}\right) \neq S(r, f)$.

From (4.2) we see that $N(r, f = a = g) \neq S(r, f)$. Therefore, by Theorem 1, it follows that f and g satisfy one of the forms (viii)–(x) of the conclusion of Theorem 2, unless

$$f = \frac{e^{\int a\gamma'} - 1}{e^{\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-\int a\gamma'} - 1}{e^{-\gamma} - 1}, \tag{4.4}$$

where γ is a nonconstant entire function and $a(z) \not\equiv -1, 1/2, 2$.

Since $\overline{N}\left(r, \dfrac{1}{f-a}\right) \neq S(r,f)$, from (4.1) we know that there exist a point z_0 and a positive integer $p(\leq 2)$ such that $f(z_0) \overset{(p)}{=} a(z_0)(\neq 0,1)$. Thus from the hypothesis of Theorem 2, we have $g(z_0) = a(z_0)$. For simplicity, we write

$$\delta(z) := e^{\int a(z)\gamma'(z)\,dz}, \tag{4.5}$$

by (4.4) and (4.5), it follows that

$$\delta(z_0) - a(z_0)e^{\gamma(z_0)} + a(z_0) - 1 = 0, \tag{4.6}$$

and

$$\frac{1}{\delta(z_0)} - a(z_0)\frac{1}{e^{\gamma(z_0)}} + a(z_0) - 1 = 0. \tag{4.7}$$

In view of $a(z_0) \neq 0,1$, by (4.6) and (4.7) we obtain

$$\delta(z_0) = 1, \quad e^{\gamma(z_0)} = 1. \tag{4.8}$$

We now assert that $\gamma'(a^2\gamma' - a\gamma' - a') \not\equiv 0$. Otherwise, we have either $\gamma' \equiv 0$ or $a^2\gamma' - a\gamma' - a' \equiv 0$. If $a^2\gamma' - a\gamma' - a' \equiv 0$, namely,

$$\gamma' \equiv \frac{a'}{a-1} - \frac{a'}{a}, \tag{4.9}$$

Integrating (4.9) we have

$$a(z) \equiv \frac{1}{1-ce^{\gamma}}, \tag{4.10}$$

where c is a nonzero constant.

Since $a(z)$ is a small function of f, by (4.10) we obtain $S(r,f) = T(r,a) = T(r,e^{\gamma}) + O(1)$, that is to say,

$$T(r,e^{\gamma}) = S(r,f). \tag{4.11}$$

Moreover, by (4.8) we see that "almost" all the zeros of $f-a$ that have multiplicity at most 2 must be the zeros of $e^{\gamma}-1$. Clearly, $e^{\gamma} \not\equiv 1$. If this is not the case, then γ is a constant, which contradicts the fact that $\gamma' \not\equiv 0$. Hence, by (4.11) and (4.1), we have

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{e^{\gamma}-1}\right) + S(r,f) = T(r,e^{\gamma}) + S(r,f) = S(r,f),$$

which contradicts the hypothesis $\overline{N}\left(r, \dfrac{1}{f-a}\right) \neq S(r,f)$. We now have shown that the above assertion is true.

Next, we shall prove

$$N\left(r, \frac{1}{f-a}\right) = N_{1)}\left(r, \frac{1}{f-a}\right) + S(r,f). \tag{4.12}$$

Suppose that z_0 is a zero of $f-a$ with multiplicity 2, and that $a(z_0) \neq 0, 1$. By (4.4) we have

$$f - a = \frac{e^{\int a\gamma'} - 1 - ae^{\gamma} + a}{e^{\gamma} - 1}, \tag{4.13}$$

Since any zero z_0 of $f-a$ with multiplicity at most 2 satisfying $a(z_0) \neq 0, 1$ is also a zero of $e^{\gamma} - 1$, thus from (4.13) we know that z_0 must be a zero of the function $G := e^{\int a\gamma'} - 1 - ae^{\gamma} + a$ with multiplicity at least 3. Differentiating G twice leads us to

$$G'' = [a'\gamma' + a\gamma'' + (a\gamma')^2]e^{\int a\gamma'} - [a'' + 2a'\gamma' + a\gamma'' + a(\gamma')^2]e^{\gamma} + a''.$$

From this, (4.5) and (4.8), we have

$$\begin{aligned} 0 = G''(z_0) &= a'(z_0)\gamma'(z_0) + a(z_0)\gamma''(z_0) + [a(z_0)\gamma'(z_0)]^2 \\ &\quad - \{a''(z_0) + 2a'(z_0)\gamma'(z_0) + a(z_0)\gamma''(z_0) + a(z_0)[\gamma'(z_0)]^2\} + a''(z_0), \end{aligned}$$

that is

$$\gamma'(z_0)[a^2(z_0)\gamma'(z_0) - a(z_0)\gamma'(z_0) - a'(z_0)] = 0. \tag{4.14}$$

Since we have proved that $\gamma'(a^2\gamma' - a\gamma' - a') \not\equiv 0$, thus from (4.14) and Lemma 2, it follows that

$$\begin{aligned} N_{(2}\left(r, \frac{1}{f-a}\right) &\leq 2N\left(r, \frac{1}{\gamma'(a^2\gamma' - a\gamma' - a')}\right) + S(r, f) \\ &\leq 2T(r, \gamma'(a^2\gamma' - a\gamma' - a')) + S(r, f) = S(r, e^{\gamma}) + S(r, f). \end{aligned} \tag{4.15}$$

Since f and g share 0, 1, ∞ CM, thus (3.2) holds. Moreover, by (4.4) we can obtain that

$$\frac{g(f-1)}{f(g-1)} = e^{\gamma}. \tag{4.16}$$

By (3.2) and (4.16) we deduce that

$$T(r, e^{\gamma}) \leq 2T(r, f) + 2T(r, g) + O(1) \leq (8 + o(1))T(r, f),$$

which when combined with (4.15) gives that $N_{(2}\left(r, \frac{1}{f-a}\right) = S(r, f)$, and so (4.12) holds.

If $T(r, f) \neq N\left(r, \frac{1}{f-a}\right) + S(r, f)$, then by Lemma 1, we deduce that $a(z)$ is an exceptional function of f. This contradicts the hypothesis of the Case 2 that $\bar{N}\left(r, \frac{1}{f-a}\right) \neq S(r, f)$. Therefore, we can suppose that

$$T(r, f) = N\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{4.17}$$

Form (4.12), (4.17) as well as the hypothesis of Theorem 2, we have

$$T(r,f) = N(r,1/f-a) + S(r,f) = N_{1)}(r,1/f-a) + S(r,f) \tag{4.18}$$
$$\leq N(r,1/g-a) + S(r,f) \leq T(r,g) + S(r,f).$$

Next, we shall show that

$$T(r,f) = T(r,g) + O(1). \tag{4.19}$$

Set

$$f_1 := \frac{f}{g}, \quad f_2 := \frac{f-1}{g-1}. \tag{4.20}$$

Since f and g share 0, 1, ∞ CM, thus from (4.20) we see that

$$N(r,f_j) + N(r,1/f_j) = 0 \quad \text{for } j = 1,2. \tag{4.21}$$

Noting that $a(z)$ is a small function of f and g, so from (4.17), (4.12), (4.2) and (4.20), we have

$$T(r,f) = N_{1)}(r,1/f-a) + S(r,f) \leq \overline{N}(r,f=a=g) + S(r,f) \tag{4.22}$$
$$\leq \overline{N}(r,f_1=1=f_2) + S(r,f).$$

From (4.20), (3.2) and the first fundamental theorem due to R. Nevanlinna, we get

$$T(r,f) = T(r,gf_1) \geq T(r,f_1) - T(r,g) + O(1) \tag{4.23}$$
$$\geq T(r,f_1) - 4T(r,f),$$

by (4.23), we obtain

$$T(r,f) \geq \frac{1}{5}T(r,f_1). \tag{4.24}$$

Similarly, from (4.20) and (3.2) we have

$$T(r,f) + O(1) = T(r,f-1) = T(r,(g-1)f_2)$$
$$\geq T(r,f_2) - T(r,g) + O(1) \geq T(r,f_2) - 4T(r,f),$$

i.e.,

$$T(r,f) \geq \frac{1}{5}T(r,f_2). \tag{4.25}$$

From (4.24) and (4.25) we obtain

$$T(r,f) \geq \frac{1}{10}(T(r,f_1) + T(r,f_2)). \tag{4.26}$$

By (4.22) and (4.26) we deduce that

$$\overline{N}(r, f_1 = 1 = f_2) \neq o(T(r, f_1) + T(r, f_2)). \tag{4.27}$$

Now we see that f_1 and f_2 satisfy all the conditions of Lemma 4, and thus there exist two integers s and t satisfying $|s| + |t| > 0$ such that $f_1^s \cdot f_2^t \equiv 1$. This and (4.20) lead to (4.19). Substitution (4.19) into (4.18) gives

$$N_{1)}\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{g-a}\right) + S(r, f). \tag{4.28}$$

Noting that f and g share 0, 1, ∞ CM, thus (3.3) holds. From Lemma 2, (4.12) and (3.3), we have

$$\begin{aligned} N\left(r, \frac{1}{g-a}\right) &= N_{2)}\left(r, \frac{1}{g-a}\right) + S(r) \\ &= N(r, g \overset{(1)}{=} a = f) + N(r, g \overset{(2)}{=} a = f) + N_{2)}(r, g = a \neq f) + S(r) \\ &= N_{1)}\left(r, \frac{1}{f-a}\right) + \overline{N}(r, g \overset{(2)}{=} a = f) + N_{2)}(r, g = a \neq f) + S(r), \end{aligned} \tag{4.29}$$

where $S(r) = S(r, f) = S(r, g)$ and the notation $N(r, g \overset{(k)}{=} a = f)$ for $k = 1, 2$ denotes the counting function of those common zeros of $f - a$ and $g - a$ with multiplicity k for g (each common zero in $N(r, g \overset{(k)}{=} a = f)$ is counted k times), $\overline{N}(r, g \overset{(k)}{=} a = f)$ denotes the corresponding reduced counting function, and $N_{2)}(r, g = a \neq f)$ denotes the counting function of those points satisfying both $f - a \neq 0$ and $g - a = 0$ with multiplicity less than or equal to 2 (each zero z^* of $g - a$ is counted p times if z^* is a zero of $g - a$ with multiplicity p). From (4.28) and (4.29), we have

$$\overline{N}(r, g \overset{(2)}{=} a = f) + N_{2)}(r, g = a \neq f) = S(r). \tag{4.30}$$

By (4.30) and Lemma 2, we can deduce that the counting function of the multiple zeros of $g - a$ is $S(r)$, i.e., $N\left(r, \frac{1}{g-a}\right) = N_{1)}\left(r, \frac{1}{g-a}\right) + S(r)$. Moreover, we can see from (4.30) and Lemma 2 that the counting function of those points satisfying both $g - a = 0$ and $f - a \neq 0$ is also $S(r)$. From this and (4.12) we deduce that f and g share $a(z)$ "CM".

If $a \equiv c_0$, where $c_0 (\neq 0, 1, -1, 1/2, 2)$ is a constant, then we set

$$F_1 := \frac{1}{2c_0 - f}, \quad G_1 := \frac{1}{2c_0 - g}. \tag{4.31}$$

From (4.31) and the hypothesis of Theorem 2, it follows that F_1 and G_1 share $\frac{1}{2c_0}$, $\frac{1}{2c_0 - 1}$, 0 CM, and share $\frac{1}{c_0}$ "CM".

If a is not a constant, then from (4.31) we see that F_1 and G_1 share $\frac{1}{2c_0 - a} (\not\equiv \infty)$ "CM". Now we have shown that F_1 and G_1 always satisfy the hypothesis of Theorem B whether a is a constant or not. Thus from Theorem C we deduce that f and g must assume one of the cases (v)–(x) of the conclusion of Theorem 2. But these are impossible because $a(z) \not\equiv -1, 1/2, 2$. This completes the proof of Theorem 2.

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Jian-Ping Wang
DEPARTMENT OF MATHEMATICS
SHAOXING COLLEGE OF ARTS AND SCIENCES
SHAOXING, ZHEJIANG 312000
P. R. CHINA
E-mail: jpwang604@hotmail.com

Ling-Di Huang
DEPARTMENT OF MATHEMATICS
SHAOXING COLLEGE OF ARTS AND SCIENCES
SHAOXING, ZHEJIANG 312000
P. R. CHINA
E-mail: hld78@zscas.edu.cn